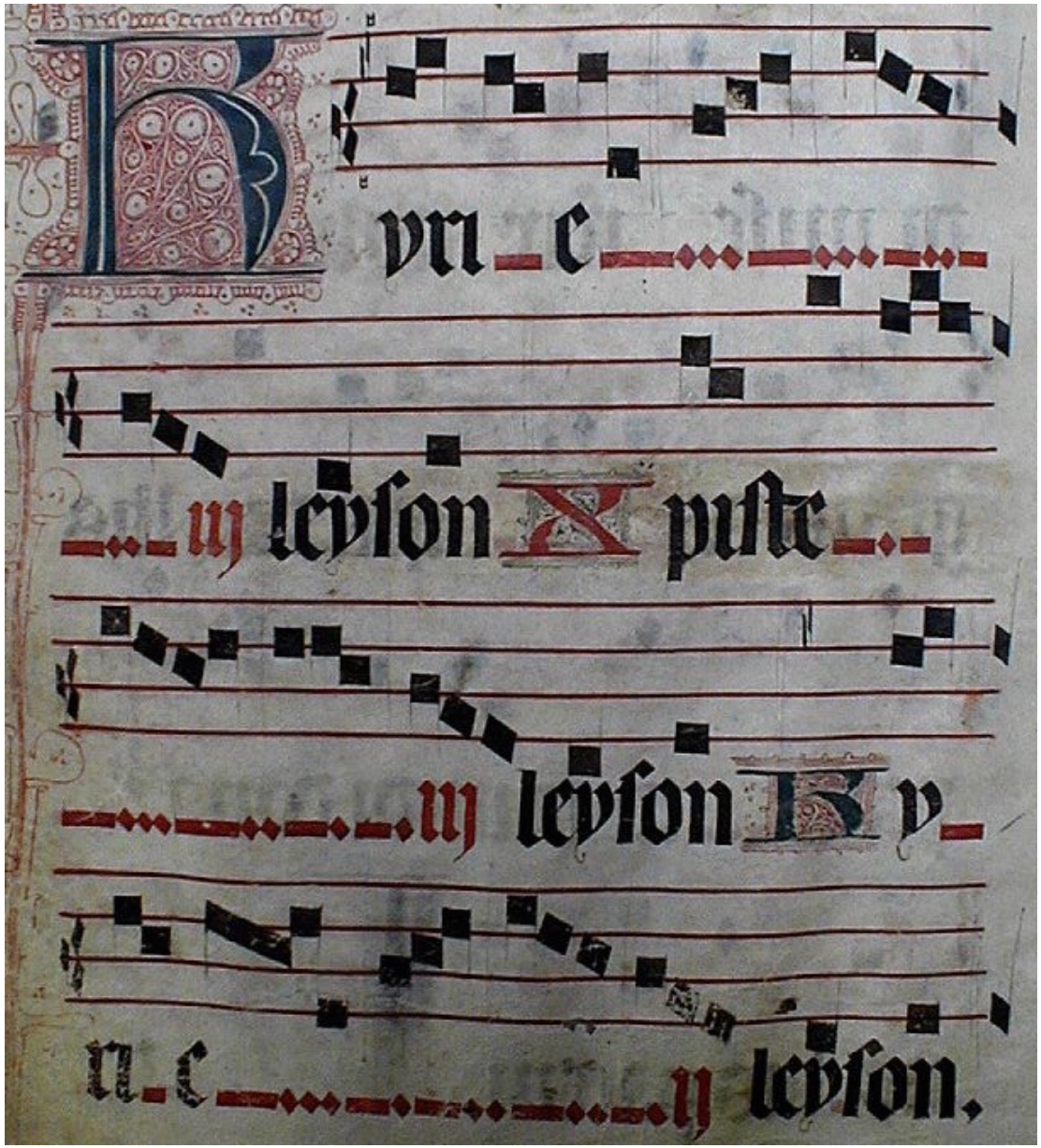
Topological surfaces in Bach's puzzle canons.

Tony Phillips

April 25, 2023

This talk is based on work with Eric Altschuler, published in *Musical Times*, Winter 2015

A musical score has basically two dimensions:



“Kyrie eleison”
(*Orbis factor*)
from a chantbook,
Miller Nichols Library
UMKC.
14th-15th century

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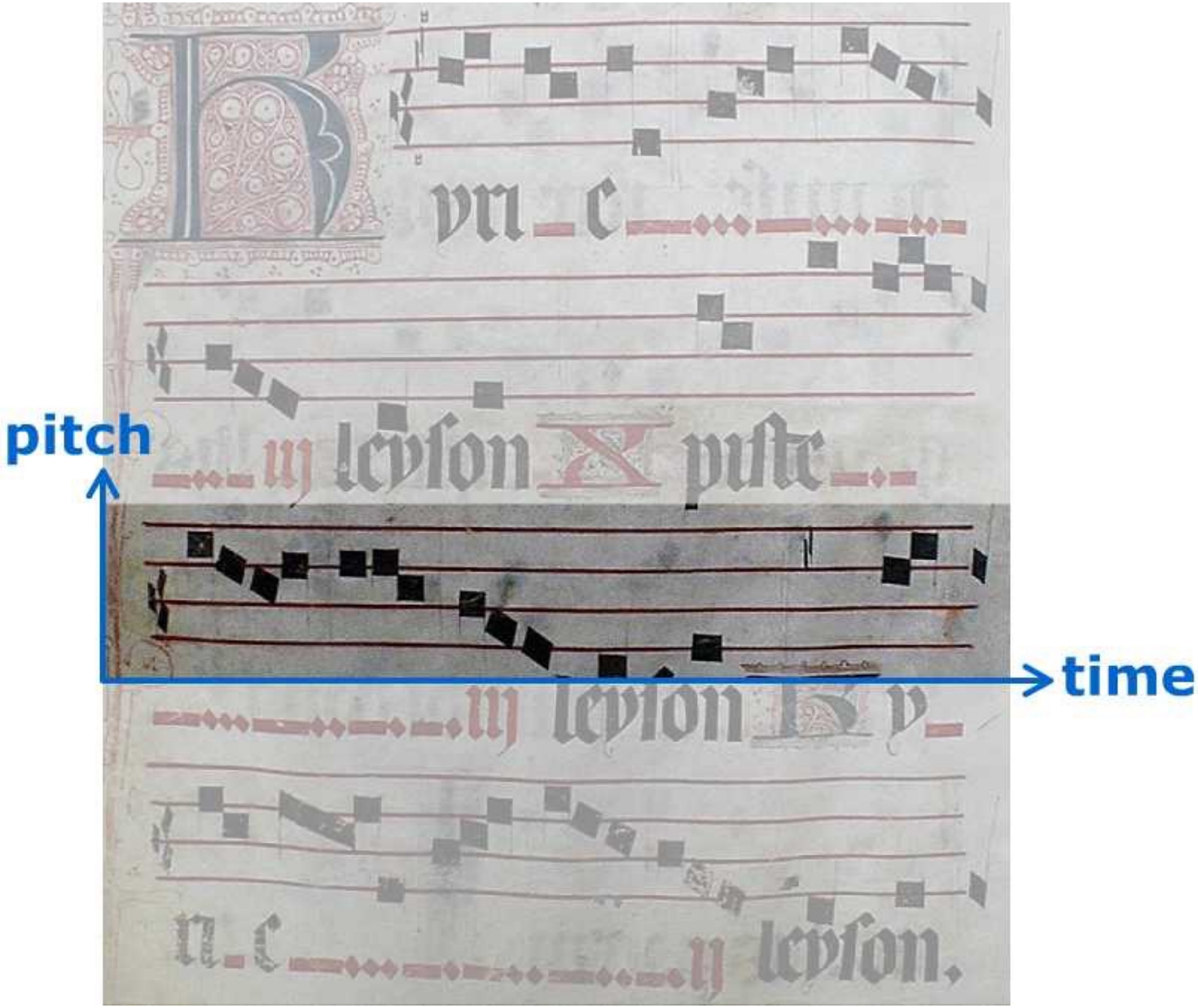

