A mathematical family of mazes

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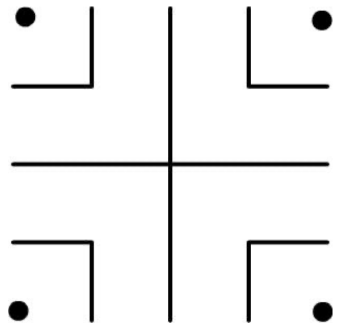
Sigma Camp
August 17, 2023

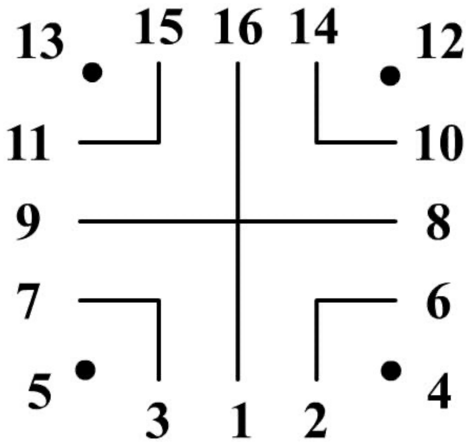
It all starts with a game.

(near the top of the page)

Draw a large plus sign,
- four L-shapes in the corners,
- four dots to make a square.



Number the free ends, including the dots.



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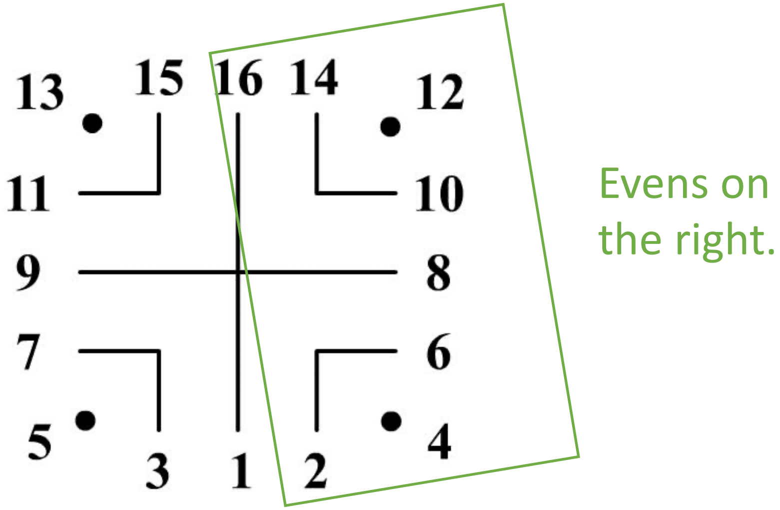

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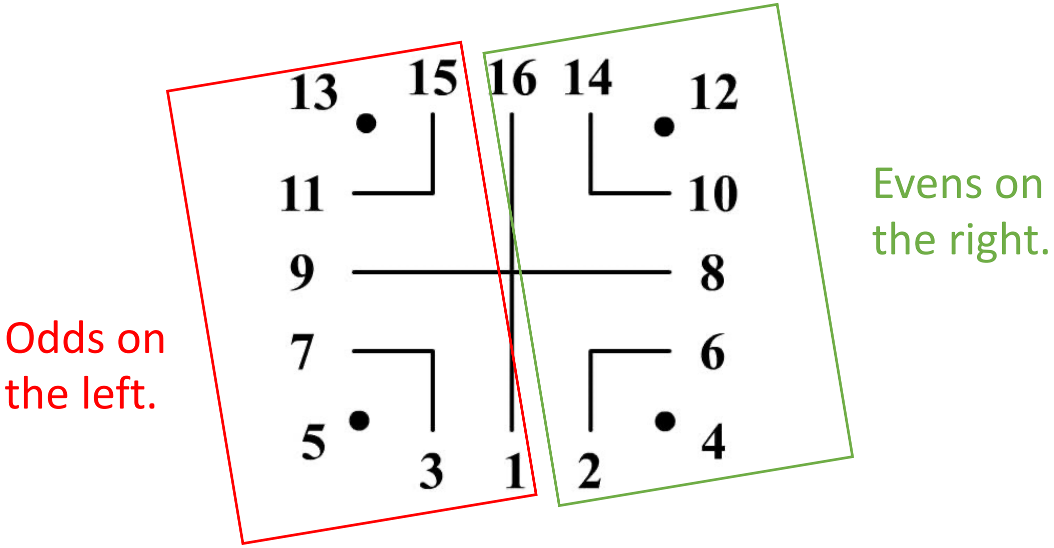
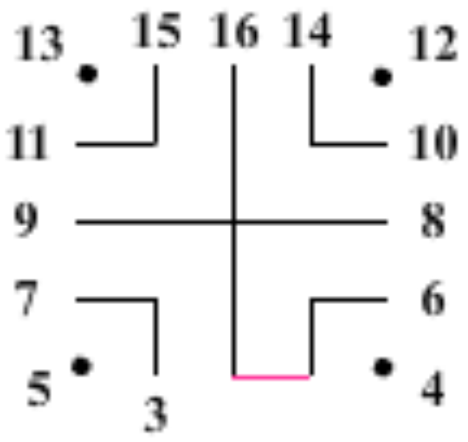
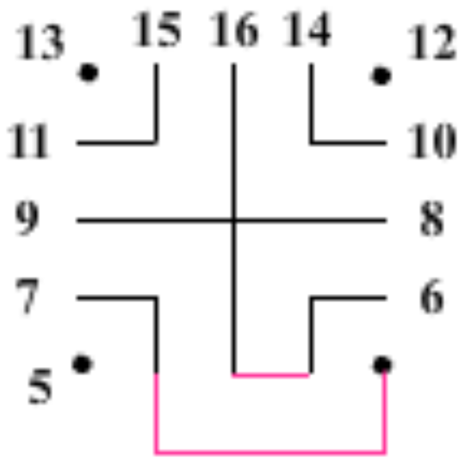


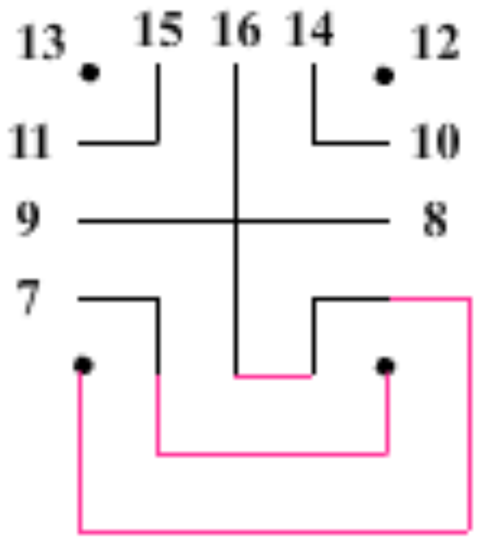
Join end 1 to end 2.



Join end 3 to end 4.

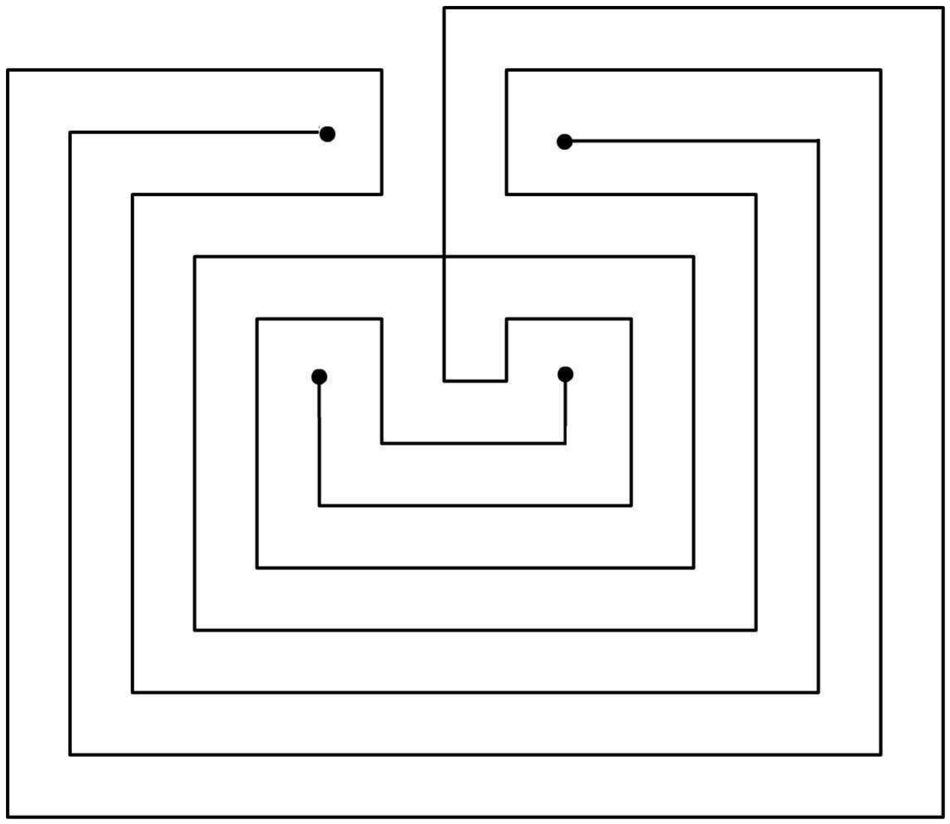


Join end 5 to end 6,
always around the bottom
of the design.



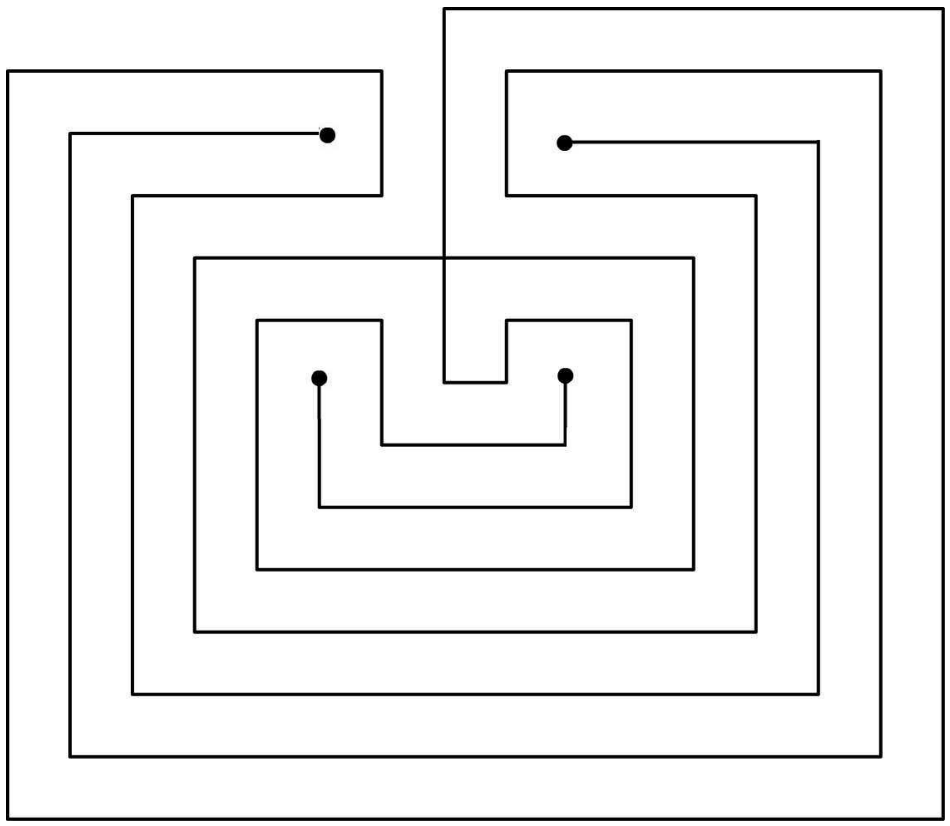
Continue:
join end 7 to end 8,
join end 9 to end 10,
…
join end 15 to end 16,
always around the bottom.

At the end, you should have this design.



At the end, you should have this design.

This design is known as "The Cretan Maze."

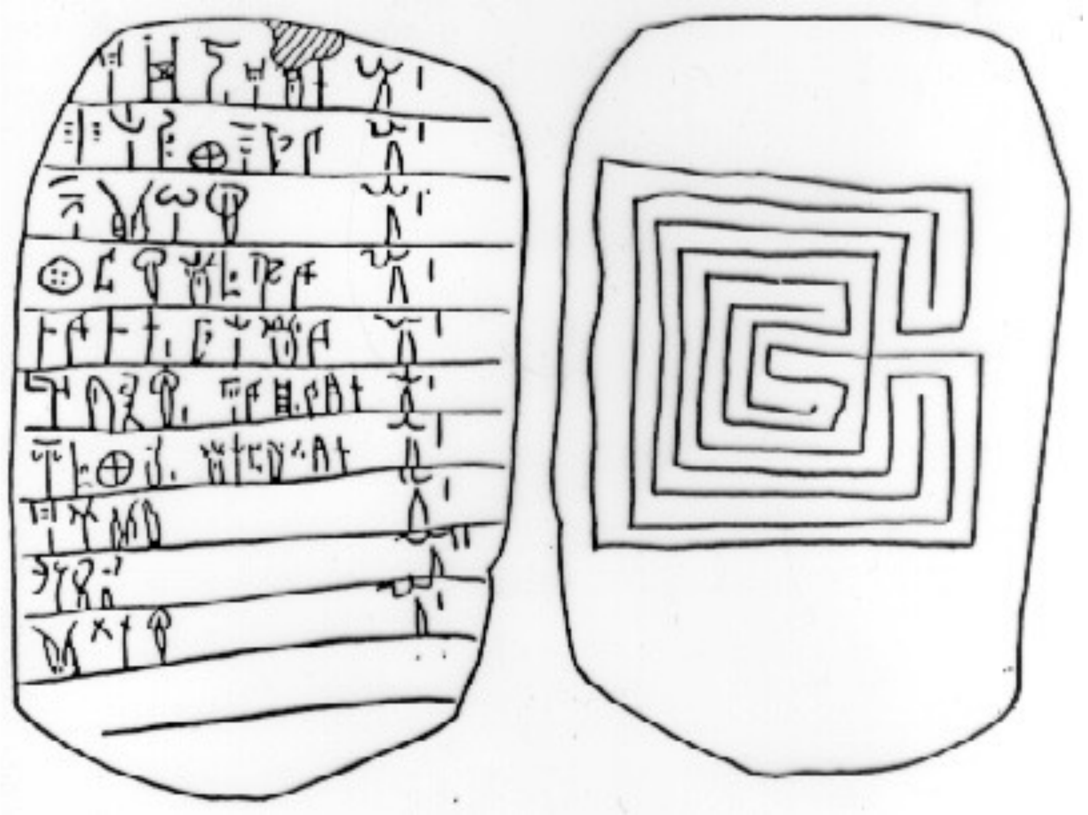


It appears on Cretan coins of the 4^{th} century BC.

ΚΝΩΣΙΩΝ = Knossos, a city in Crete.

It must be very old, because
it appears on the back of
a clay tablet from King Nestor's
Palace in Pylos (western Greece).

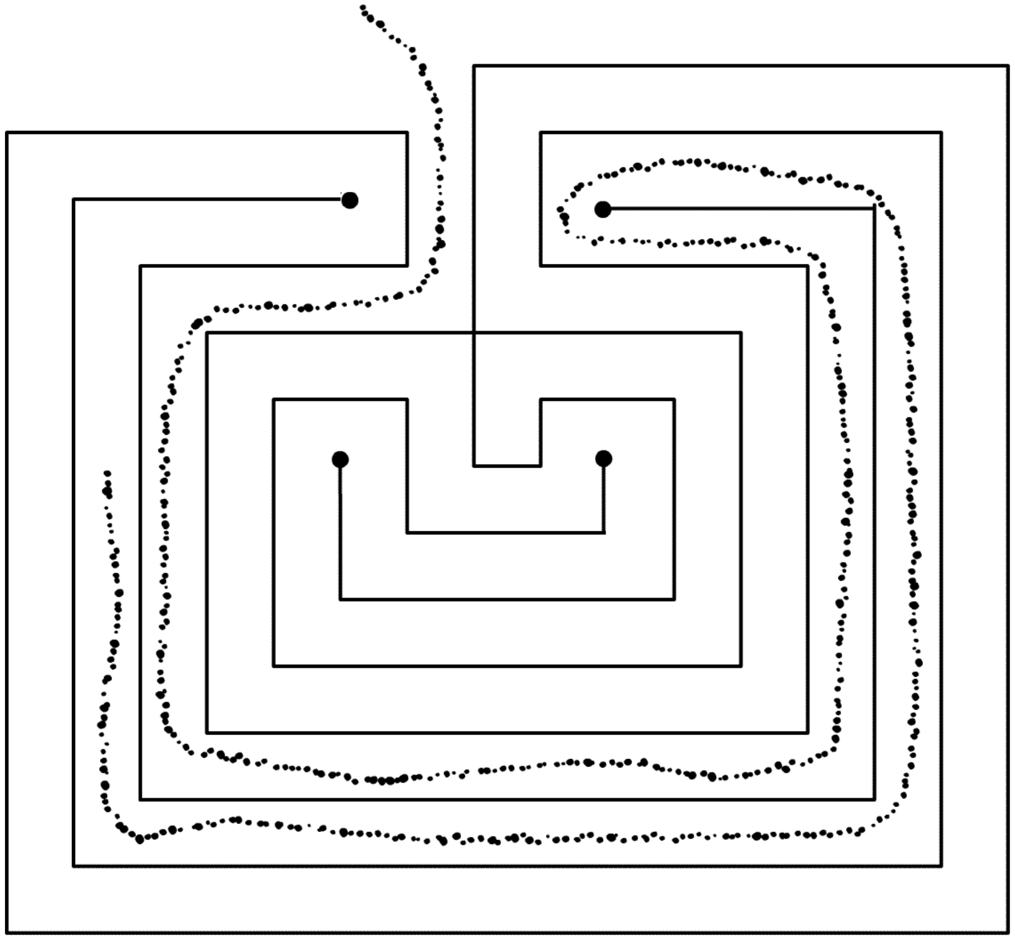
The tablet was baked when
the palace burned down
in 1200 BC.