"Kyrie eleison"
(*Orbis factor*)
from a chantbook,
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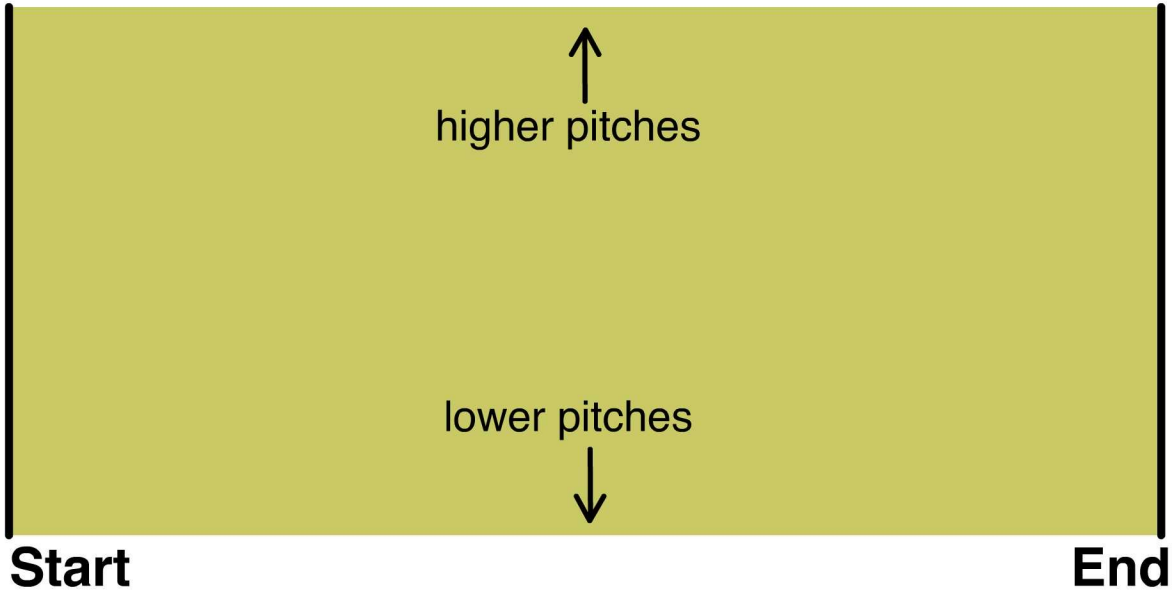
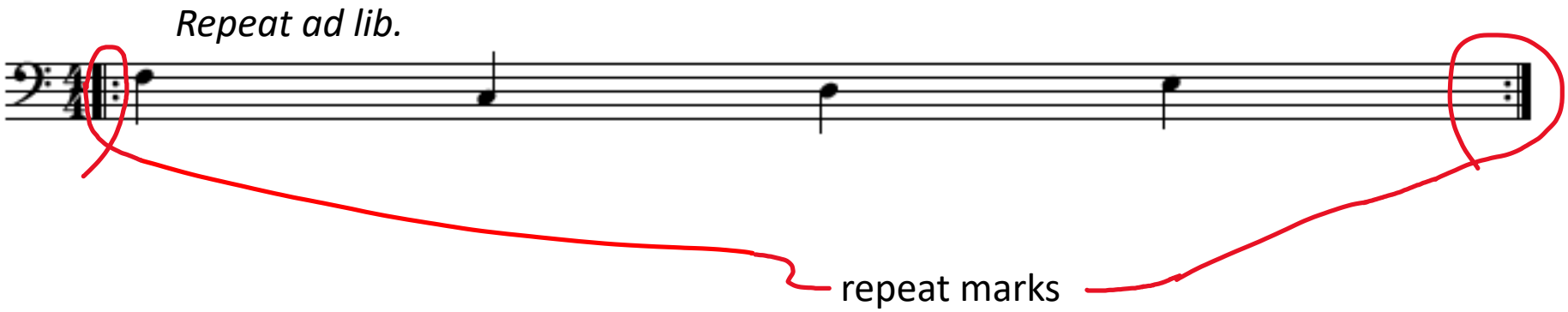
14th-15th century

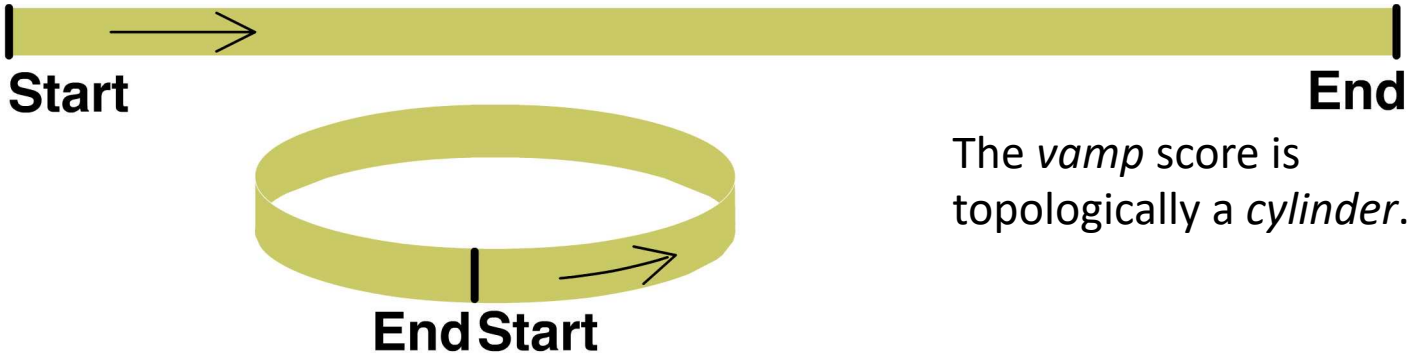
So topologically a one-voice musical score is a 2-dimensional strip.



SCORES WITH SYMMETRY

For example, suppose the score repeats. E.g. a *vamp*:

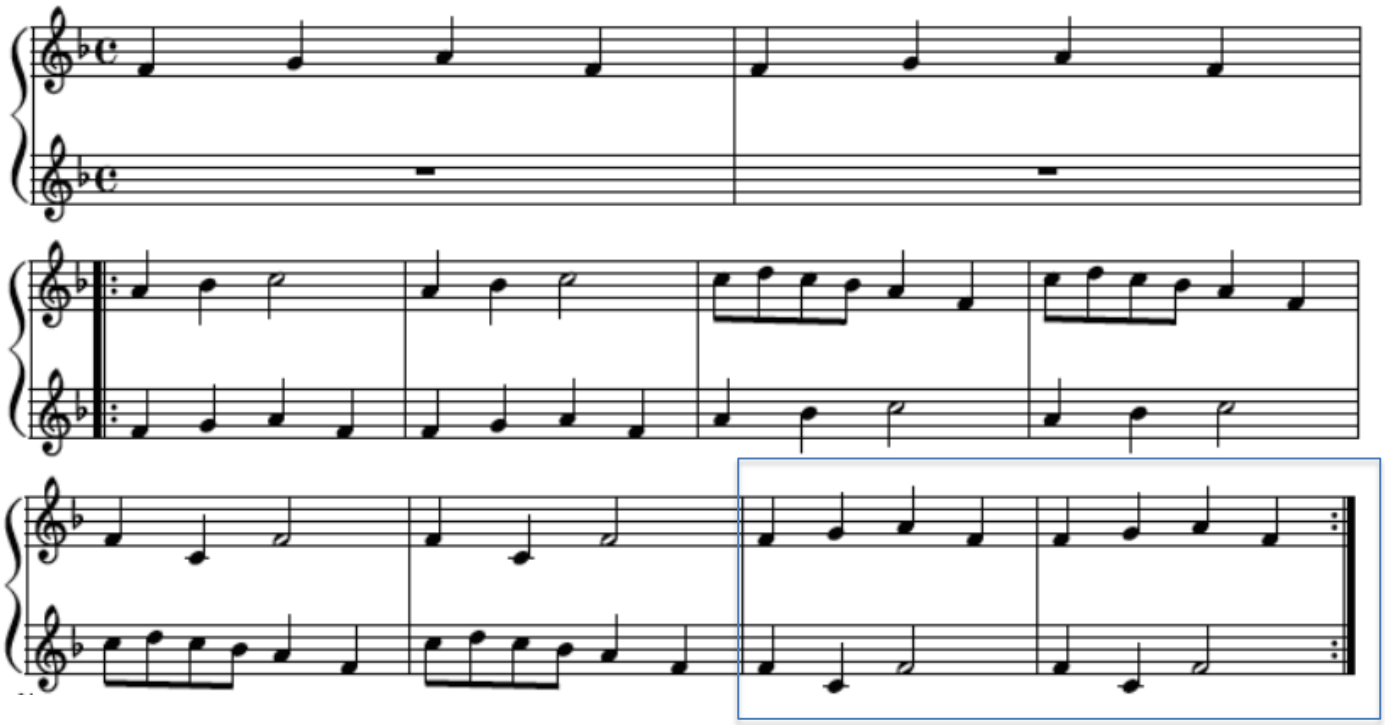




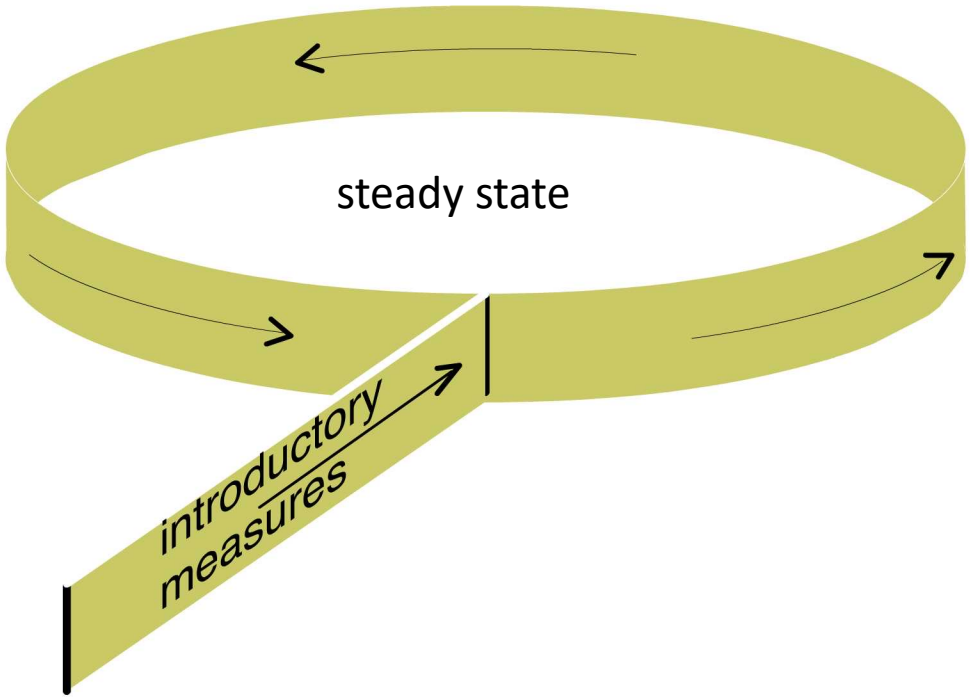
The *vamp* score is topologically a *cylinder*.