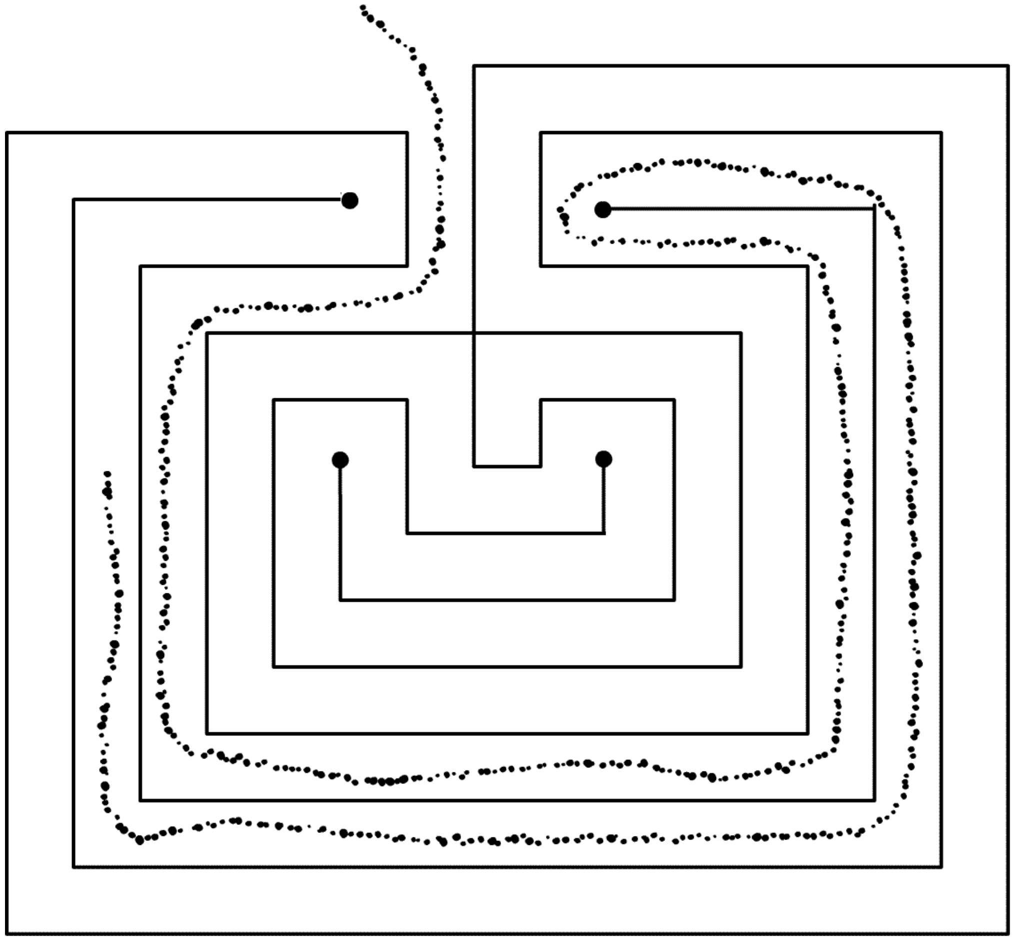
7 x 5.7cm. Text in Linear B lists
people's names and numbers
of goats.

It’s a maze because a path runs from the outside to the center, traversing it completely.

For the Cretans, it was a symbol of the mythical labyrinth where the Minotaur was kept.



Try it!



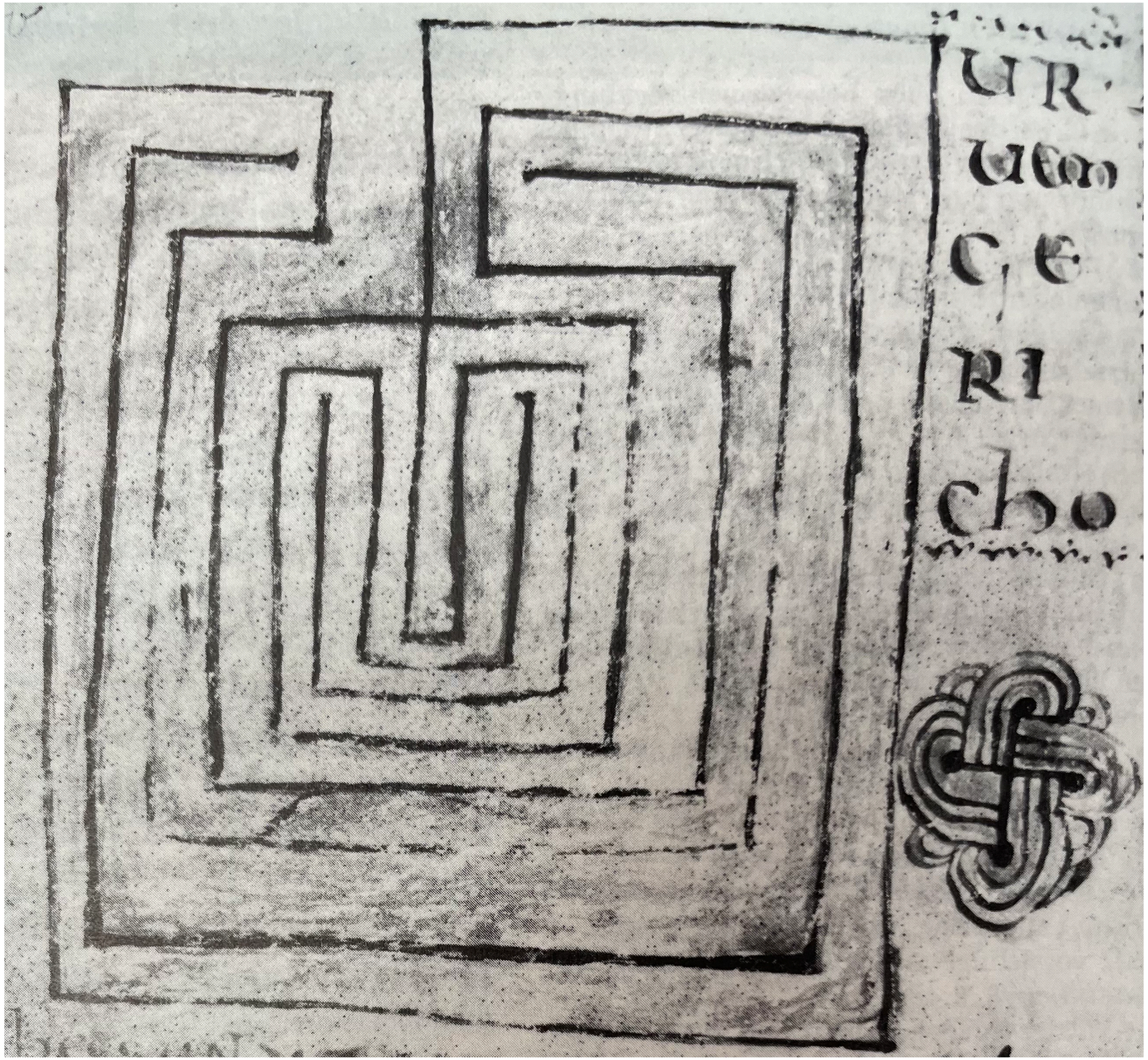
Here it is scratched on a wall in Pompeii.

Presumably shortly before the eruption of Vesuvius in 79 AD.

Text: “The Labyrinth. Here lives the Minotaur.”

Here it is in a medieval manuscript from central Italy, written between 806 and 822 AD.

Text: “The City of Jericho.”

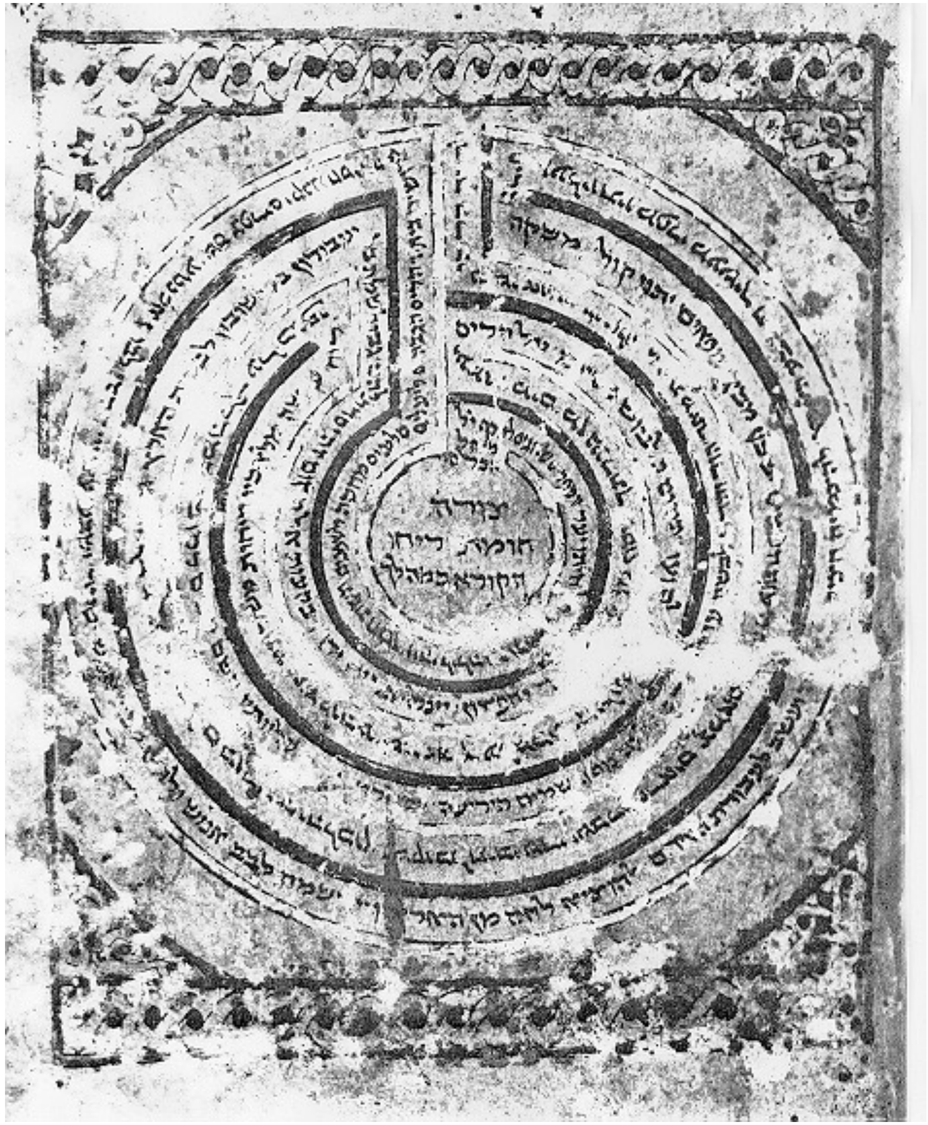


Here's another example.

A page from a medieval *Sefer Haftorot* (a Hebrew Prayer-book).

The words of Psalm 104 run along the maze path.

The center reads: "The image of the wall of Jericho. The reader is as if walking."



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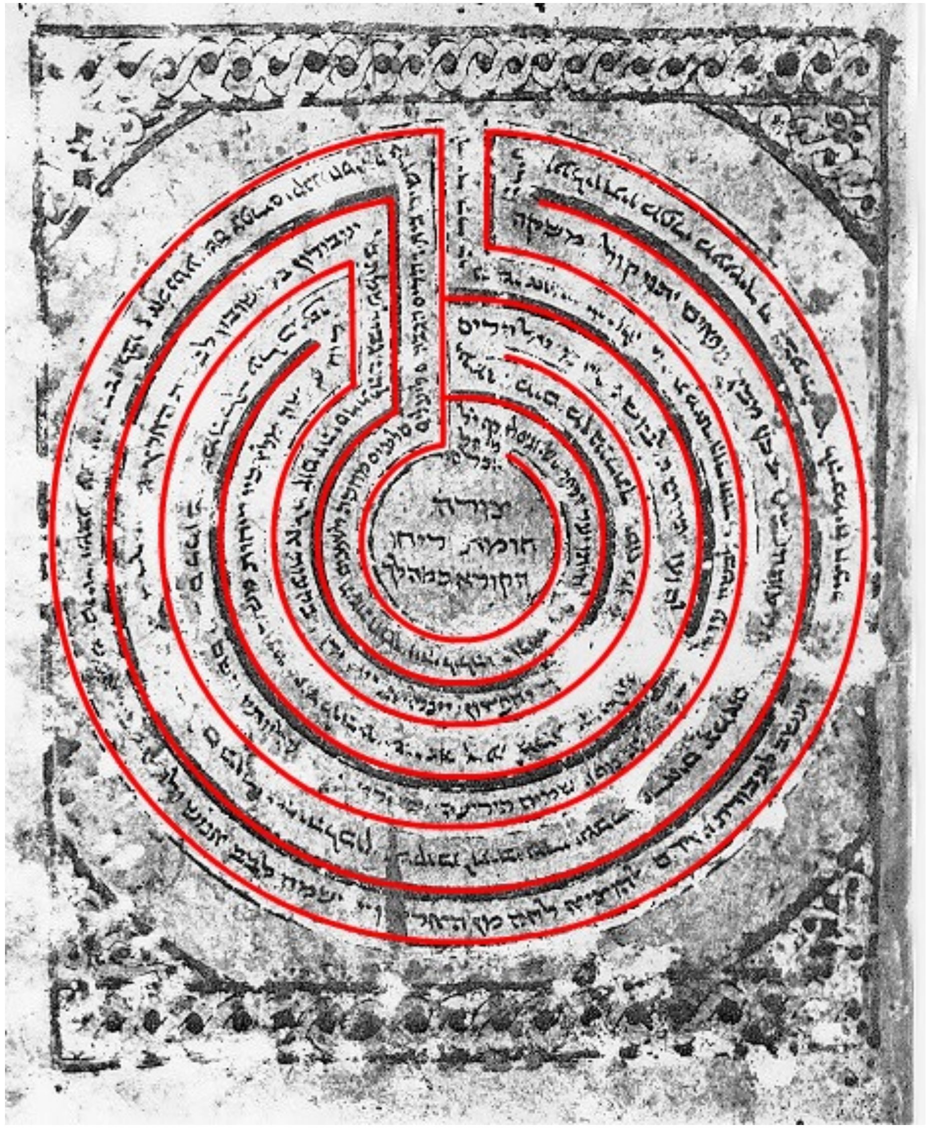
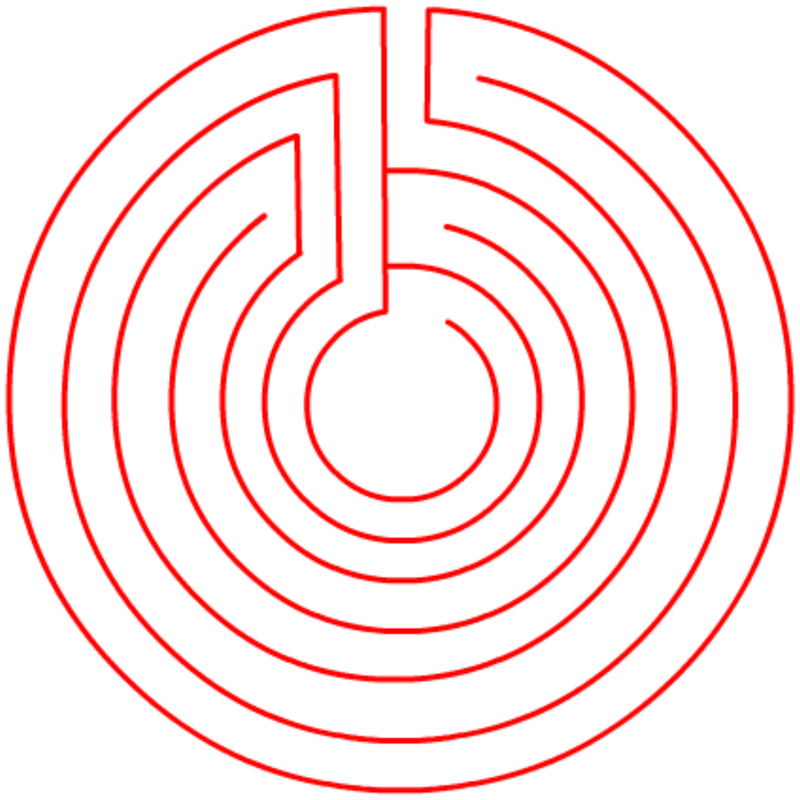
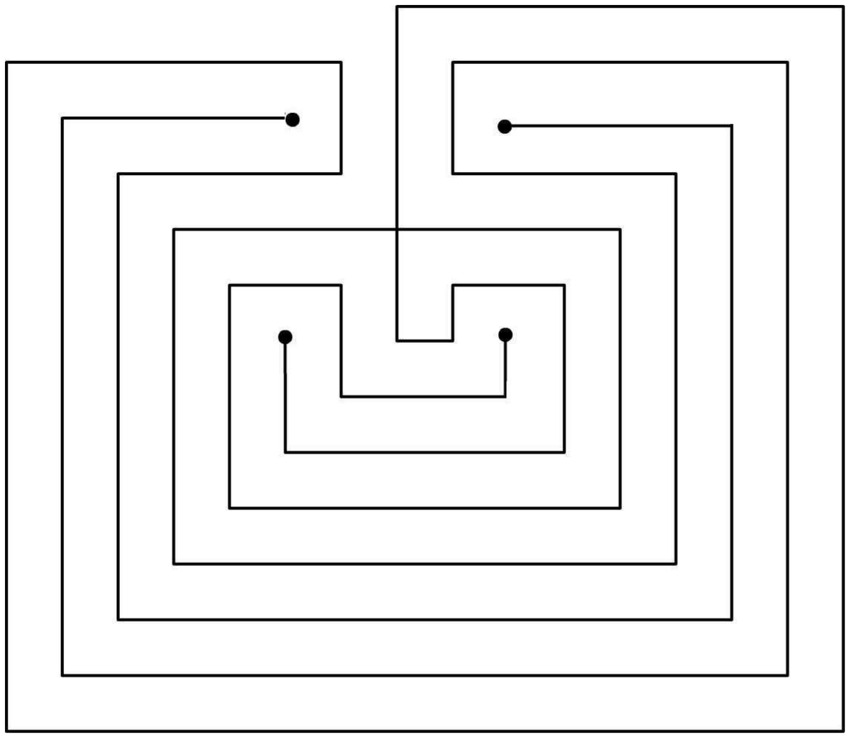
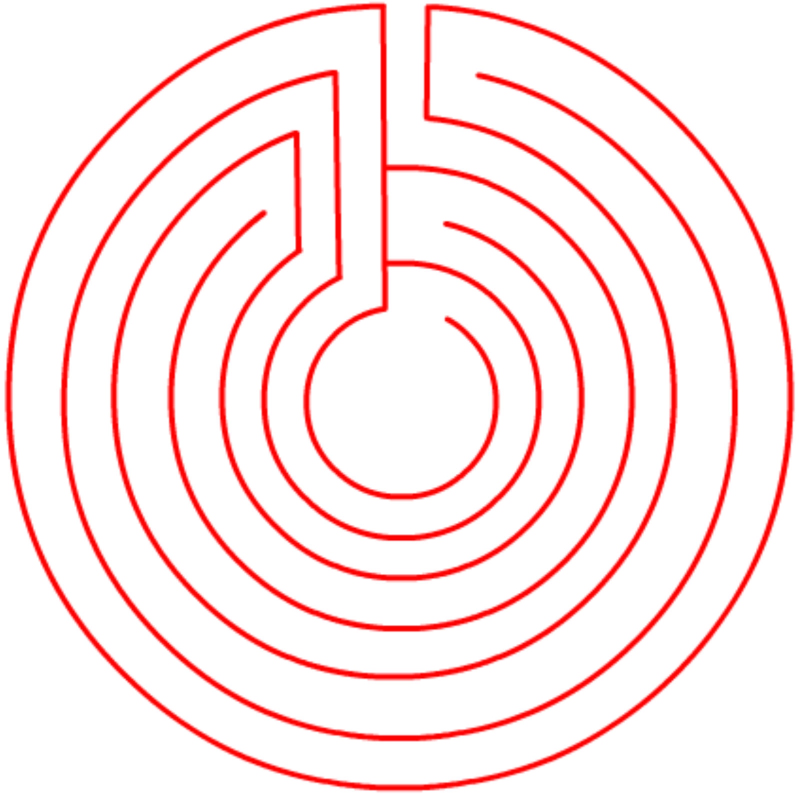
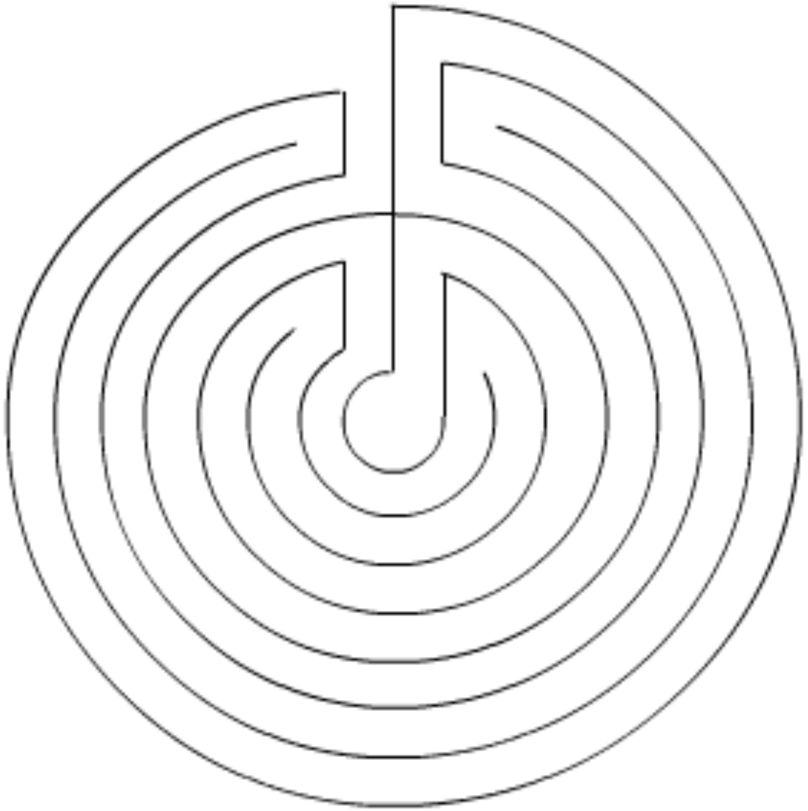
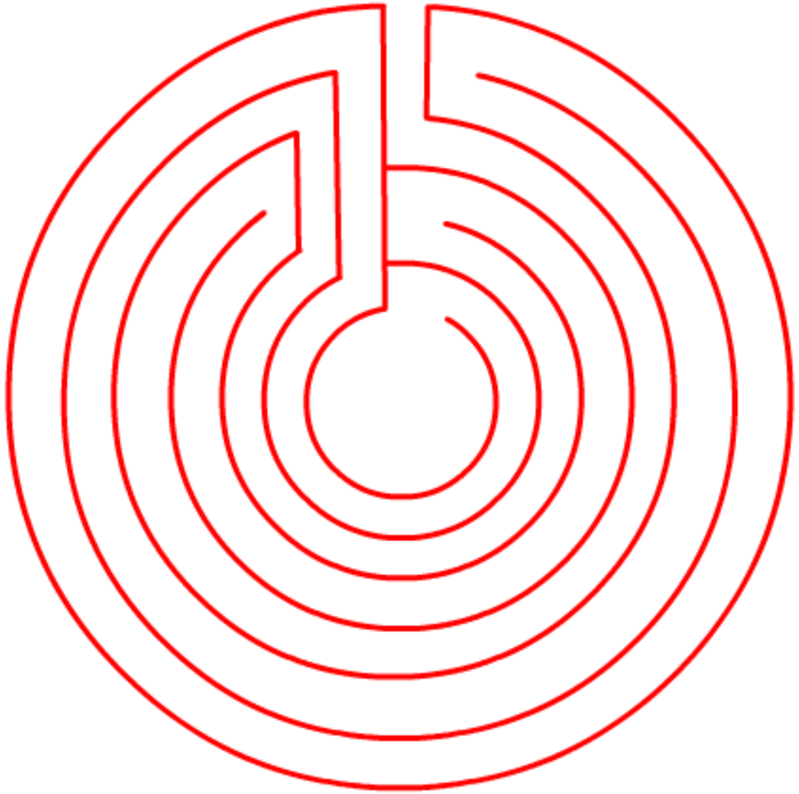




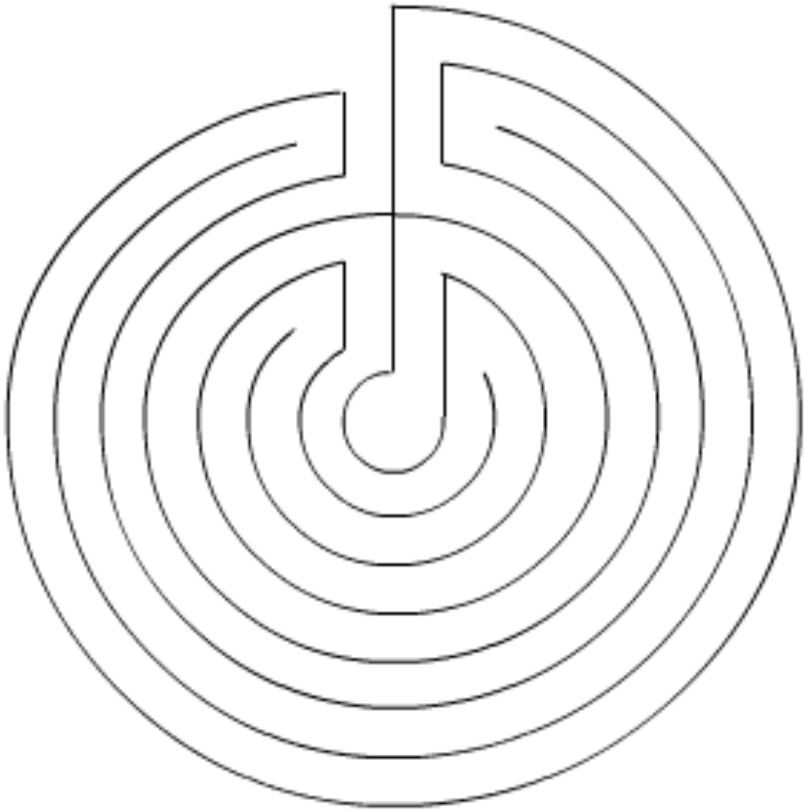
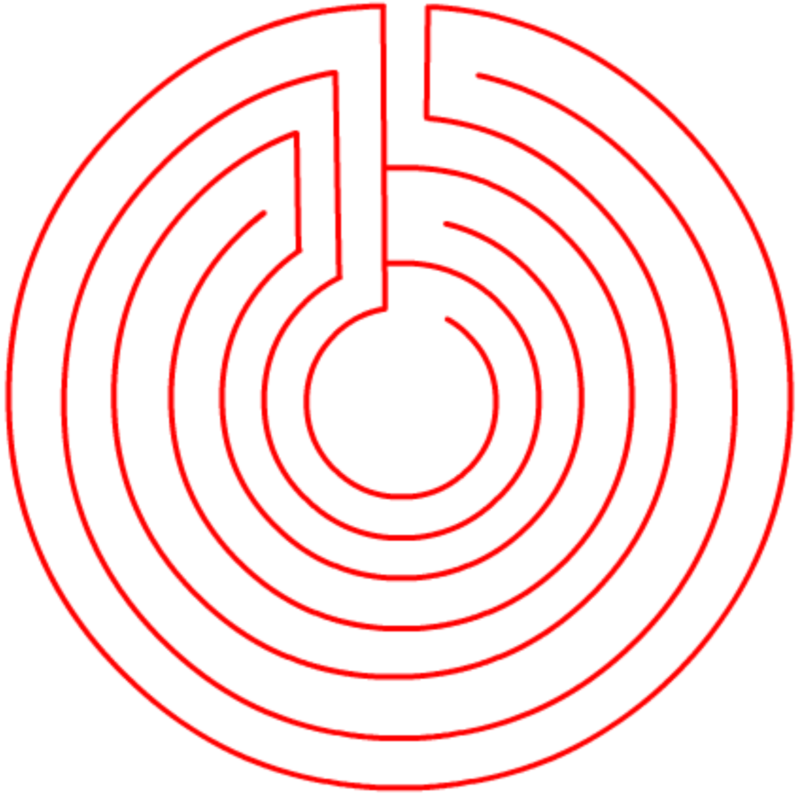
 

Cretan maze and Haftorot maze.

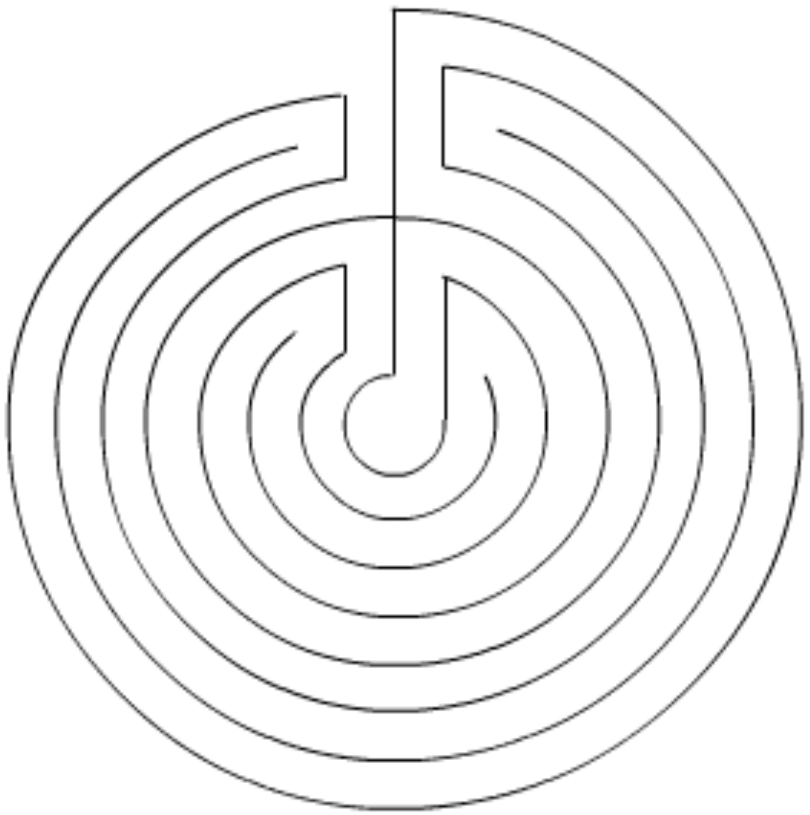


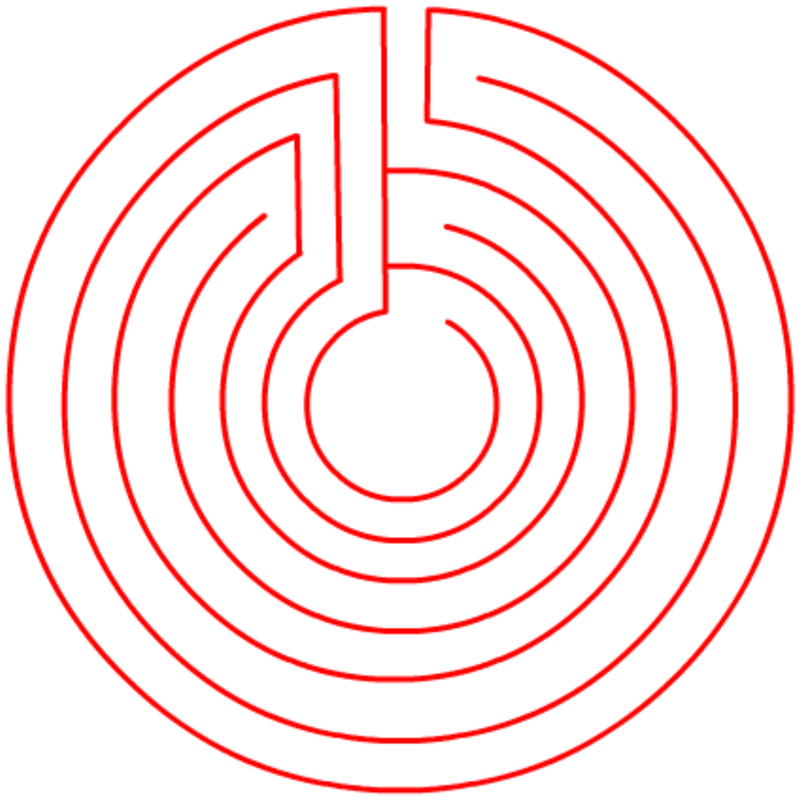


Cretan maze and Haftorot maze.
Redraw Cretan maze in circular form.

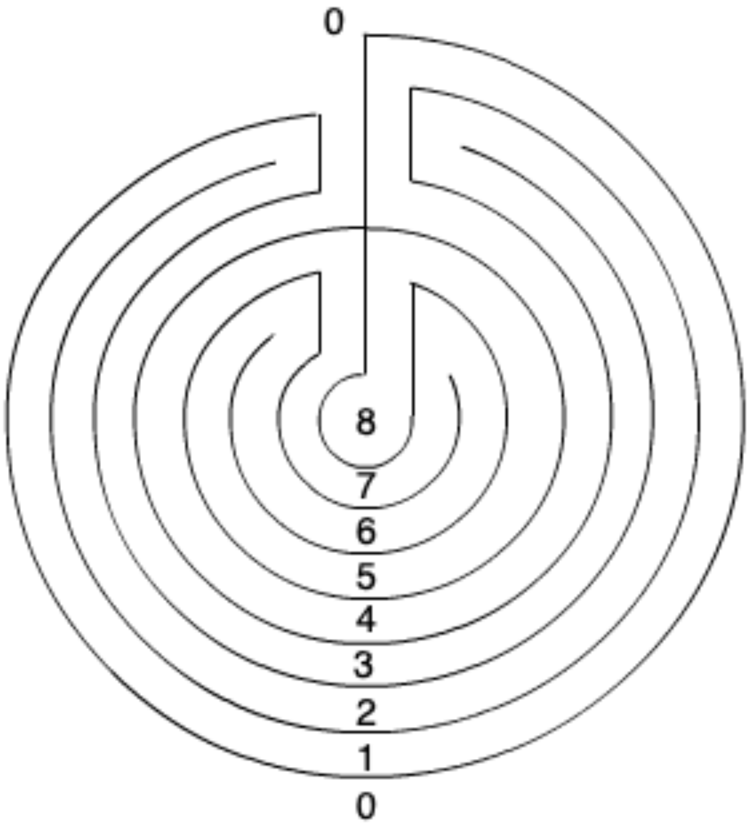
Cretan maze and Haftorot maze. Can there be others?

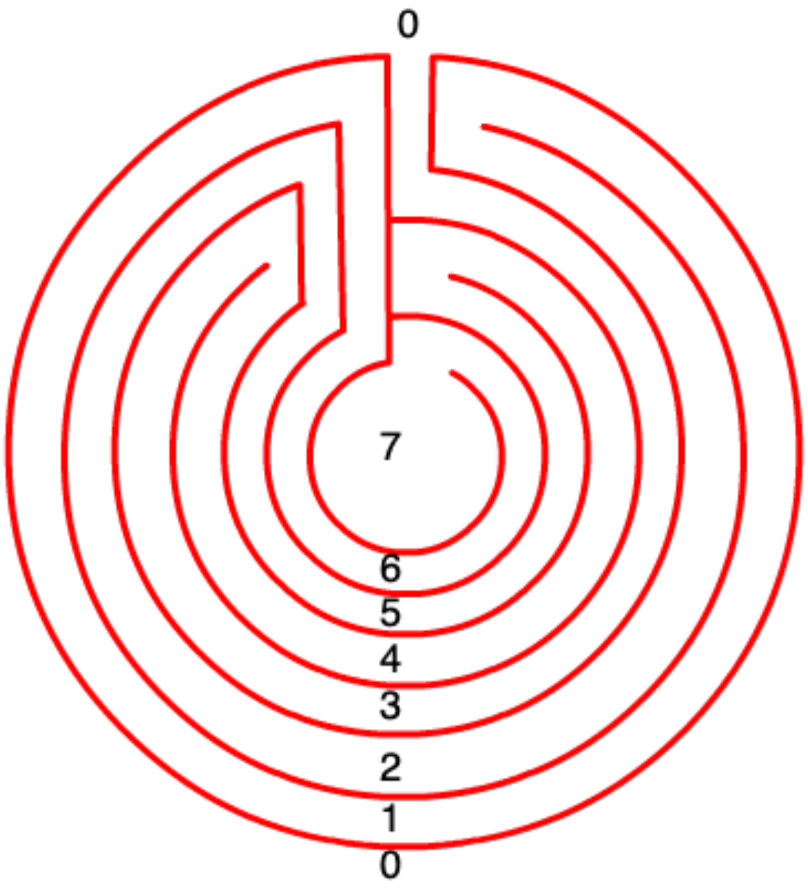




How are they similar?

1. Each has a single path leading from the outside to the center.
2. Each path traverses a number of concentric *levels*.
3. Each level is reached exactly once; level changes only occur at a central axis.
4. Path changes direction at each level change.