CANONS

A (2-part) ***canon*** is a score in which a second voice imitates the first voice after a delay. One of the best known is “Frère Jacques.”



Note: The end of the second statement harmonizes with the beginning of the first, so the two voices dovetail to form a ***steady state*** that can be repeated *ad lib*.



Topologically, in two-part canons, after the introductory measures (the start of the lead voice) the score is a *cylinder*.

Canons were a speciality
of J. S. Bach (1685-1750).

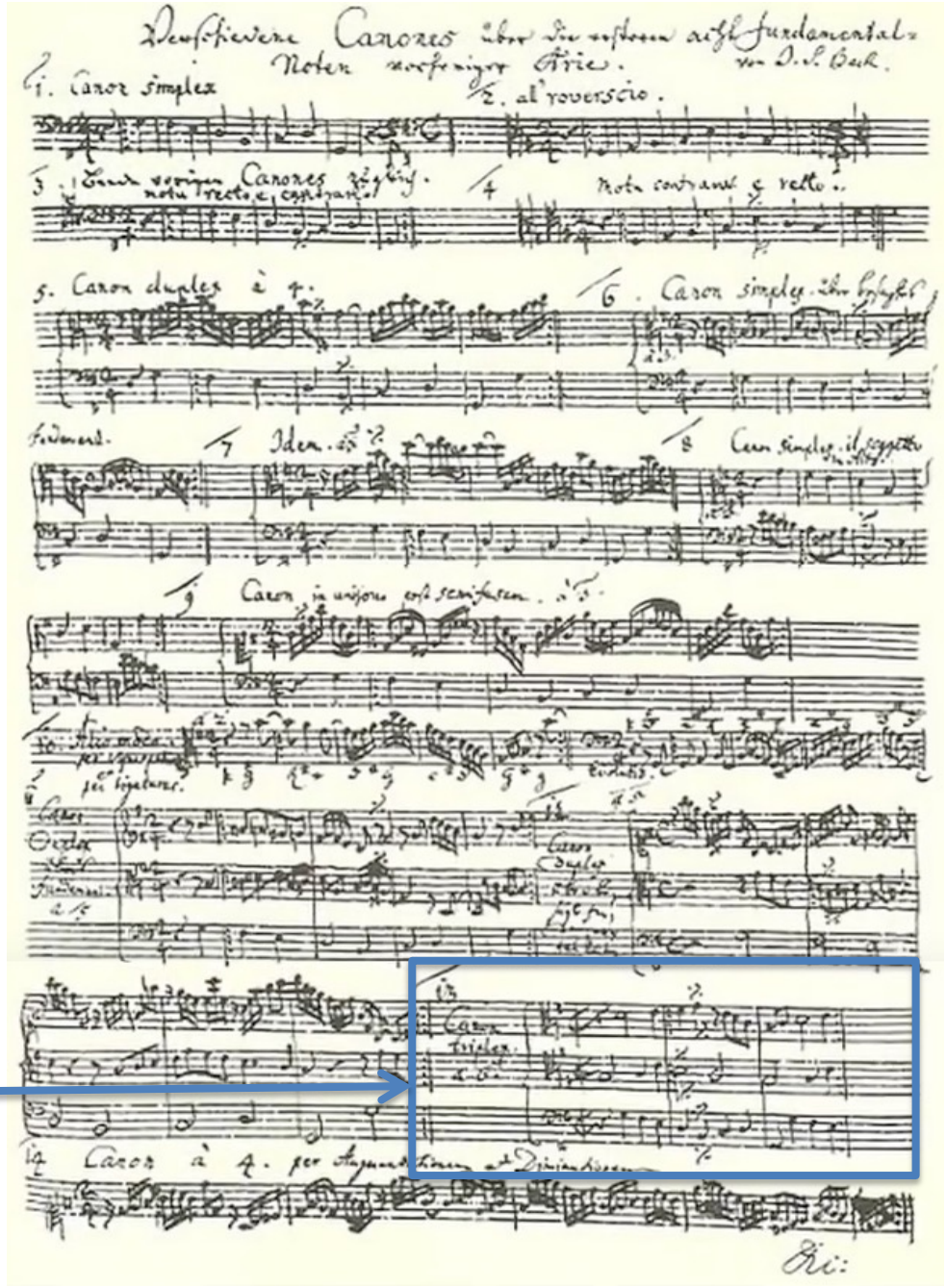
In this portrait, from 1746,
he is holding the score of
a canon:

one of the set of 14 canons,
BWV 1087, on the bass line
of the Goldberg Variations.

14 Canons on the Goldberg Ground.

This manuscript was discovered in 1974 in the back of Bach's own copy of the Goldberg Variations.

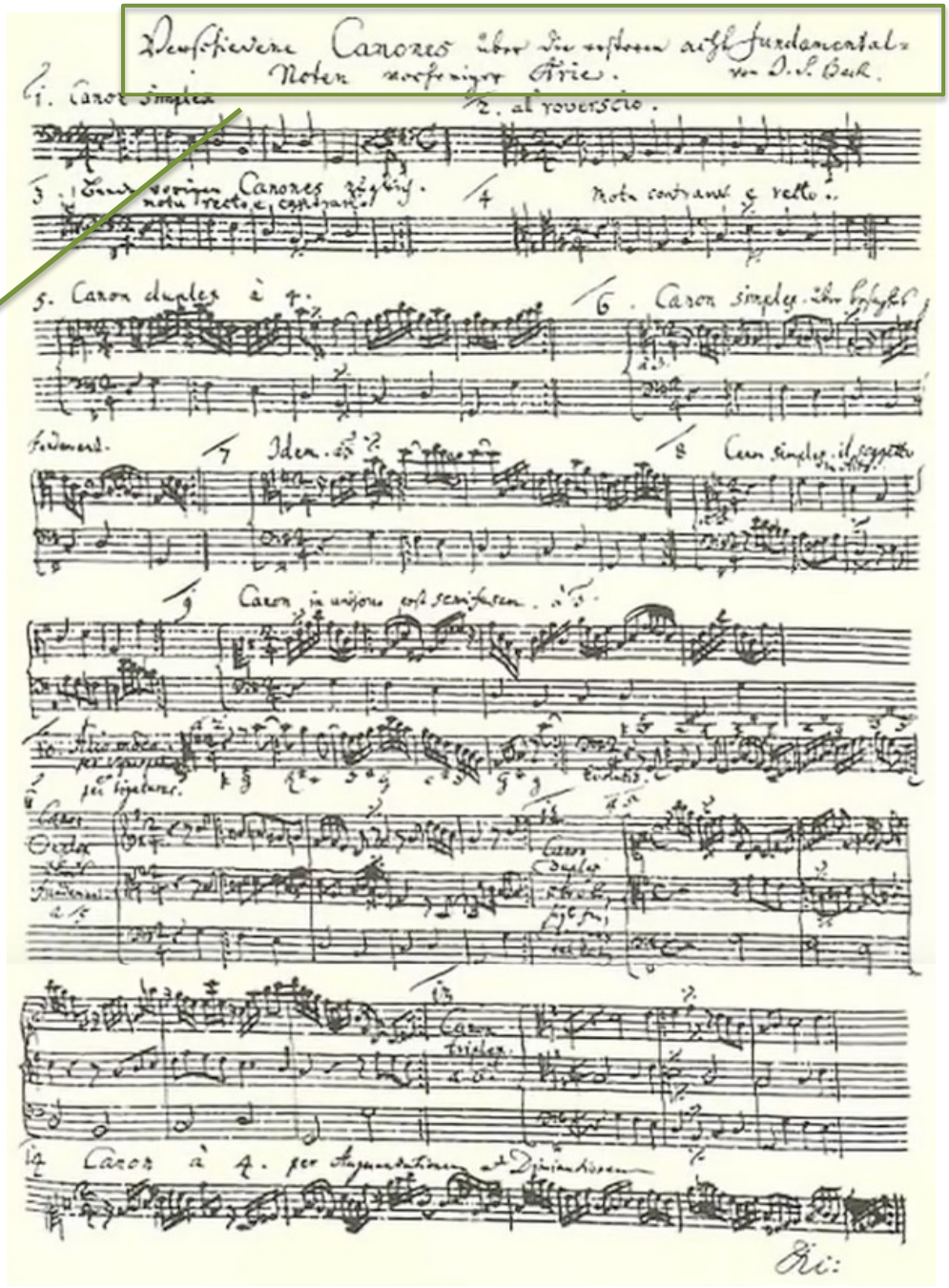
Canon 13 (*triplex*) –the canon in the painting– is one of the only two that were known before then.

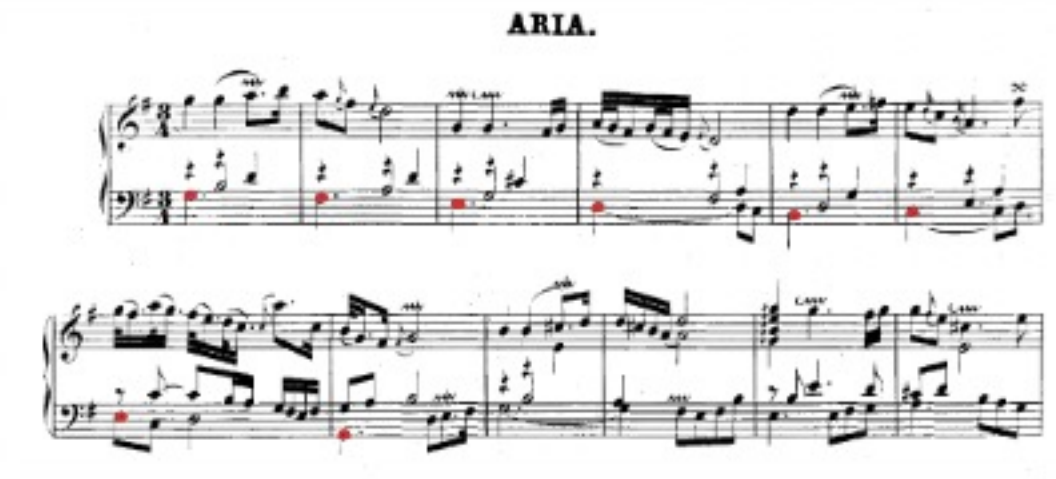


14 Canons on the Goldberg Ground.

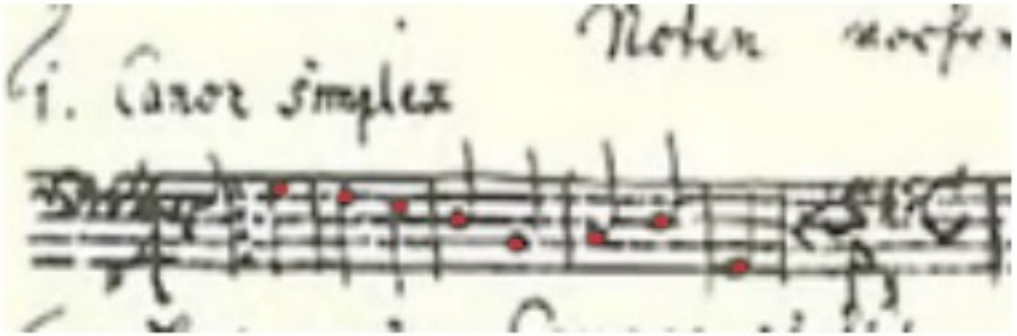
"Verschiedene Canones über die ersten acht fundamentalnoten vorheriger Arie." von J. S. Bach.

(Various canons based on the first eight bass-line notes of the foregoing aria.)





The beginning of the Goldberg-variations Aria
with the first eight notes of the bass-line highlighted in red.

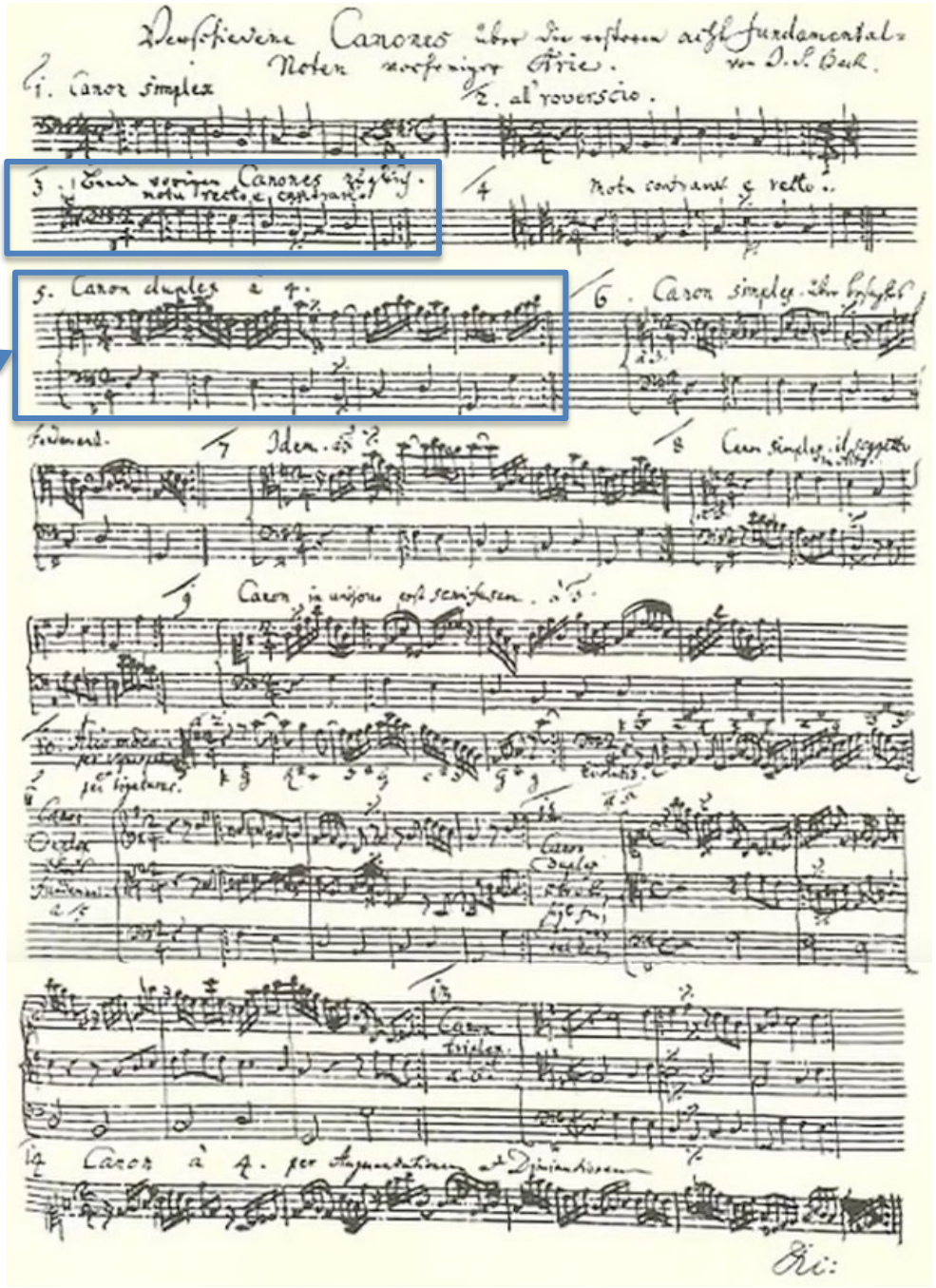


Canon 1 from the set of fourteen.

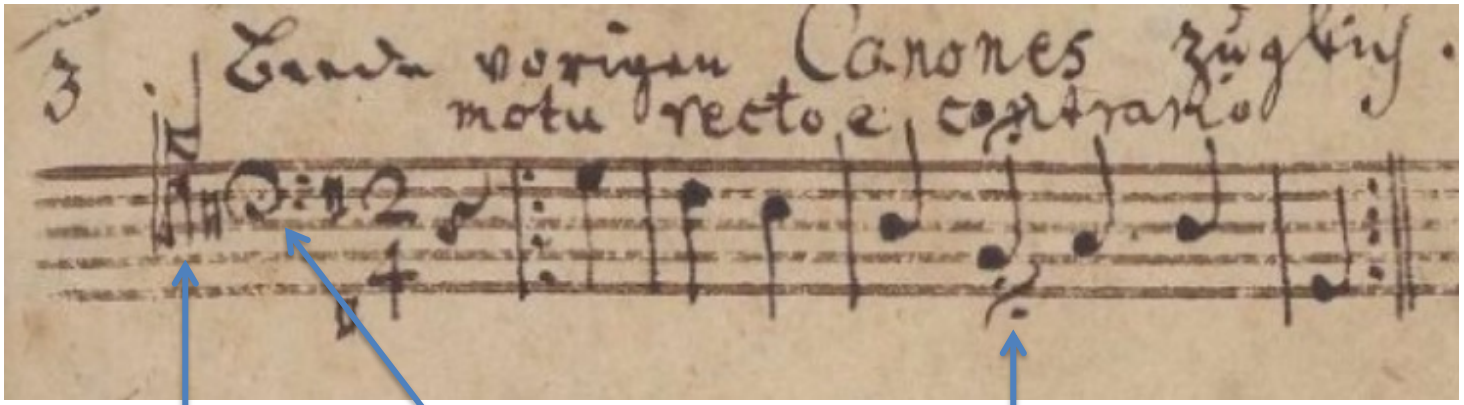
14 Canons on the Goldberg Ground.

Canons 3 and 5 are the ones we will be considering.

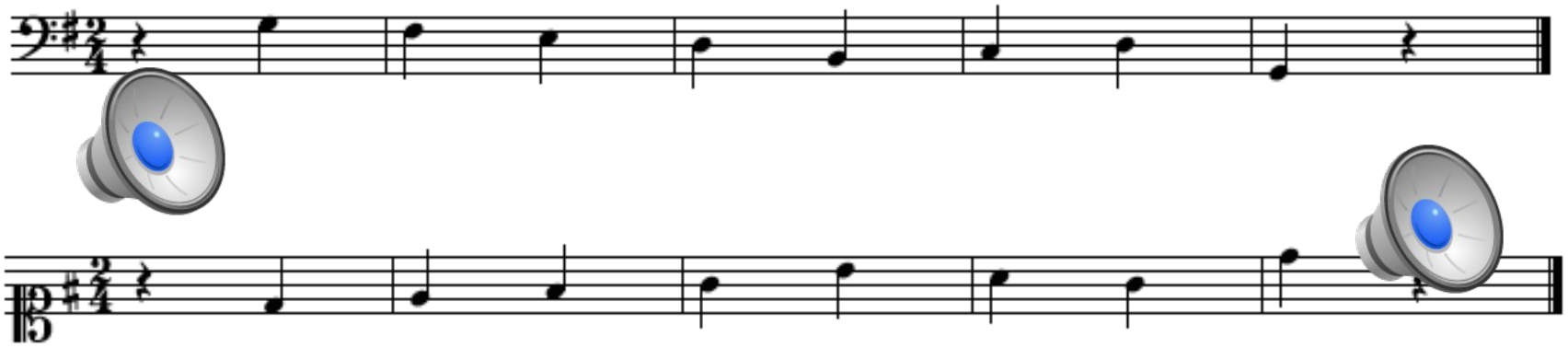
Canons 3 and 5 are part of a set labeled “motu recto et contrario.” They use both upright and contrary (upside-down) motion.