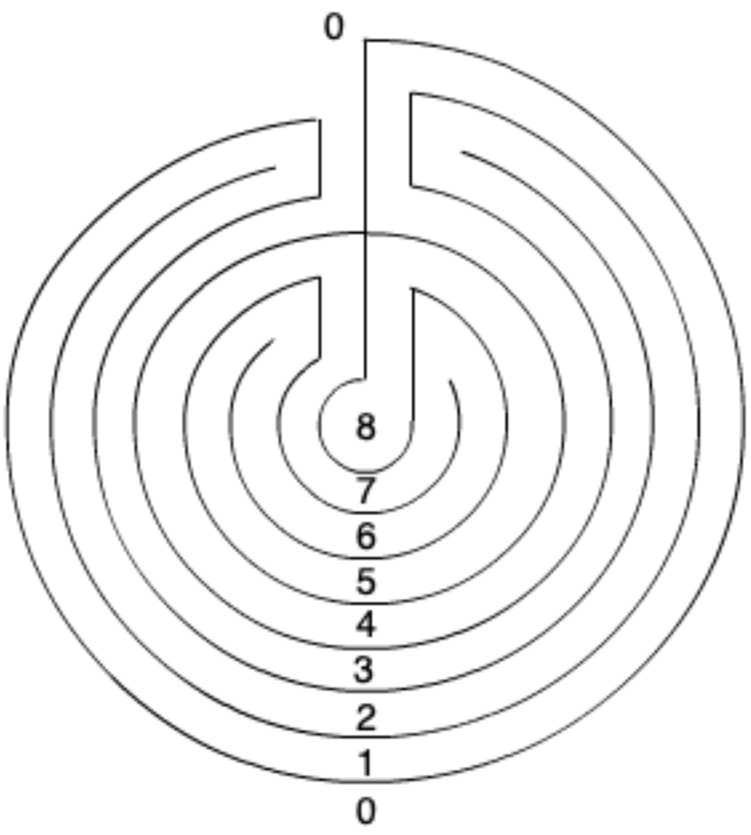


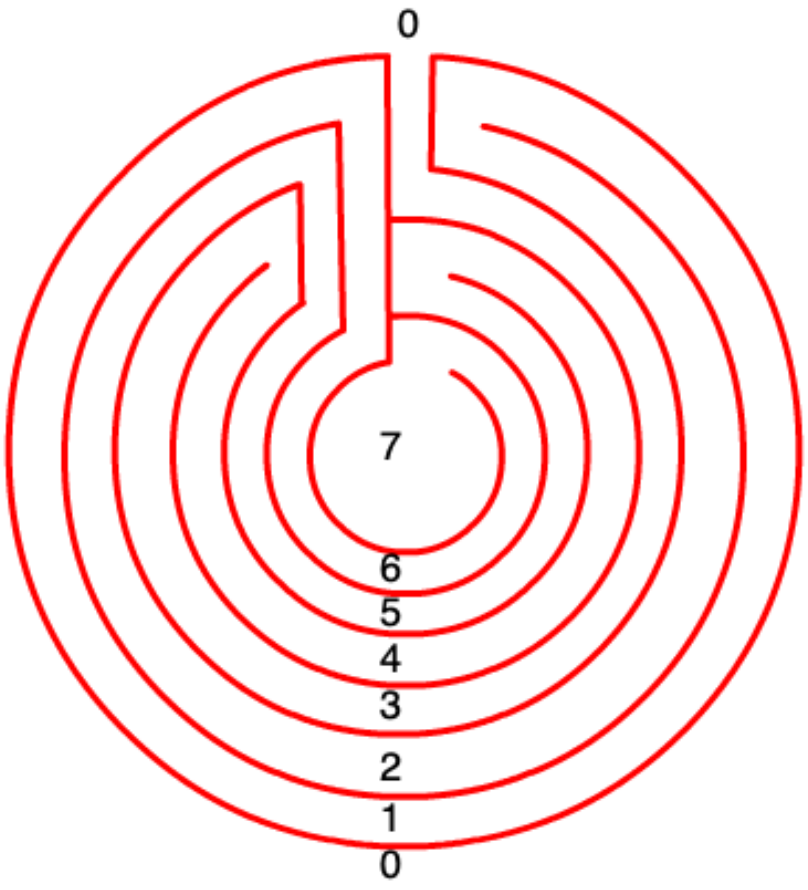


Number the levels. Then each maze has a *level sequence:* the list of level numbers you meet along the maze path.

Cretan maze level sequence is

Haftorot level sequence is





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Cretan maze level sequence is 0 3 2 1 4 7 6 5 8,

Haftorot level sequence is 0 3 4 5 2 1 6 7.

What the Cretan maze and the Haftorot maze had in common:

1. Each has a single path leading from the outside to the center.
2. Each path traverses a number of concentric *levels*.
3. Each level is reached exactly once; level changes only occur at a central axis.
4. Path changes direction at each level change.

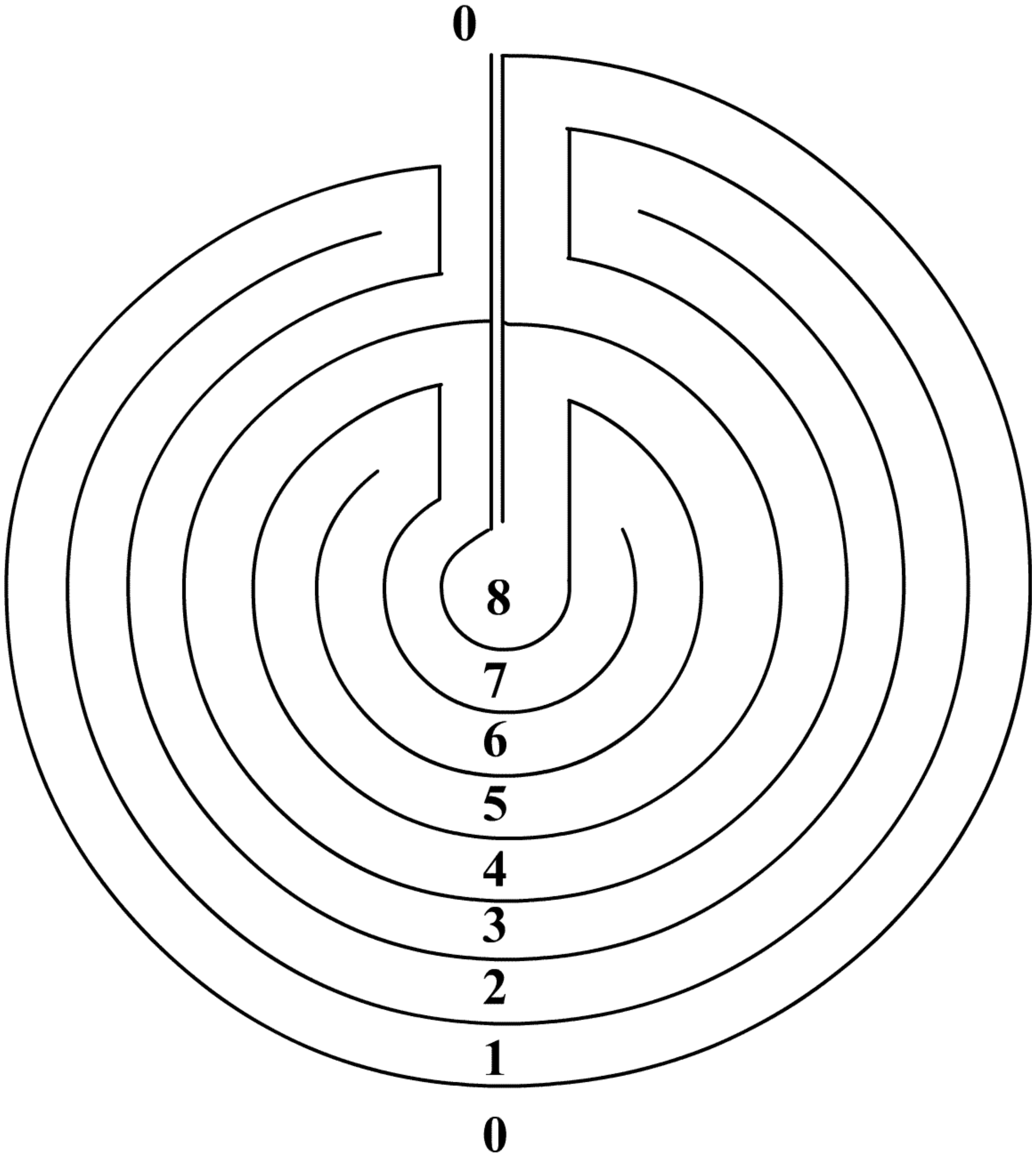
We will call a maze satisfying properties 1-4 a *simple, alternating transit maze*, or *SAT maze*. We can prove:

The topology of an SAT maze is entirely determined by its level sequence.

(If two SAT mazes have the same level sequence, there is a continuous. level-preserving deformation taking one to the other.)

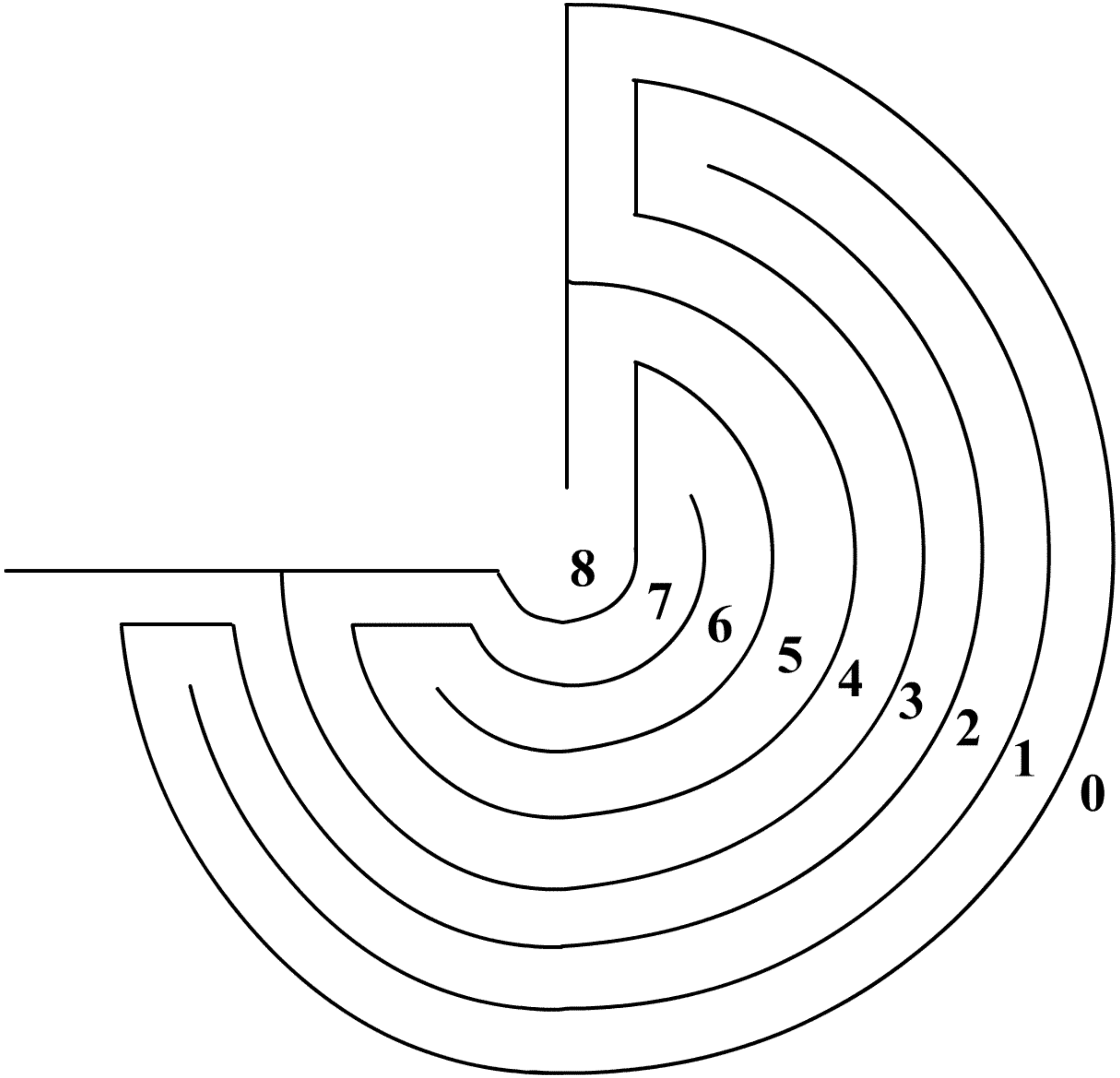
An SAT maze can be put into *rectangular form*.

1. Split the central axis.



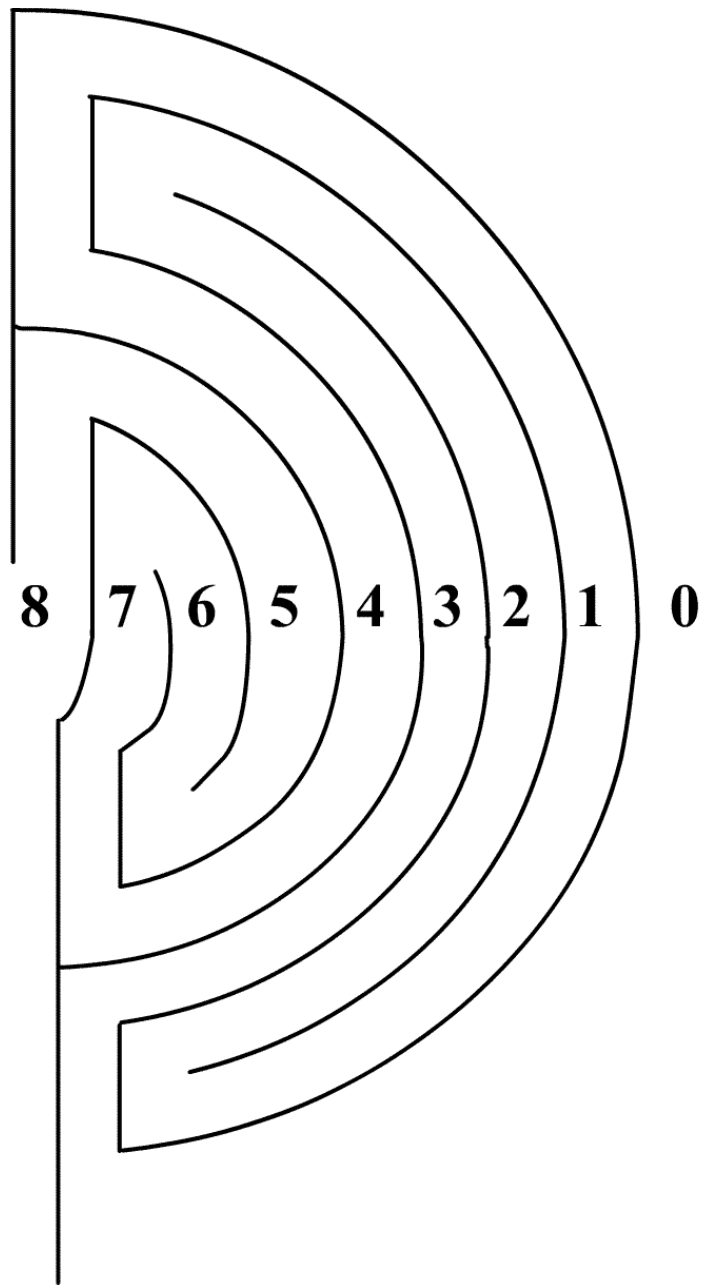
An SAT maze can be put into *rectangular form*.

1. Split the central axis.
2. Unroll the left side: 90^o



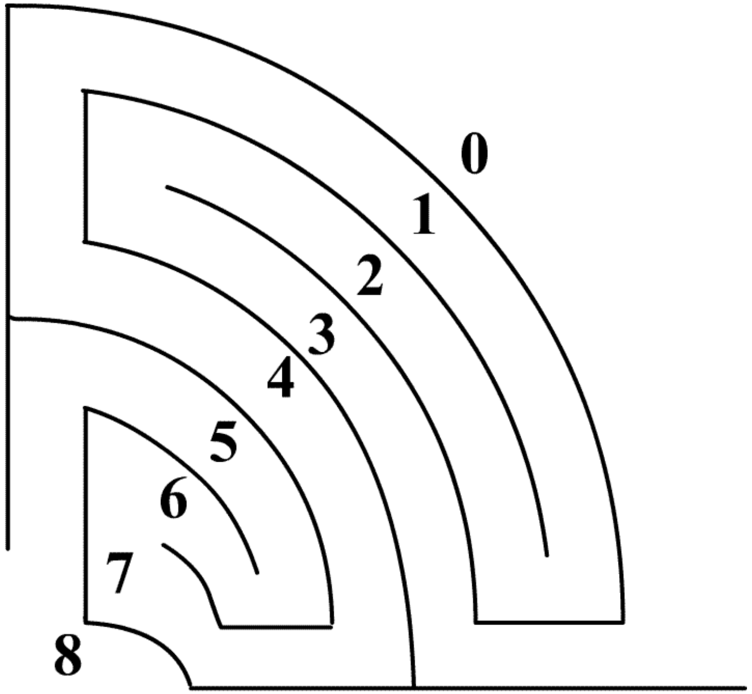
An SAT maze can be put into *rectangular form*.

1. Split the central axis.
2. Unroll the left side: 90^{o}
3. 180^{o}



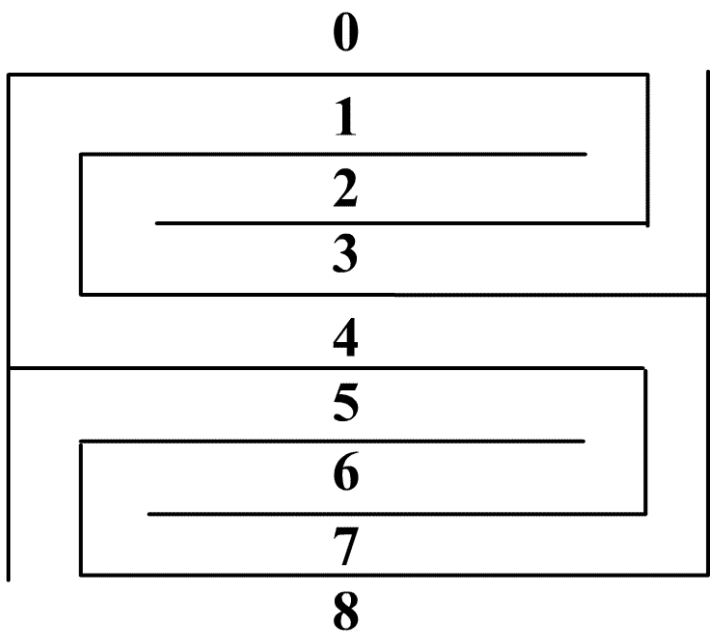
An SAT maze can be put into *rectangular form*.