Canon 3.



Upside-down alto clef Bass clef “Segno” where second voice comes in.



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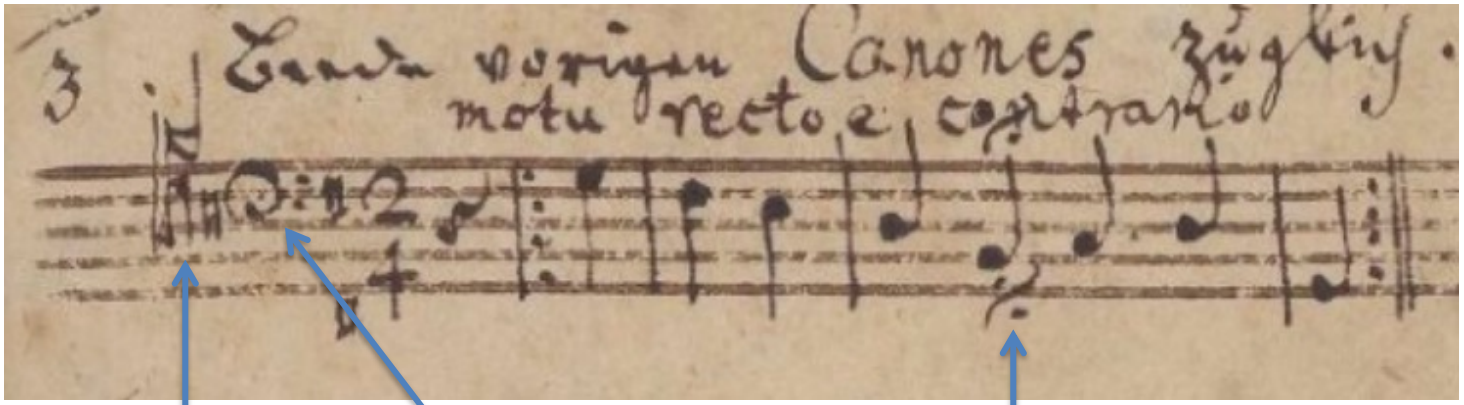

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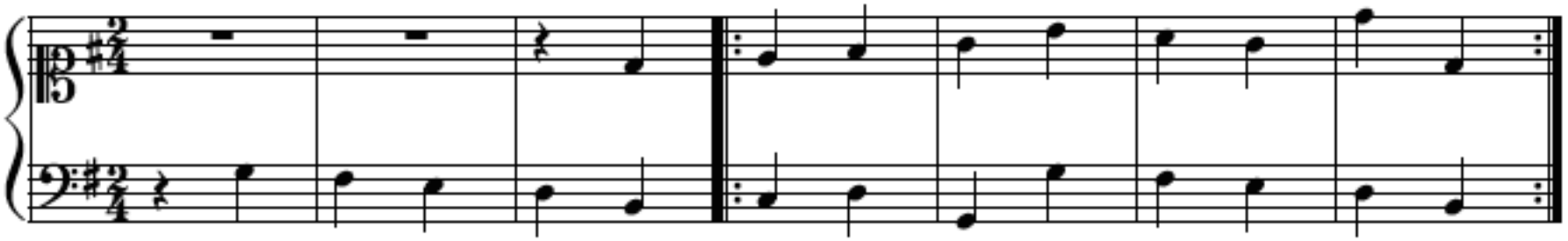
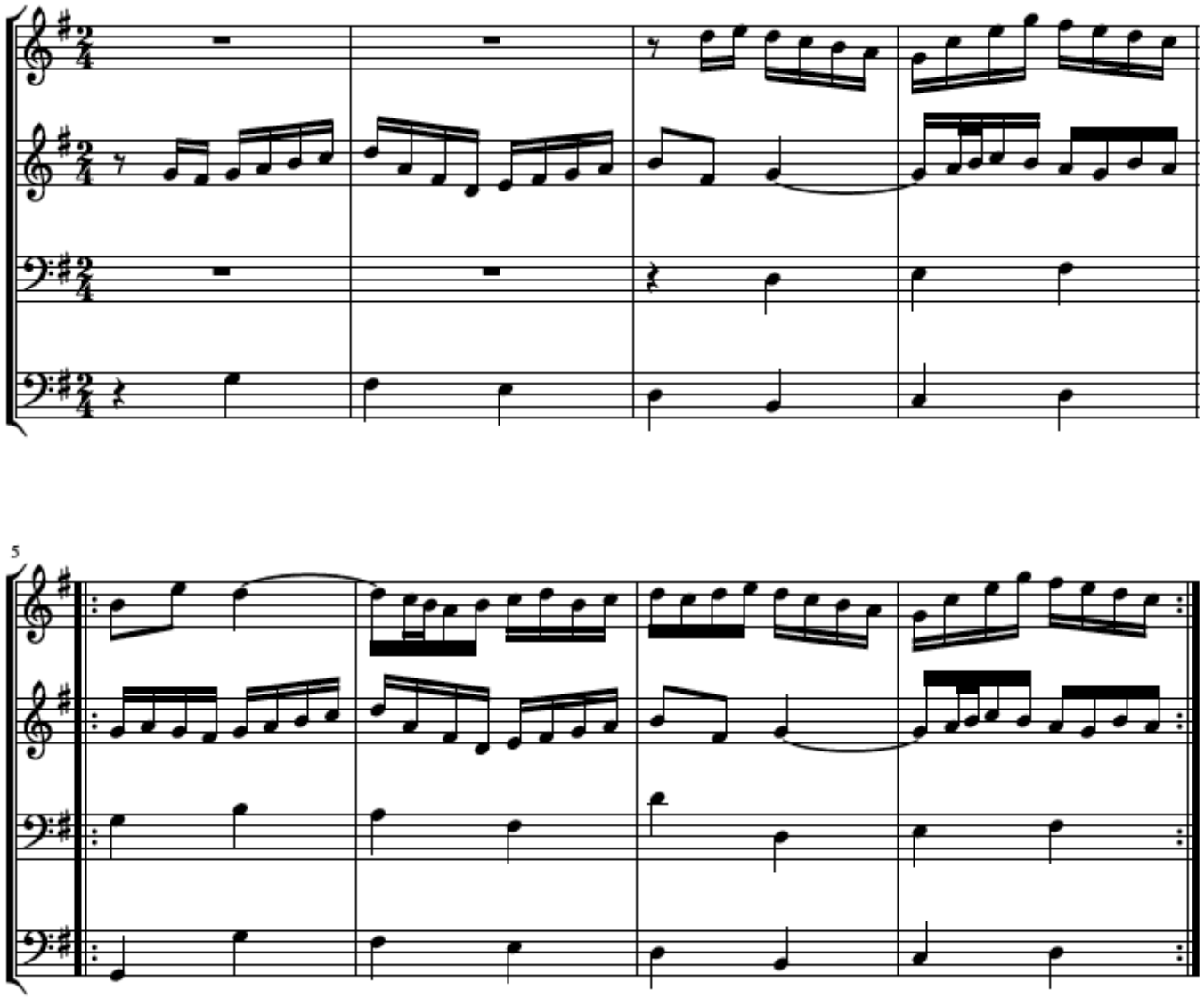


Canon 5. This canon has 4 voices. The lowest two sing **Canon 3**.