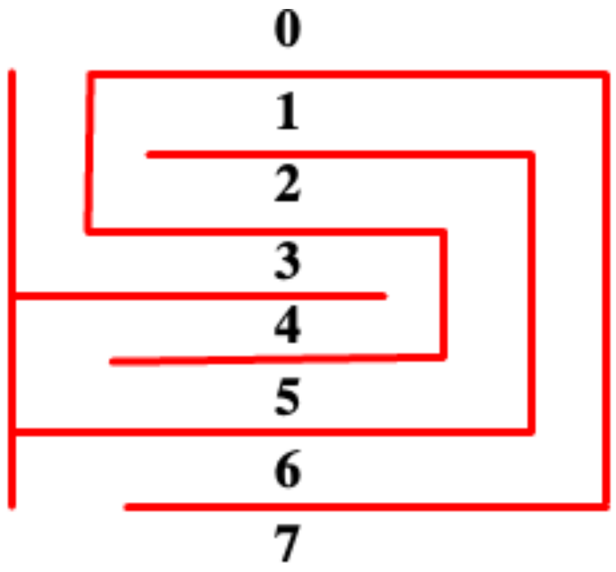
1. Split the central axis.
2. Unroll the left side: 90^o
3. 180^o
4. 270^o

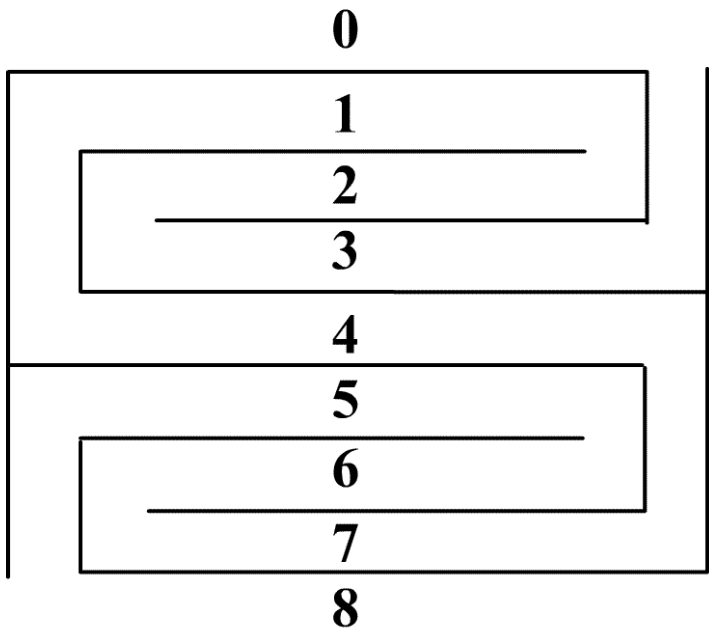


An SAT maze can be put into *rectangular form*.

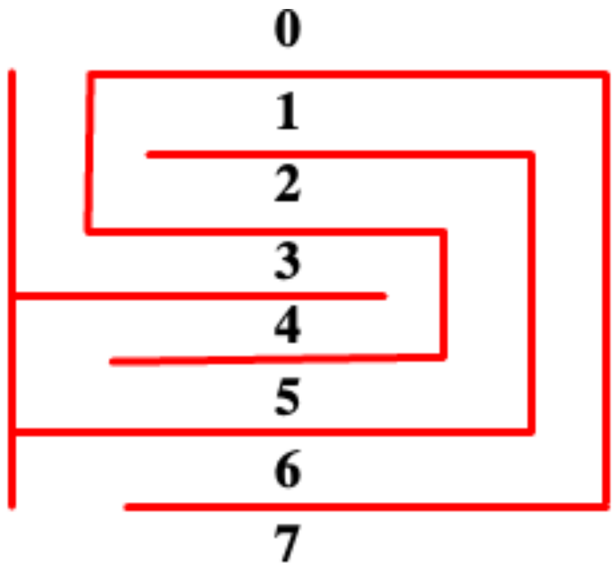
1. Split the central axis.
2. Unroll the left side: 90^o
3. 180^o
4. 270^o
5. 360^o

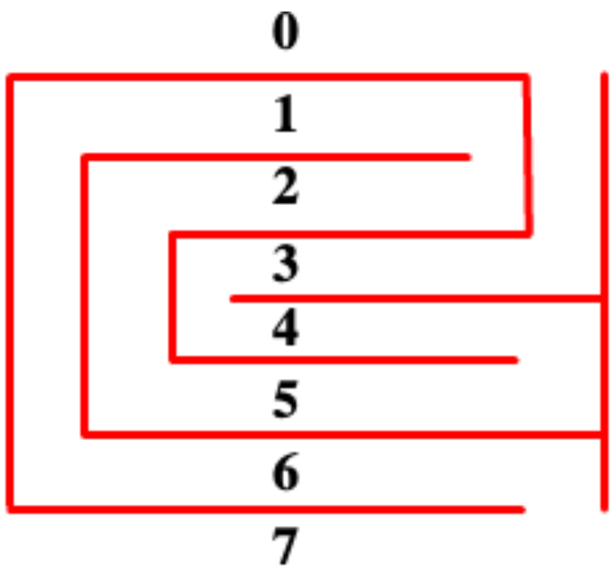




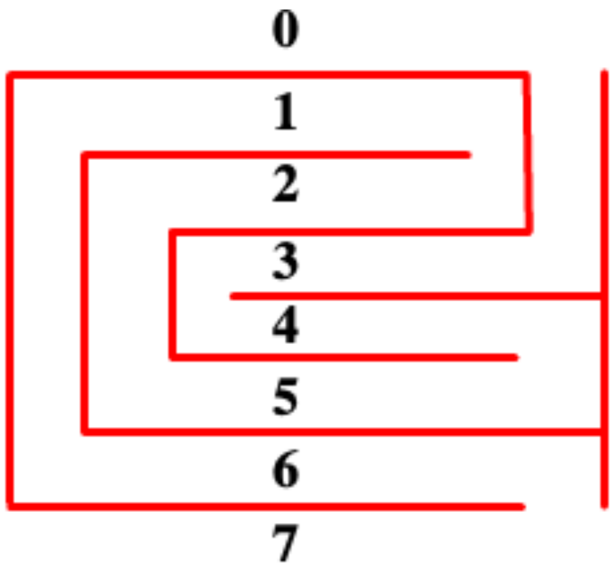


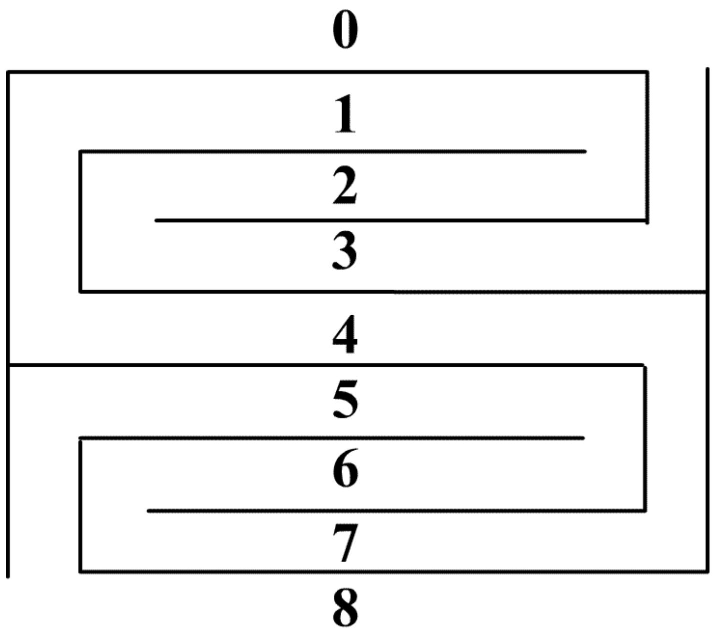
We can do the same for the Haftorot maze.





It will be convenient to draw all rectangular mazes with entrance on the *right*.
Reflection does not change the topological nature of the maze.



X

Which permutations of 0 1 2 … *n* can be the level sequence for an *n*-level SAT maze? Examples are 0 3 2 1 4 7 6 5 8 and 0 3 4 5 2 1 6 7. What do they have in common?

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2. Odds and evens must alternate. Why?

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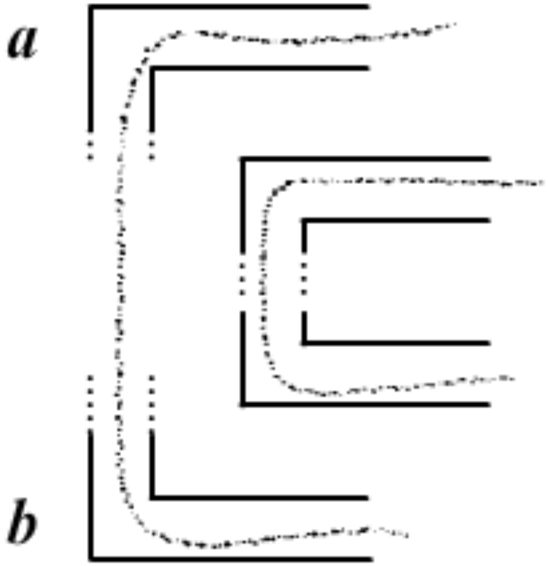


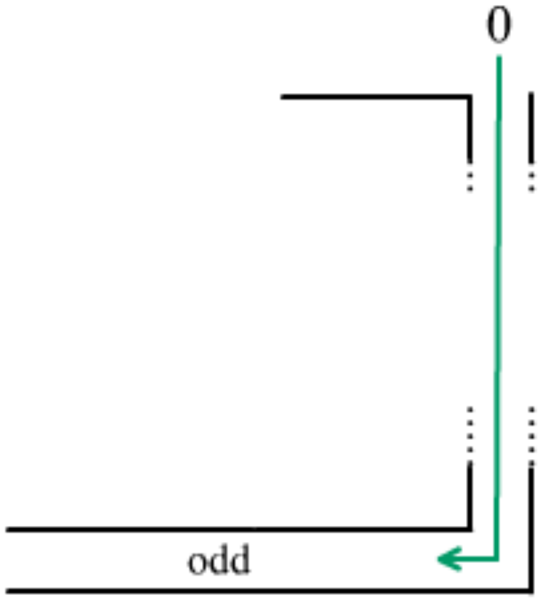
Suppose *a* and *b* are consecutive levels in the sequence.

← If the path enters between *a* and *b* it must also exit.

So there is an *even* number 0, 2, 4, … of levels between *a* and *b*.

So *a* and *b* must have opposite parity: if *a* is odd *b* must be even, and vice-versa.

Since the path starts at level 0 (even) on the right side, the next level must be odd. That level is traversed from right to left.



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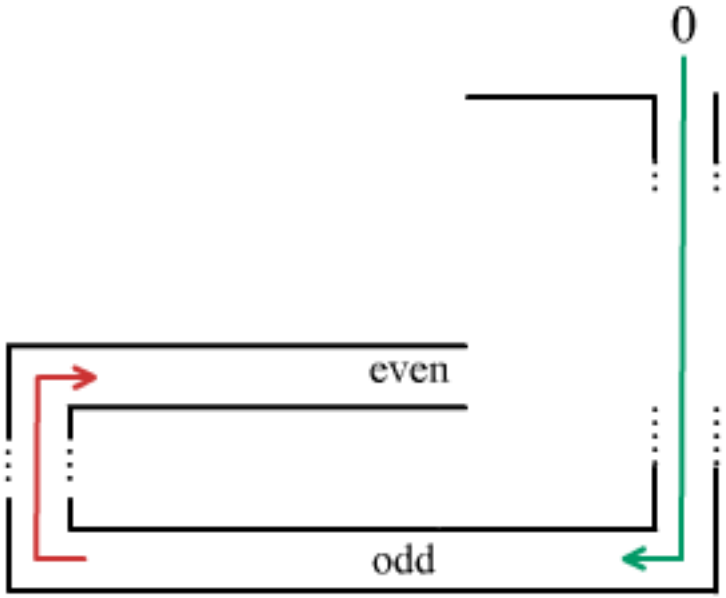
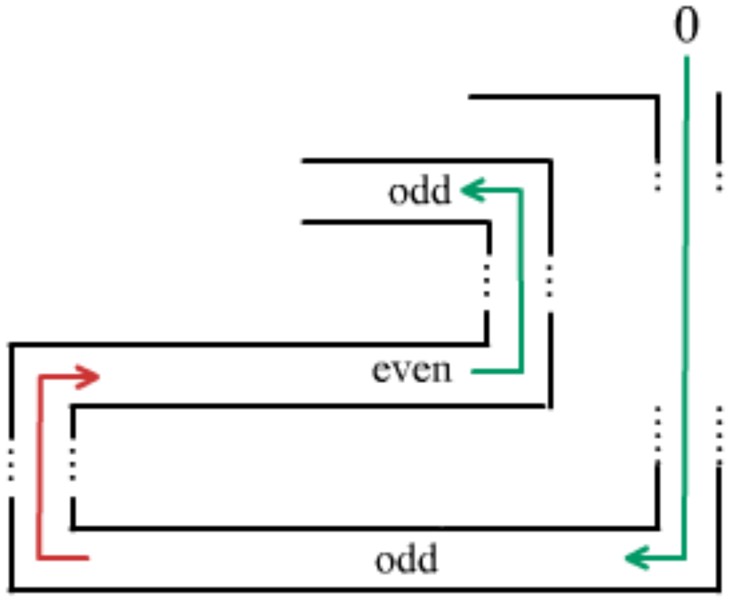
On the left side there will be a change to an even level, traversed from left to right.



Since the path starts at level 0 (even) on the right side, the next level must be odd. That level is traversed from right to left.

On the left side there will be a change to an even level, traversed from left to right.

Then on the right side, another change to an odd level.



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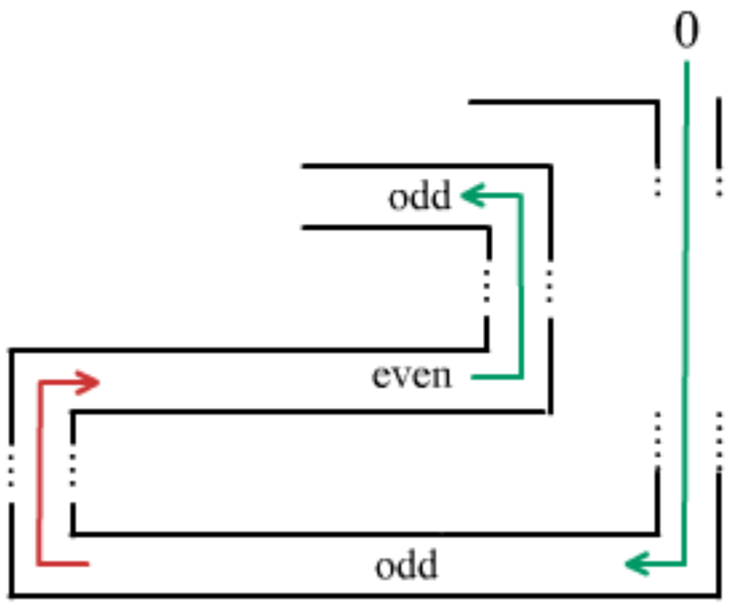

All level changes on the right (entrance) side are *even* to *odd*.

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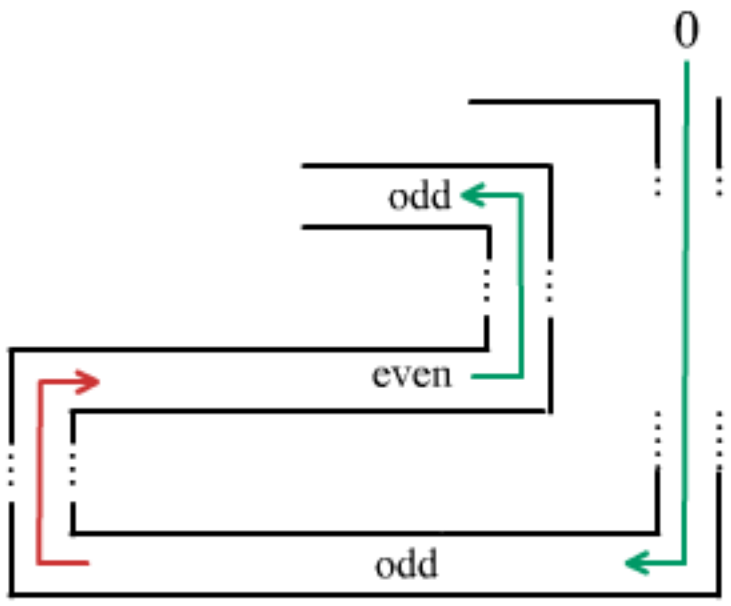
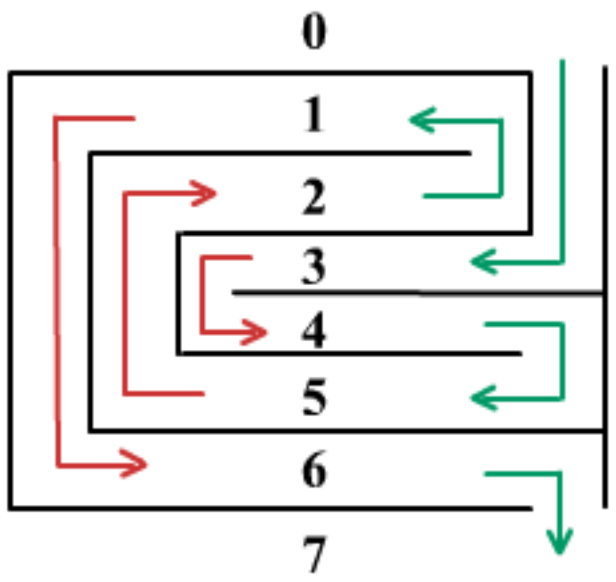
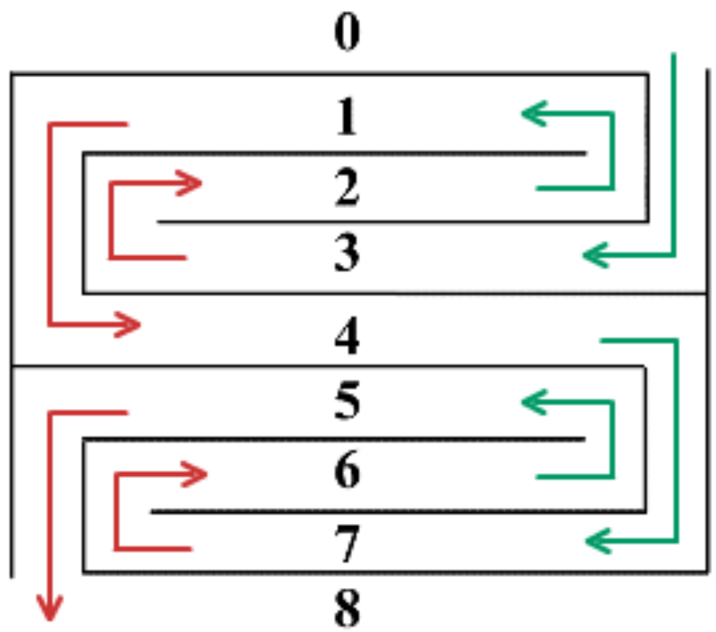


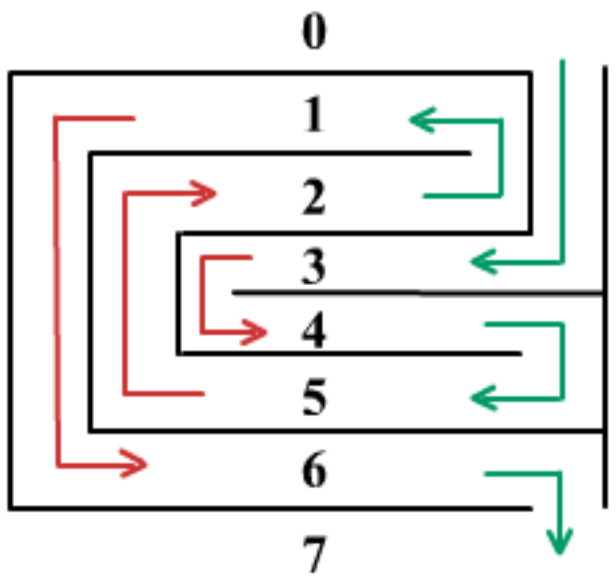
All level changes on the right (entrance) side are *even* to *odd*.
All level changes on the left side are *odd* to *even*.