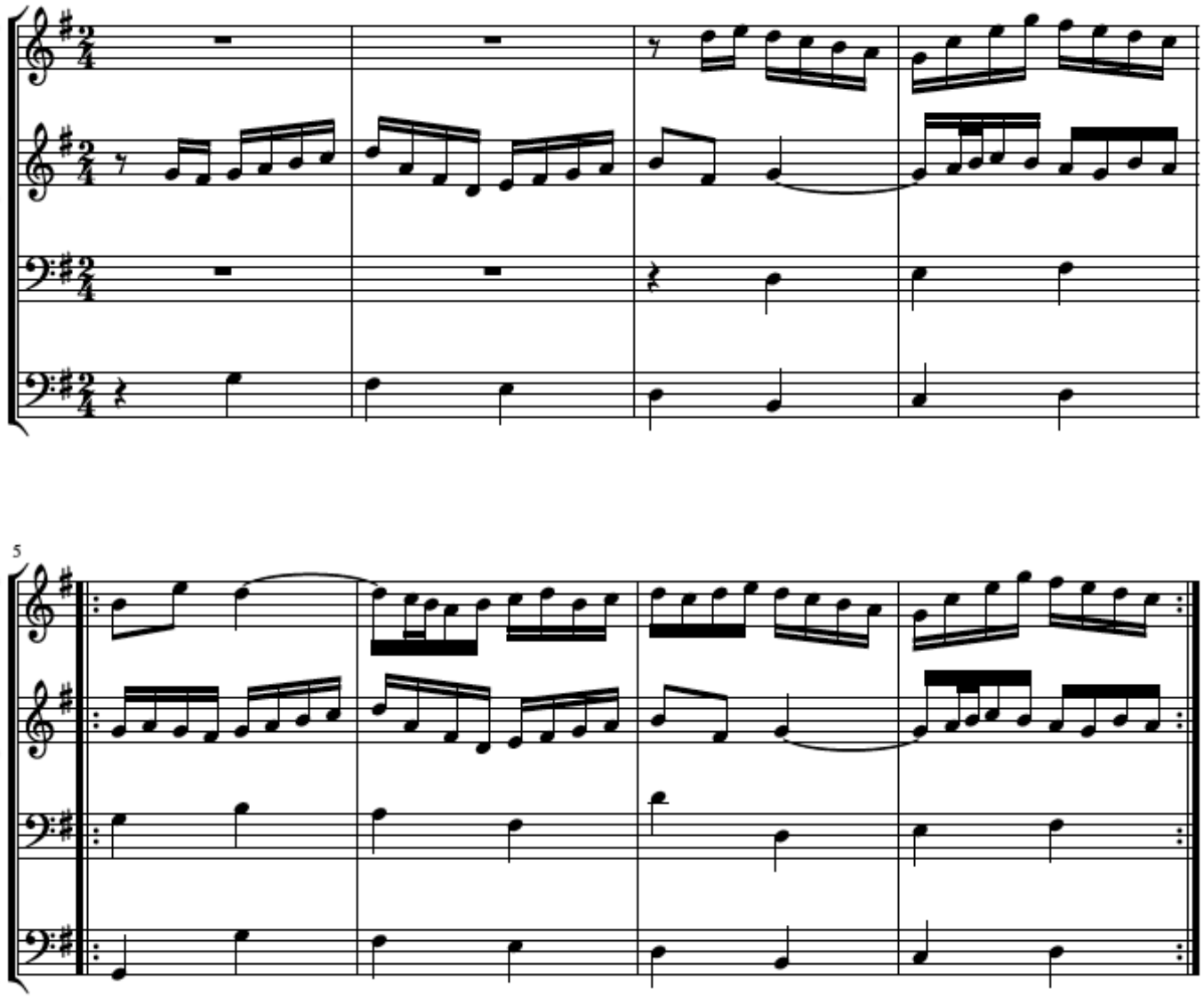
Canon 3



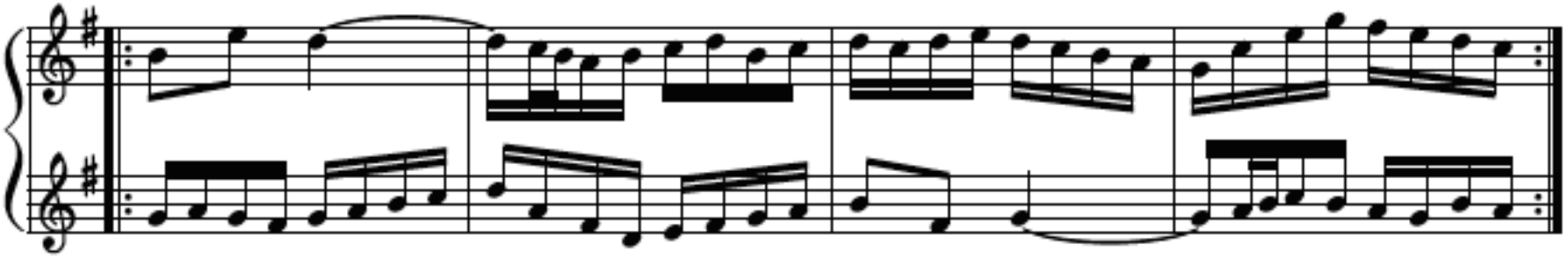
Canon 5. This canon has 4 voices. The lowest two sing **Canon 3**.

Top 2 voices are also inverses.

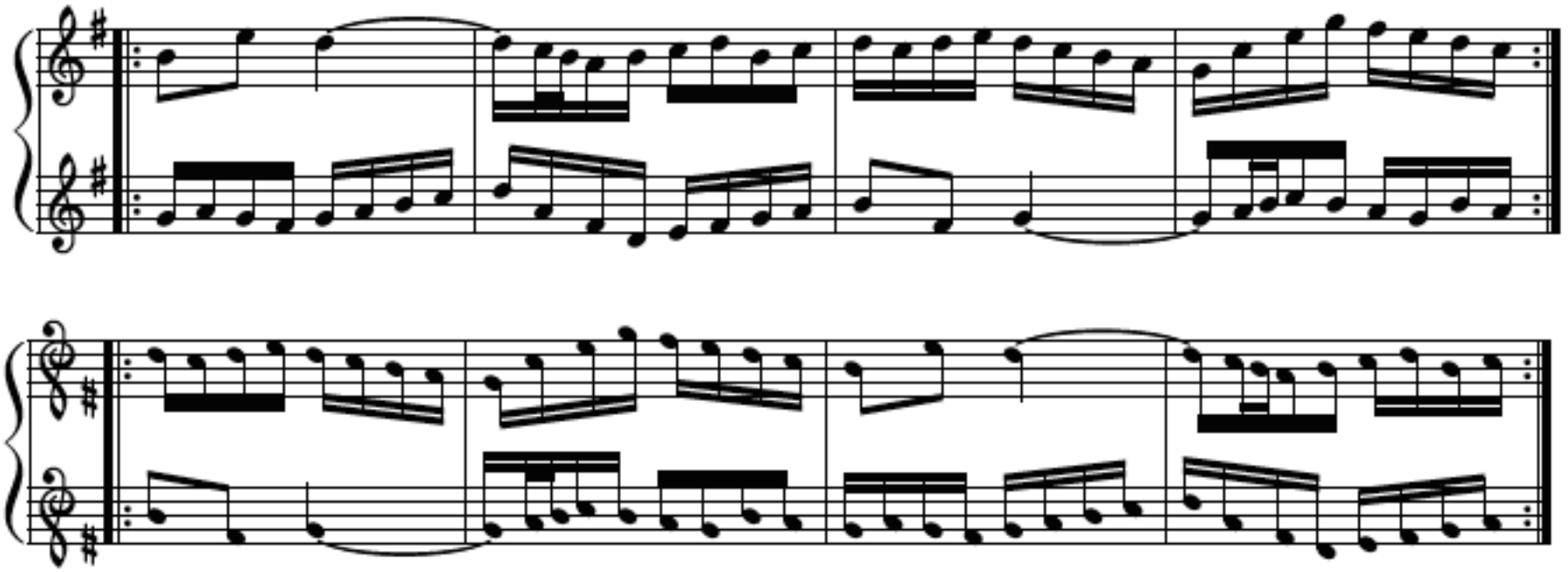
Canon 3



Analysis of the top two voices in the steady state.

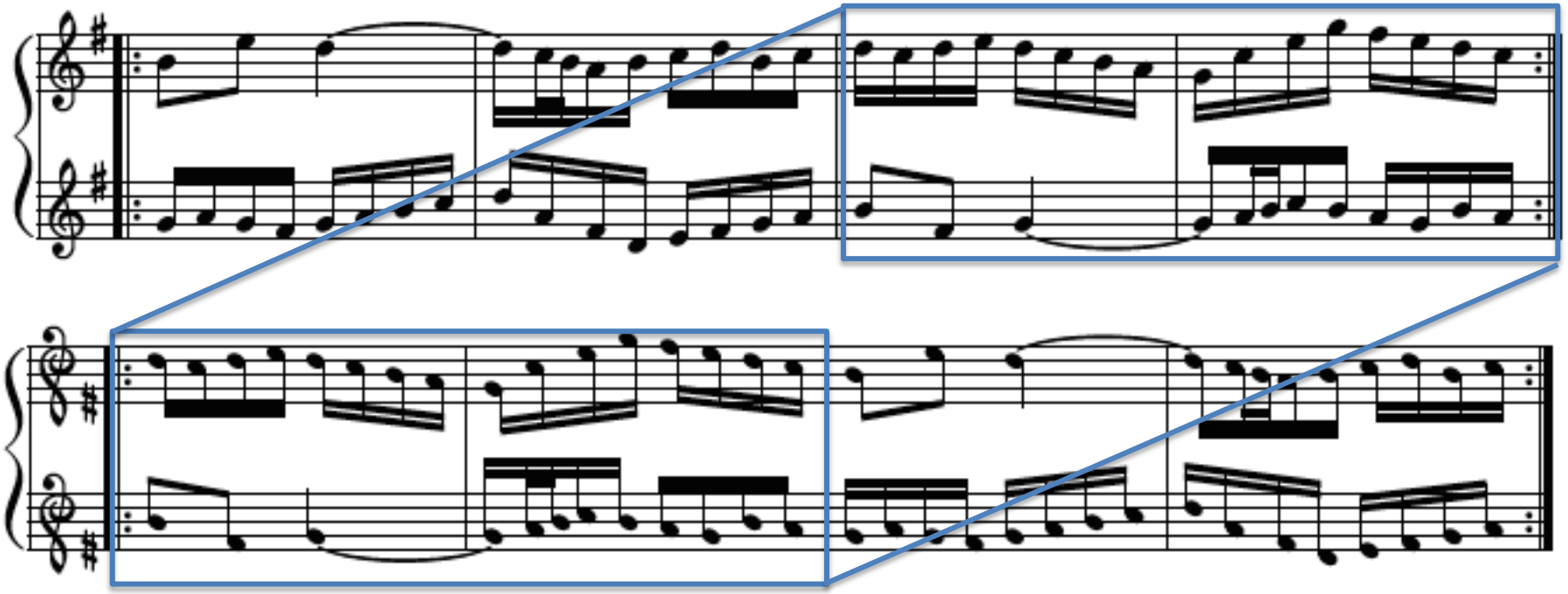


Analysis of the top two voices in the steady state.