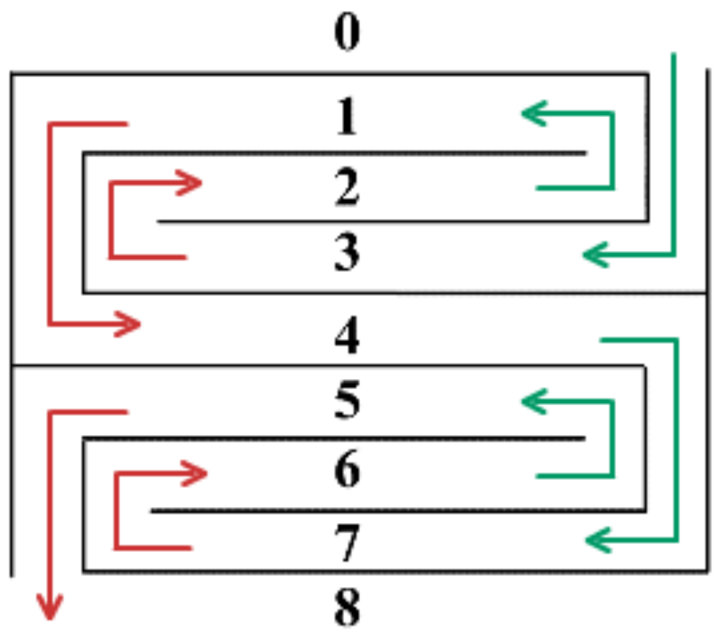




Notice: if two green intervals overlap, one must be nested inside the other. Same for red. (Otherwise the path would intersect itself).

What does this mean for the level sequence?





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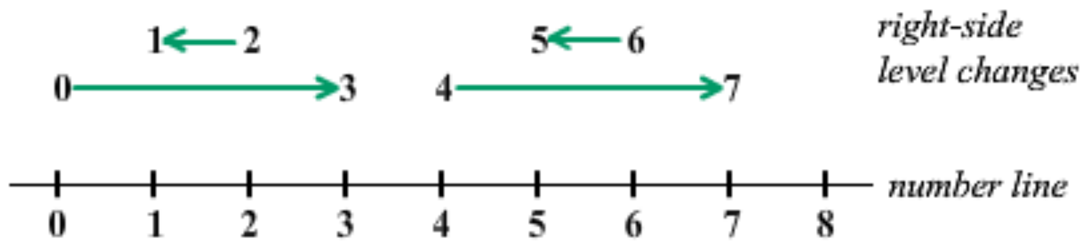
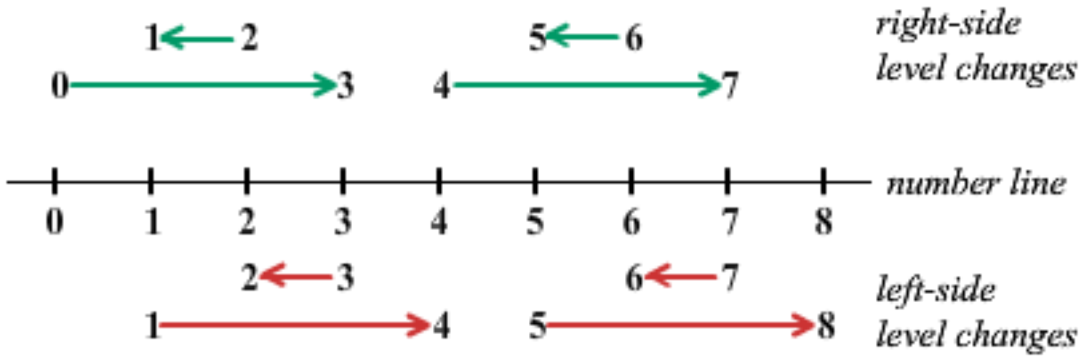
On the number line, plot all right-side (even→odd) level changes.



If two of them overlap, one must be nested inside the other.

What does this mean for the level sequence? With 0 3 2 1 4 7 6 5 8 as an example.

On the number line, plot all right-side (even→odd) level changes.
Plot all left-side (odd→even) level changes.



If two of them overlap, one must be nested inside the other.

Same procedure for the Jericho maze 0 3 4 5 2 1 6 7.

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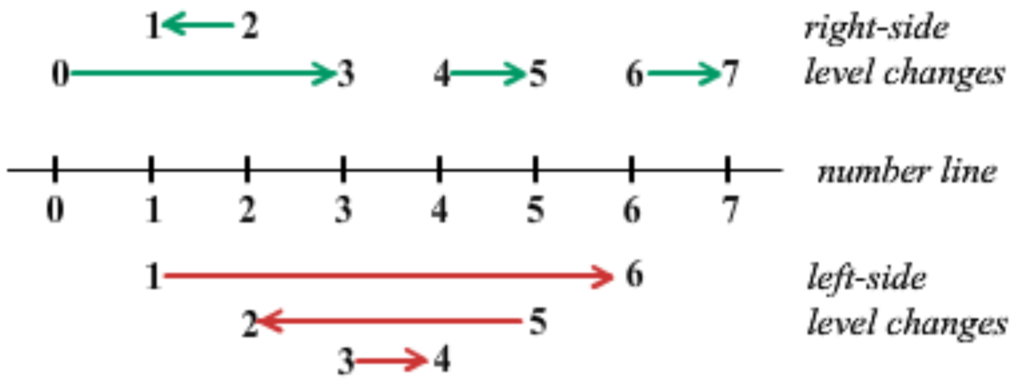
Plot all left-side (odd→even) level changes.



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Conditions for a permutation of 0 1 2 … *n* to be the level sequence of an SAT maze.

1. Start with 0, end with *n*.
2. Odds and evens alternate.
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Necessary and sufficient!

Some more examples.

Some more examples.

From a late 12th century manuscript in Munich.

Should have been the
SAT maze
0.3.2.1.4.7.6.5.8.11.10.9.12
but level 11 got eaten
by the picture.

Text: “Theseus fights with the Minotaur in the Labyrinth.”

Some more examples.

Rock mazes in Scandinavia.

There are several hundred of them.

We will visit one on the island of Gotland in the Gulf of Finland.

Courtesy of Google Earth.

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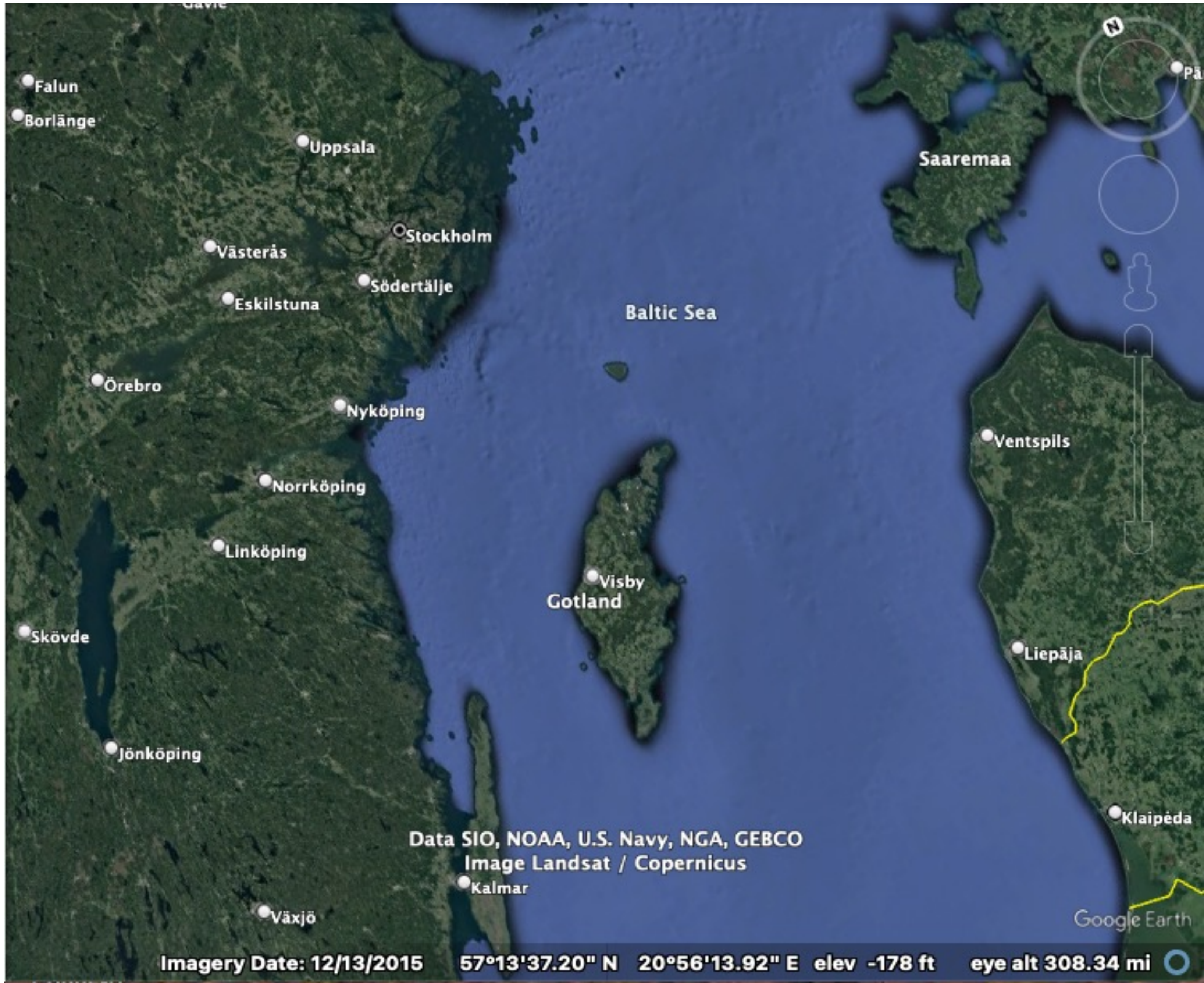

Some more examples.

Rock mazes in Scandinavia.

There are several hundred of them.

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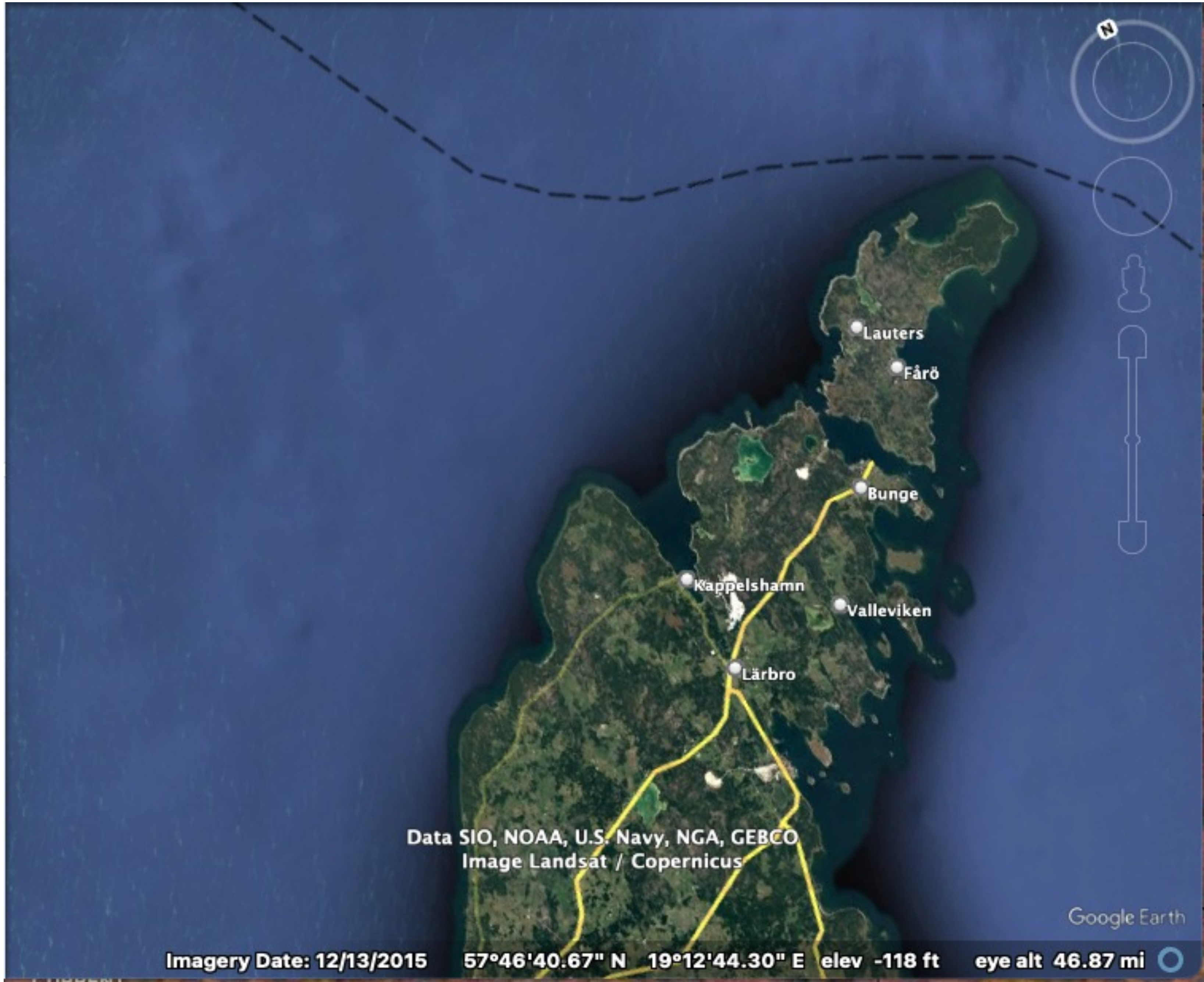

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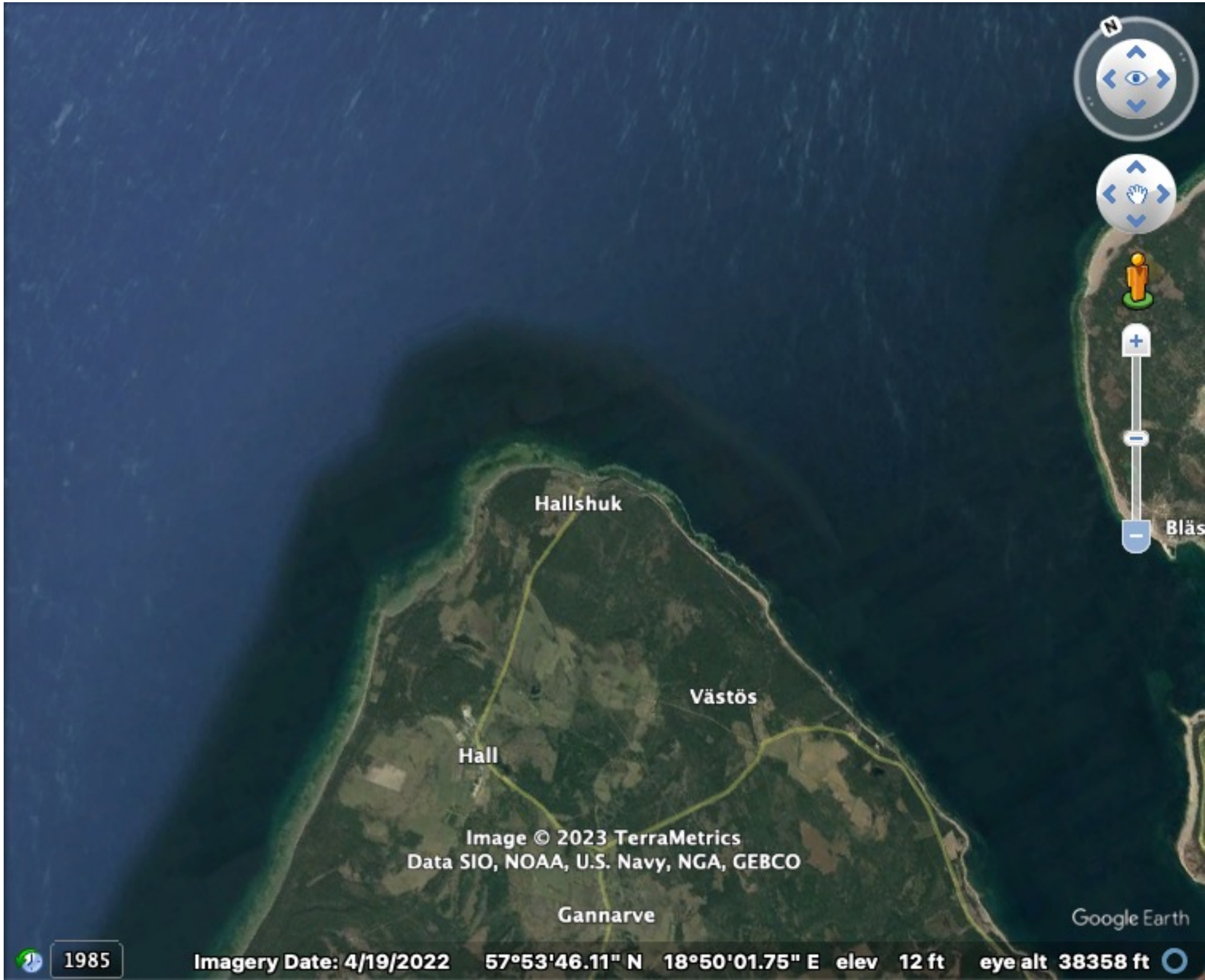