Mirror image

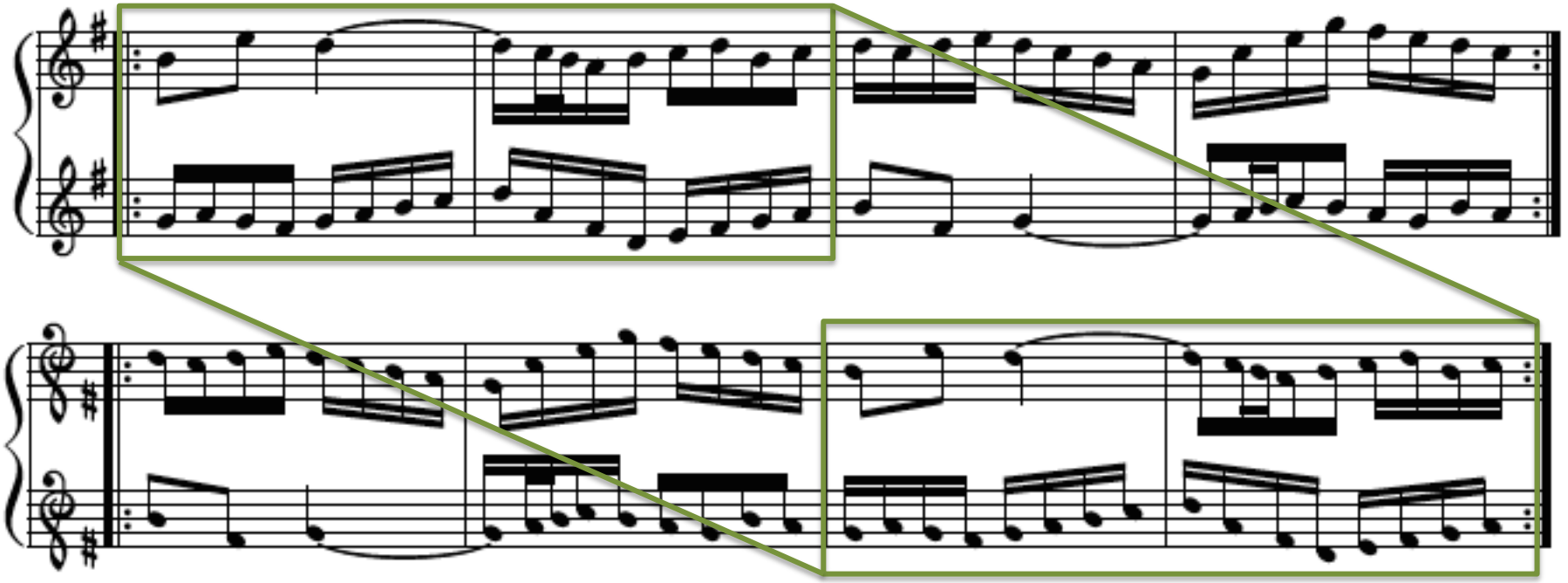
Analysis of the top two voices in the steady state.



“Optical” mirror image (inversion).

The last two measures are the upside-down mirror image of the first two.

Analysis of the top two voices in the steady state.



"Optical" mirror image (inversion).

The last two measures are the upside-down mirror image of the first two.
(And, necessarily, vice-versa).

This symmetry is a special case of a ***glide reflection***.

A periodic text in which the second half is a glide reflection of the first

can be encoded in a Möbius strip.

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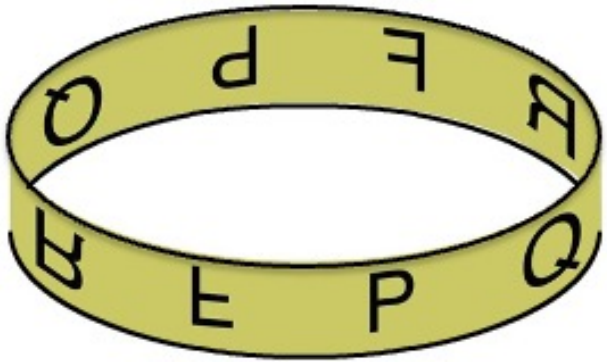
can be encoded in a Möbius strip.



1. Periodicity means it can be read from a cylinder.

A periodic text in which the second half is a glide reflection of the first

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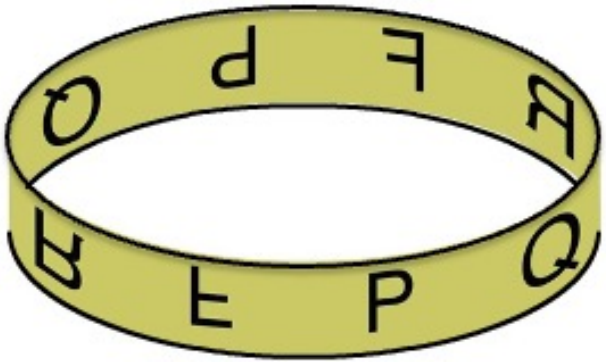

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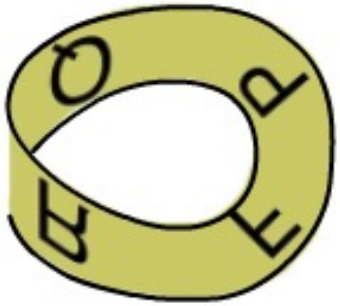

2. Half-and-half glide-reflection symmetry means the cylinder can be rolled up on itself, with a twist, to form a Möbius strip.

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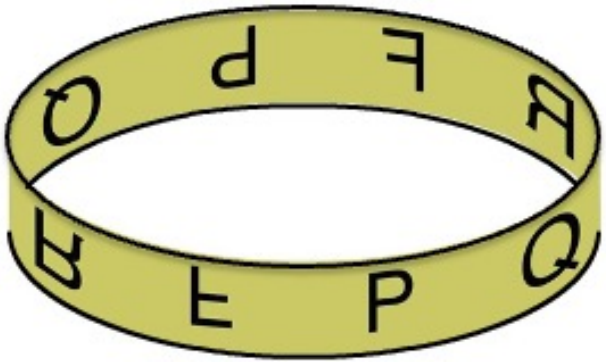

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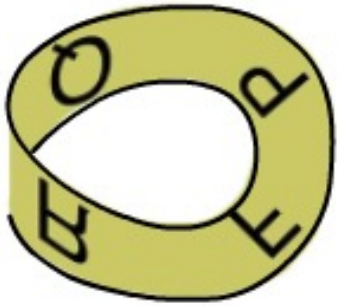
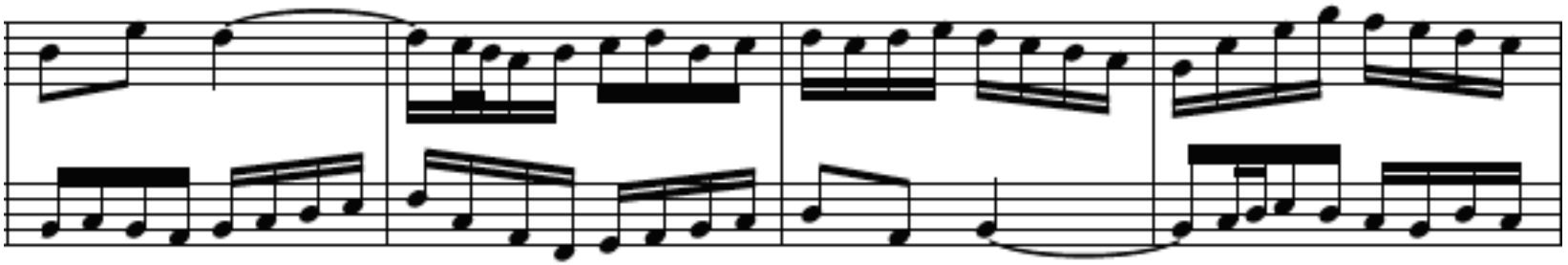


2. Half-and-half glide-reflection symmetry means the cylinder can be rolled up on itself, with a twist, to form a Möbius strip.

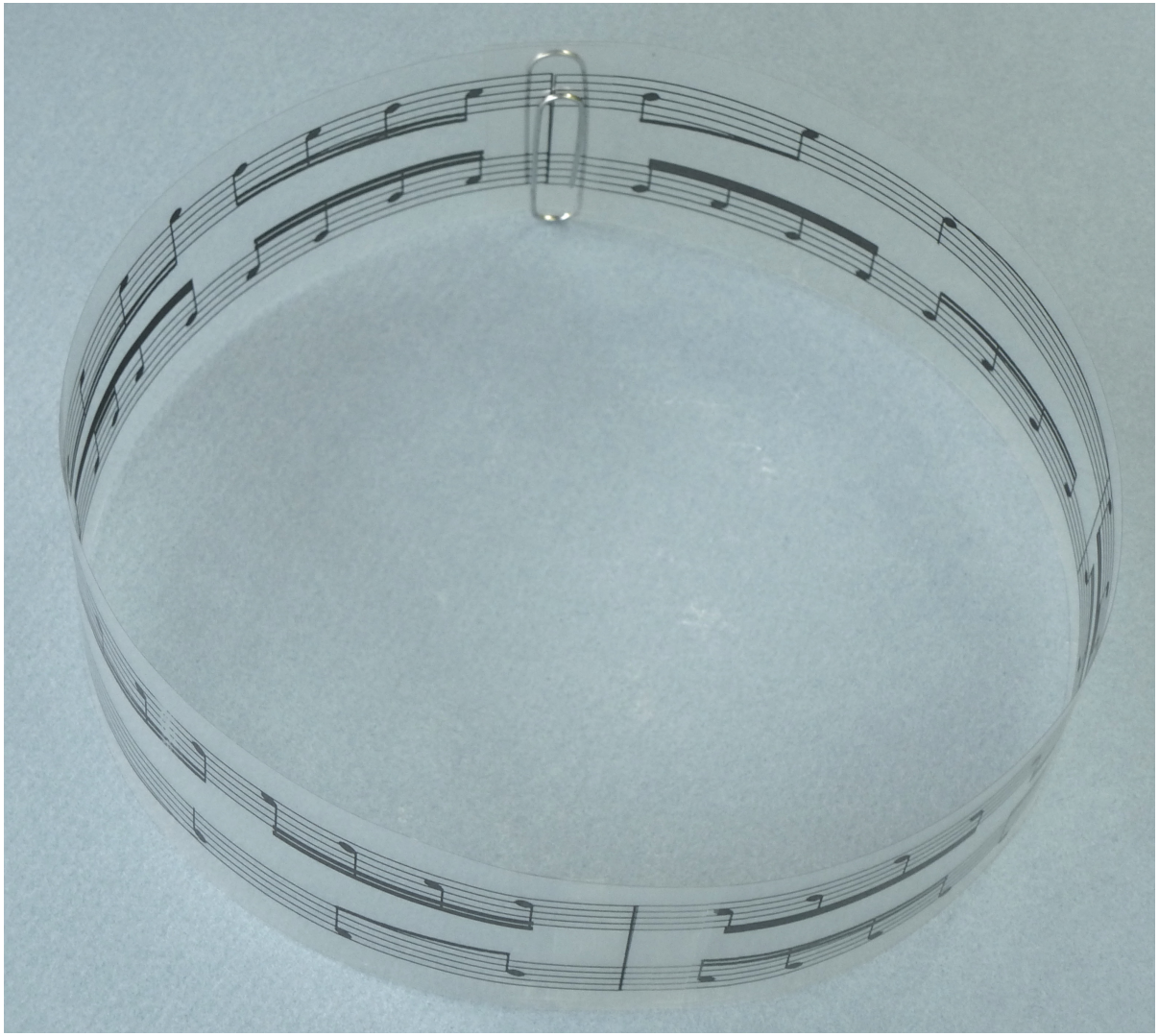
Same Möbius strip, flipped over.