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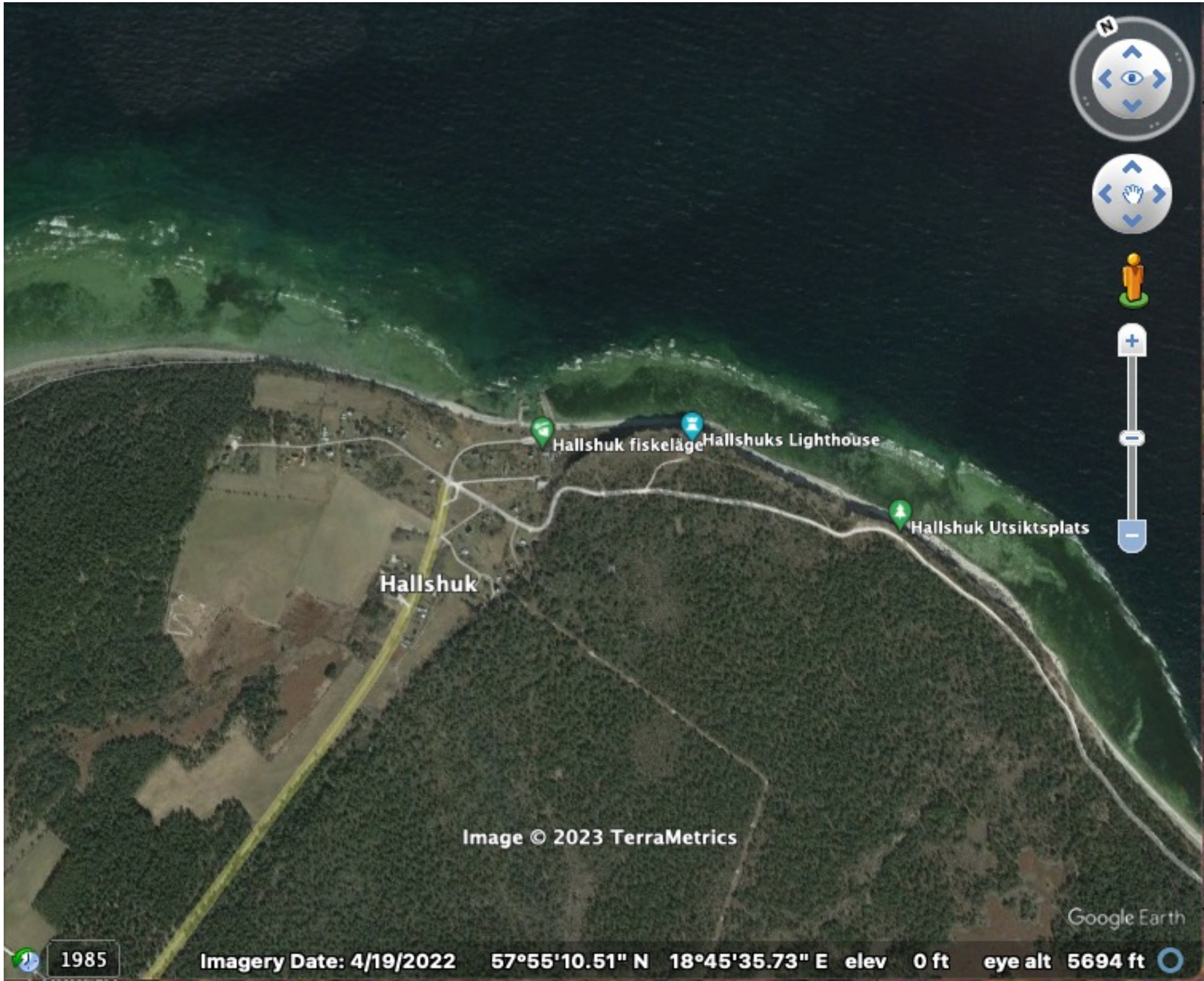

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Rock mazes in the Solovetsky Archipelago in the White Sea, Russia.

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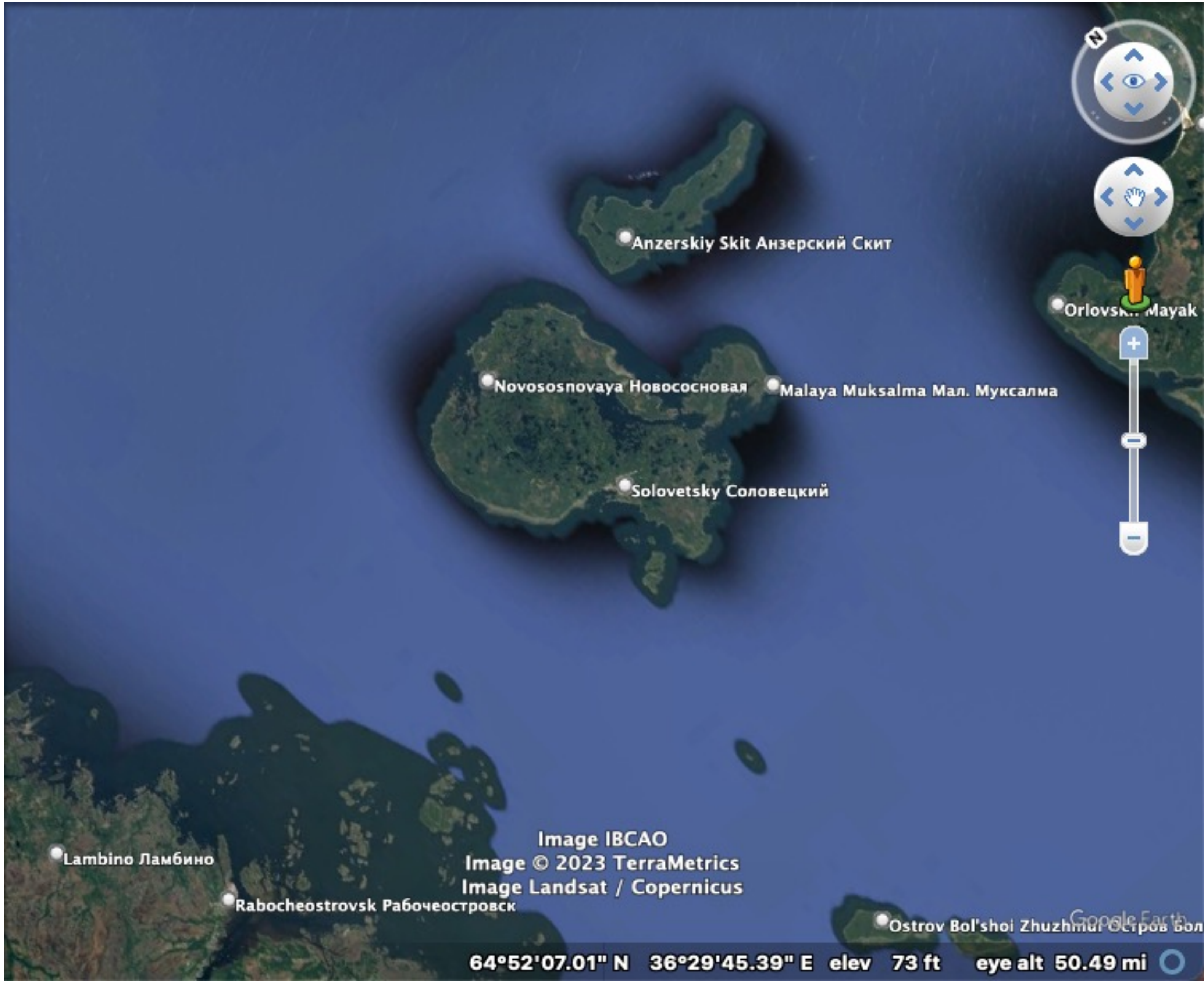

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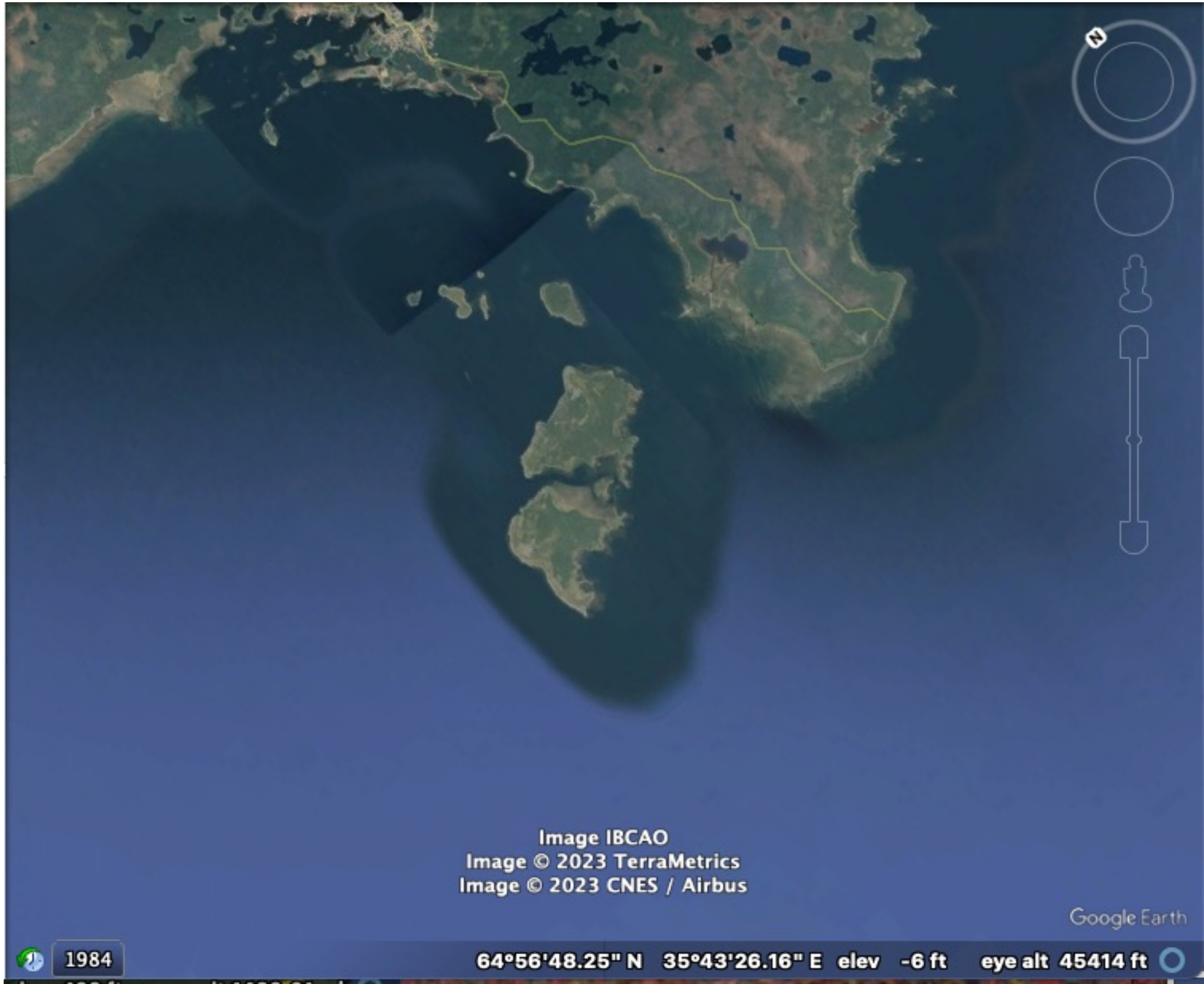

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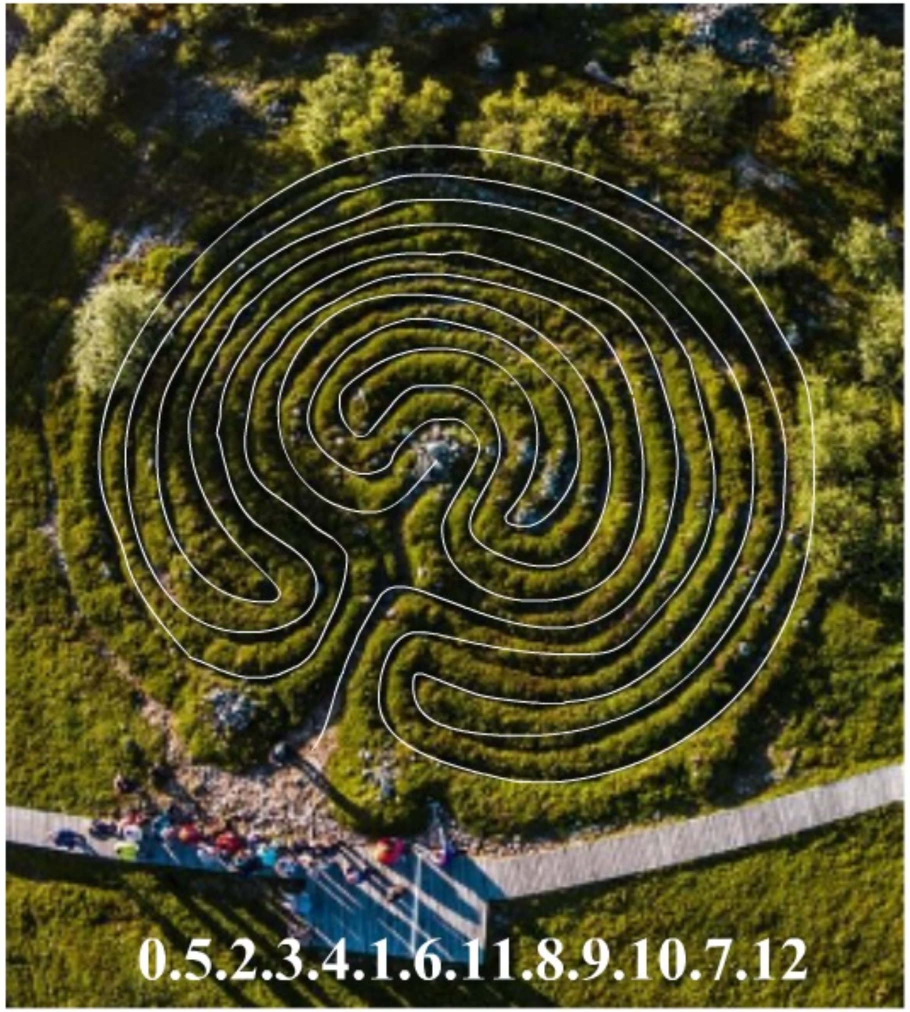

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Level sequence:
0.7.10.9.8.11.6.1.4.3.2.5.12

Counting mazes.

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We have seen three different 12-level SAT mazes:

12th Century manuscript.

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How many can there be?

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Write $M(n)$ for the number of different n-level SAT mazes.

1. $M(n)$ increases exponentially with n.

2. There is no way to compute $M(n)$ directly from n.

You have to generate them all and count them.

n	M(n)
1	1
2	1
3	1
4	2
5	3
6	8
7	14
8	42
9	81
10	262
11	538
12	1828
13	3926
14	13820
15	30694
16	110954
17	252939
18	933458
19	2172830
20	8152860
21	19304190
22	73424650

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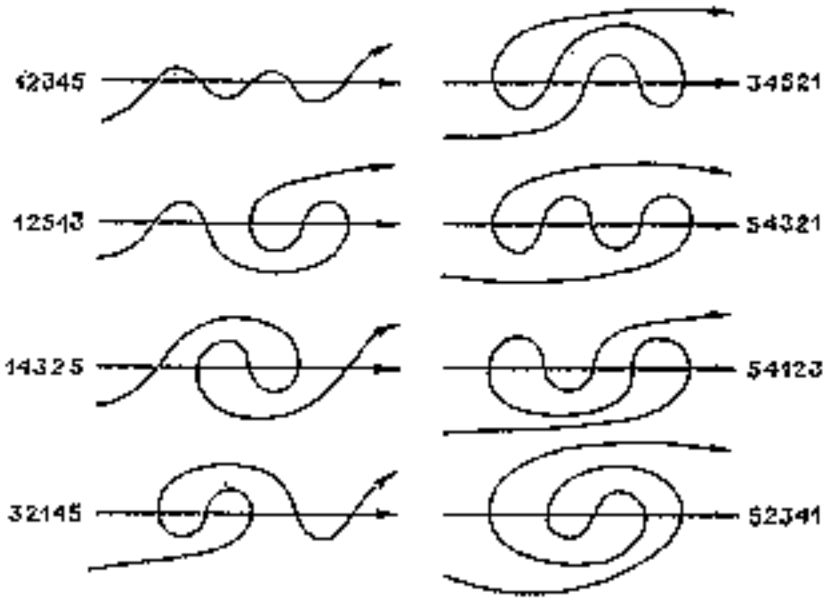
Meanwhile …

V. I. Arnol’d
Sibirskii Matematicheskii Zhurnal
29 36-67 (1988)
“meanders”



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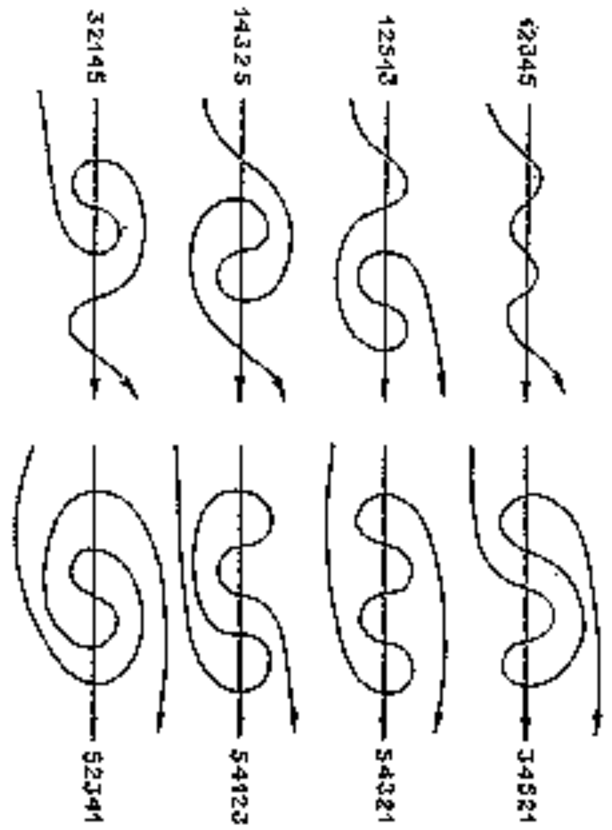

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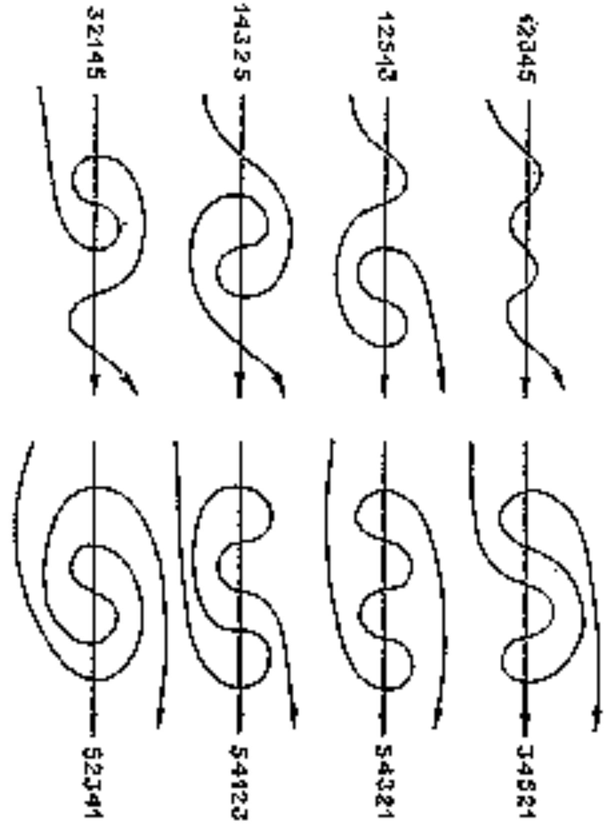

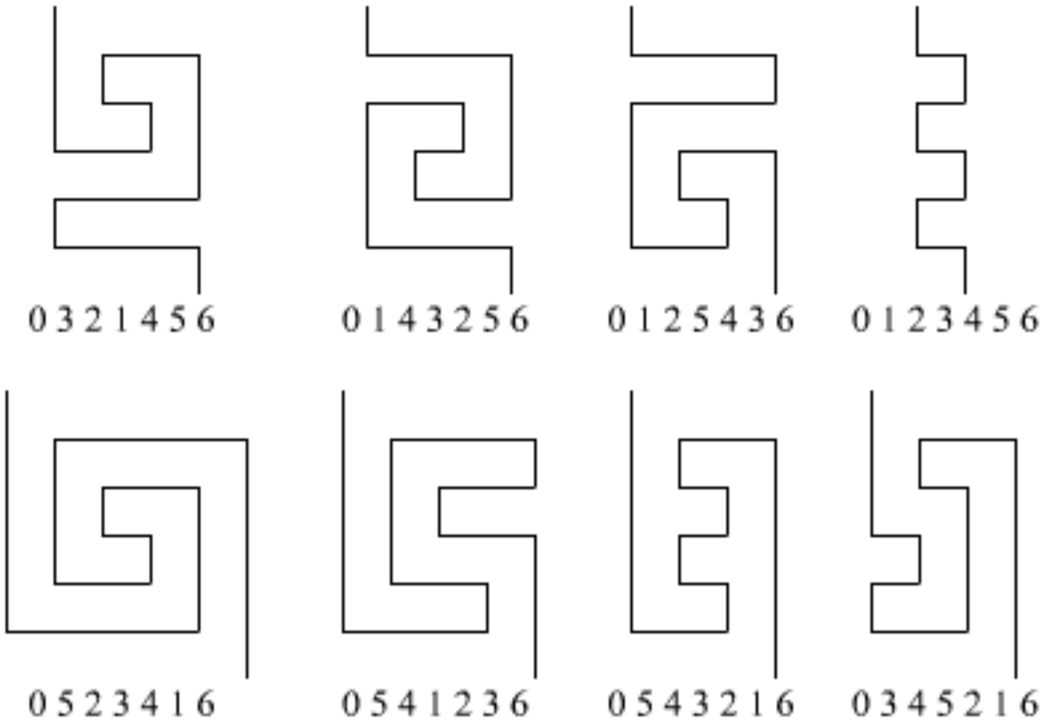

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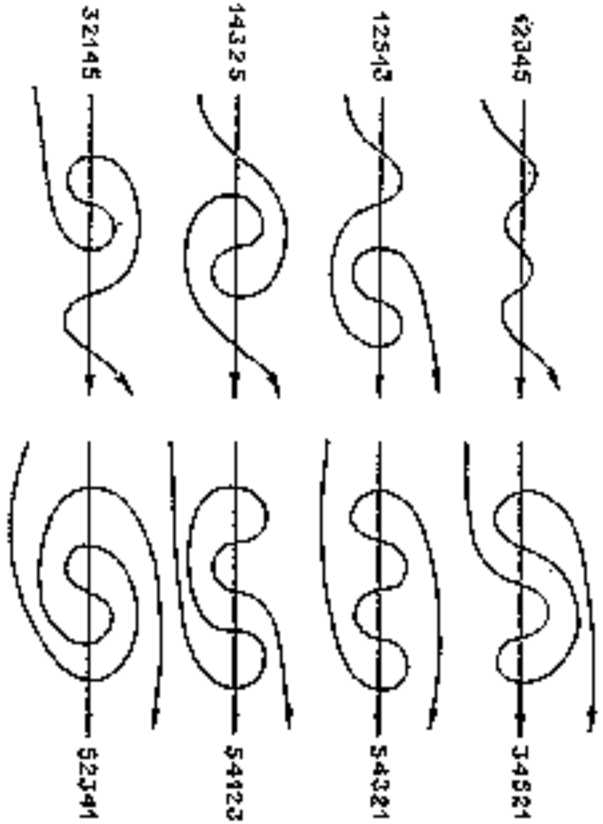

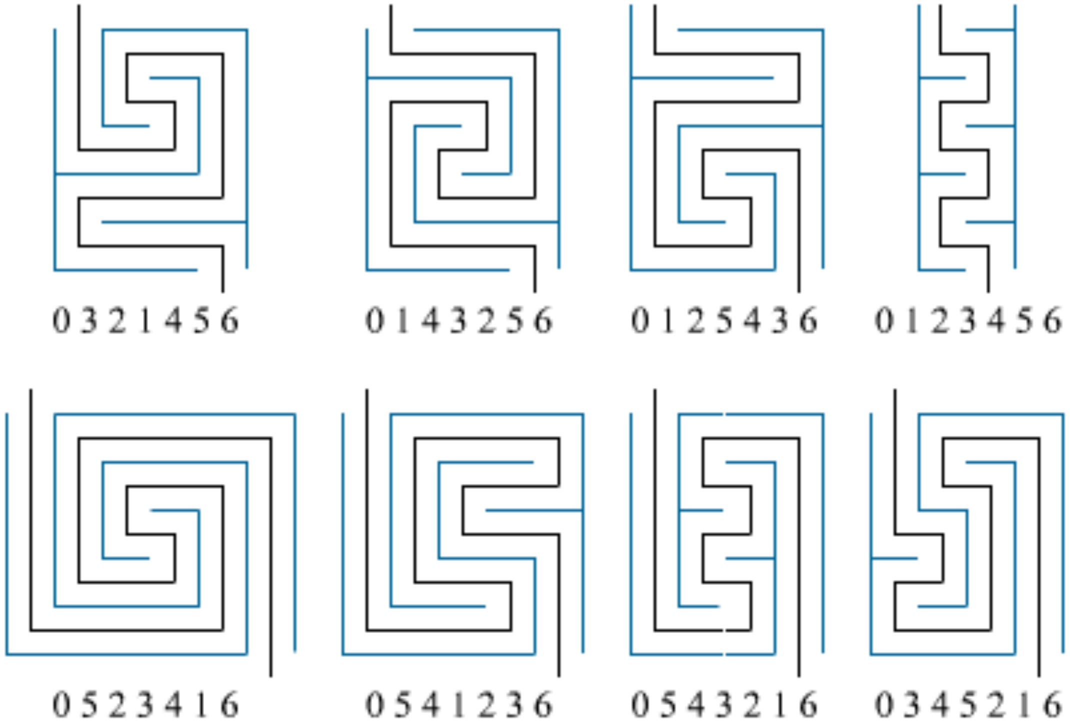

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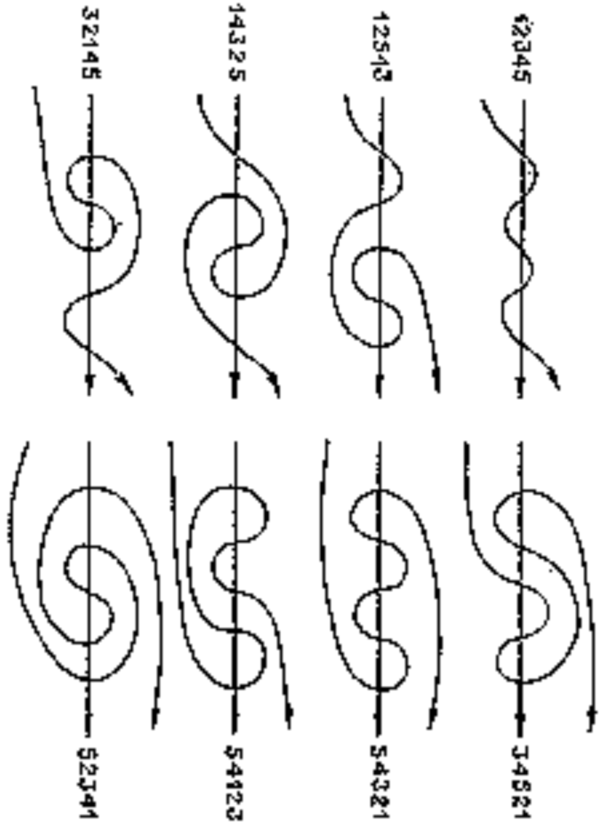

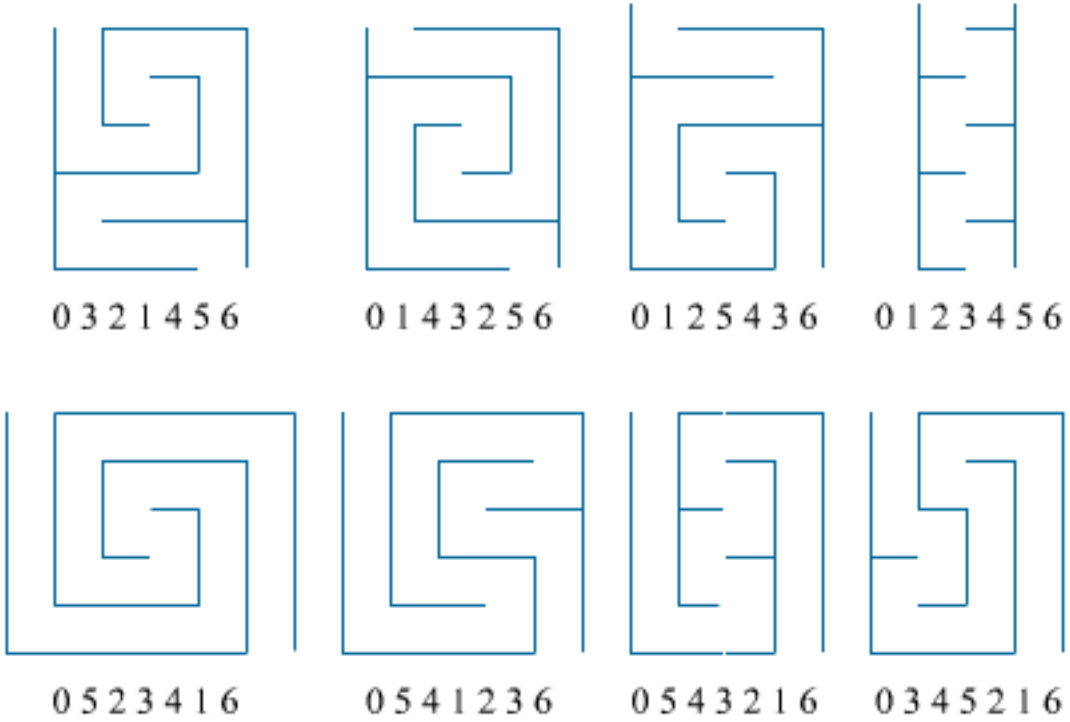

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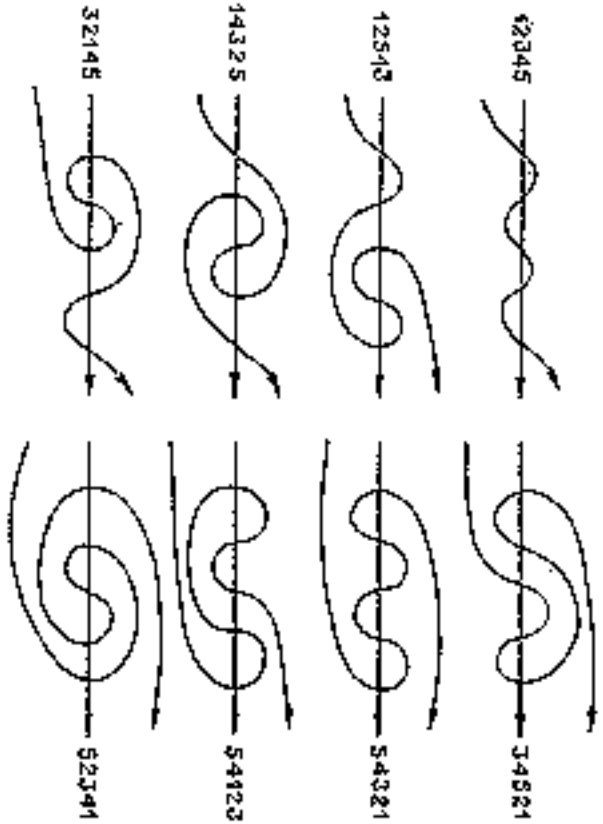
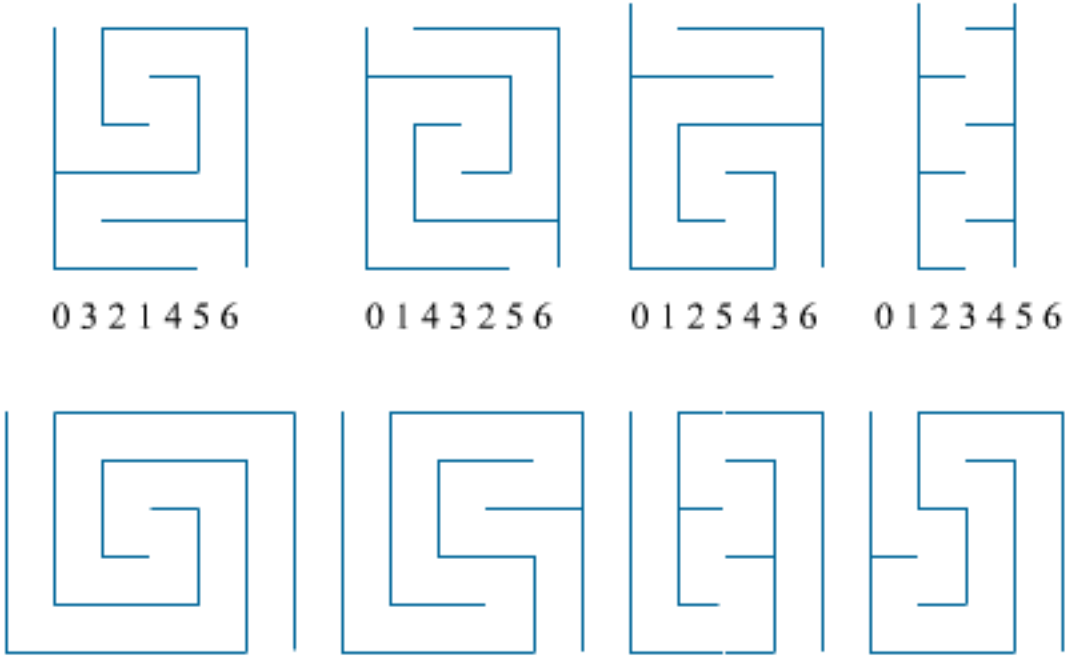
Arnol’d’s 8 meanders with 5 intersections ←→ the 8 six-level SAT mazes.





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Arnol'd calculated up to M(17) = 252939.

Meanwhile …

S. K. Lando & A. K. Zvonkin
Theoretical Computer Science
117 227-241 (1993)
Plane and projective meanders

Table of meandric numbers

n	0	1	2	3	4	5	6	7	8	9	10
m_n	1	1	1	2	3	8	14	42	81	262	538

n	11	12	13	14	15	16
m_n	1828	3926	13 820	30 694	110 954	252 939

n	17	18	19	20	21
m_n	933 458	2 172 830	8 152 860	19 304 190	73 424 650

n	22	23	24	25	26	27
m_n	?	678 390 116	?	6 405 031 050	?	61 606 881 612

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M(28)

I. Jensen and A. J. Guttmann,
Critical exponents of plane meanders,
J. Phys. A 33 (2000) L187-L192

Meanwhile …

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24	678390116
26	6405031050
28	61606881612
30	602188541928
32	5969806669034
34	59923200729046
36	608188709574124
38	6234277838531806
40	64477712119584604
42	672265814872772972
44	7060941974458061392
46	74661728661167809752
48	794337831754564188184


That's all Folks!

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Thus, the problem of meanders seems to belong to the simply formulated but fairly difficult problems of combinatorial analysis related to different sections of mathematics and is a touchstone for various methods of enumerative combinatorics.