Note that the text is ***in the strip*** (not on the surface) so it can be read from both sides.

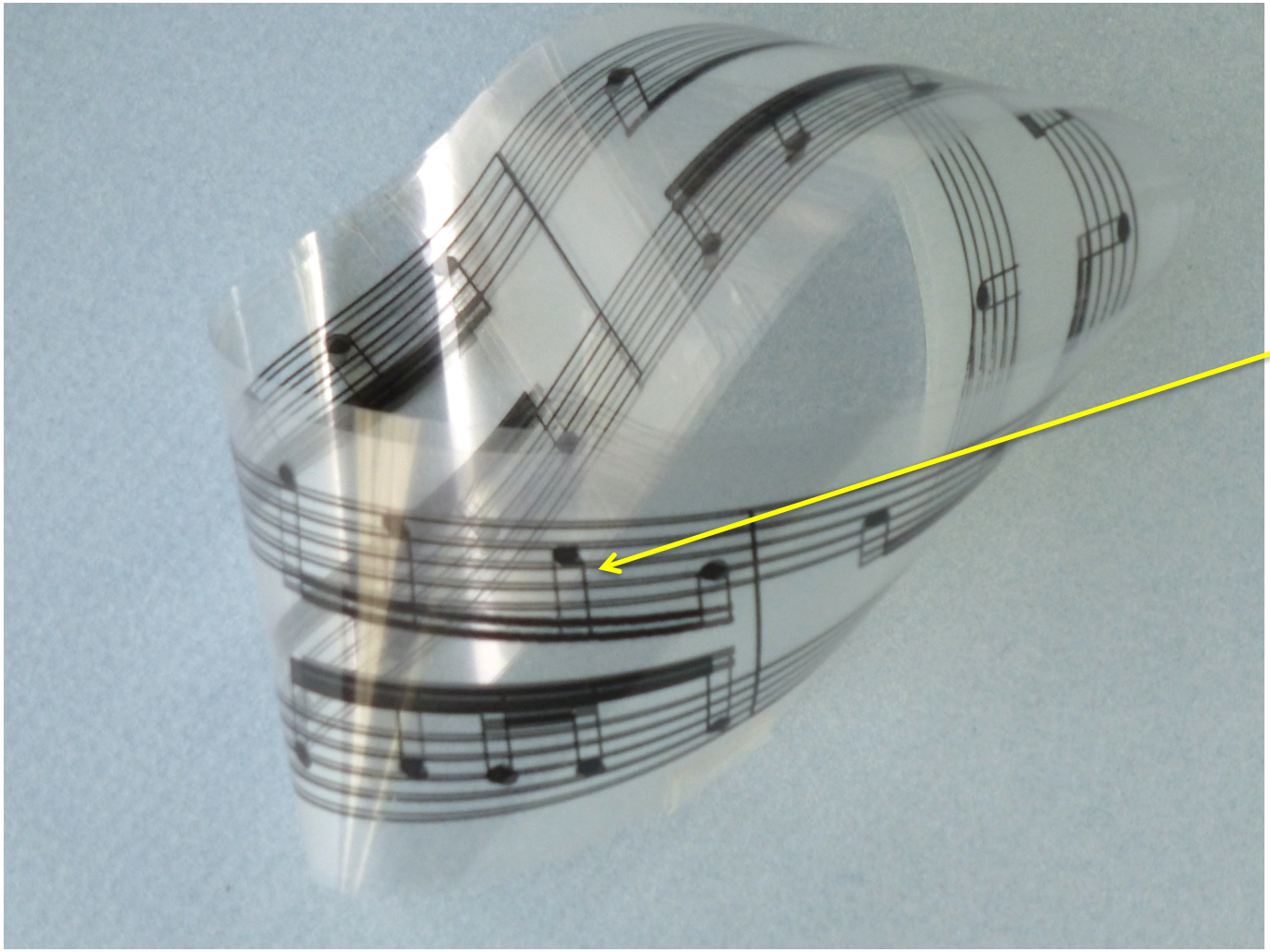
The score of the steady state of Canon 5
(upper two voices, without the clefs and time markings)
can be rolled into a Möbius strip so that the notes match exactly.



First as a cylinder (the notes are *in* the surface)



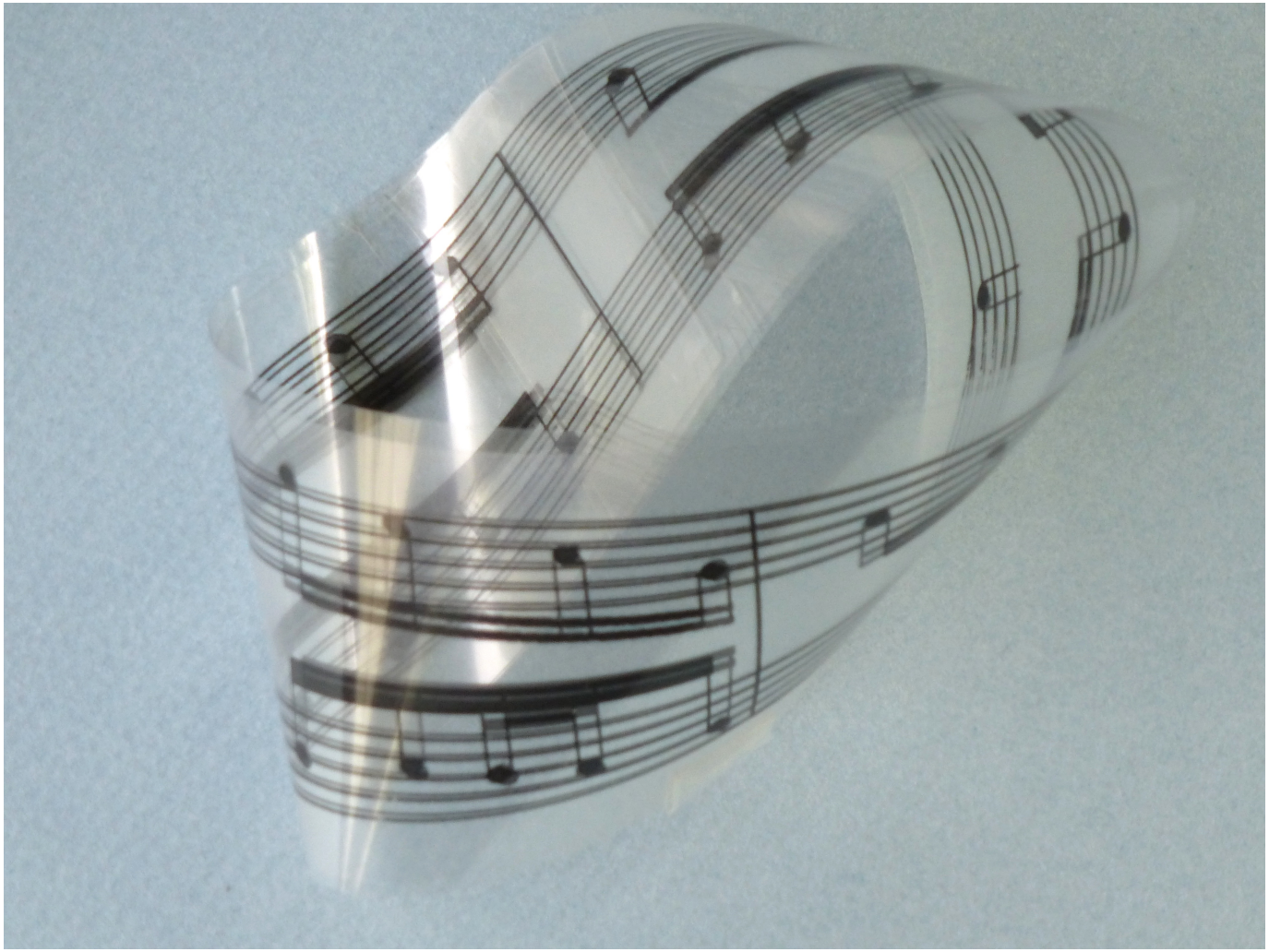
Then rolled into a Möbius strip.



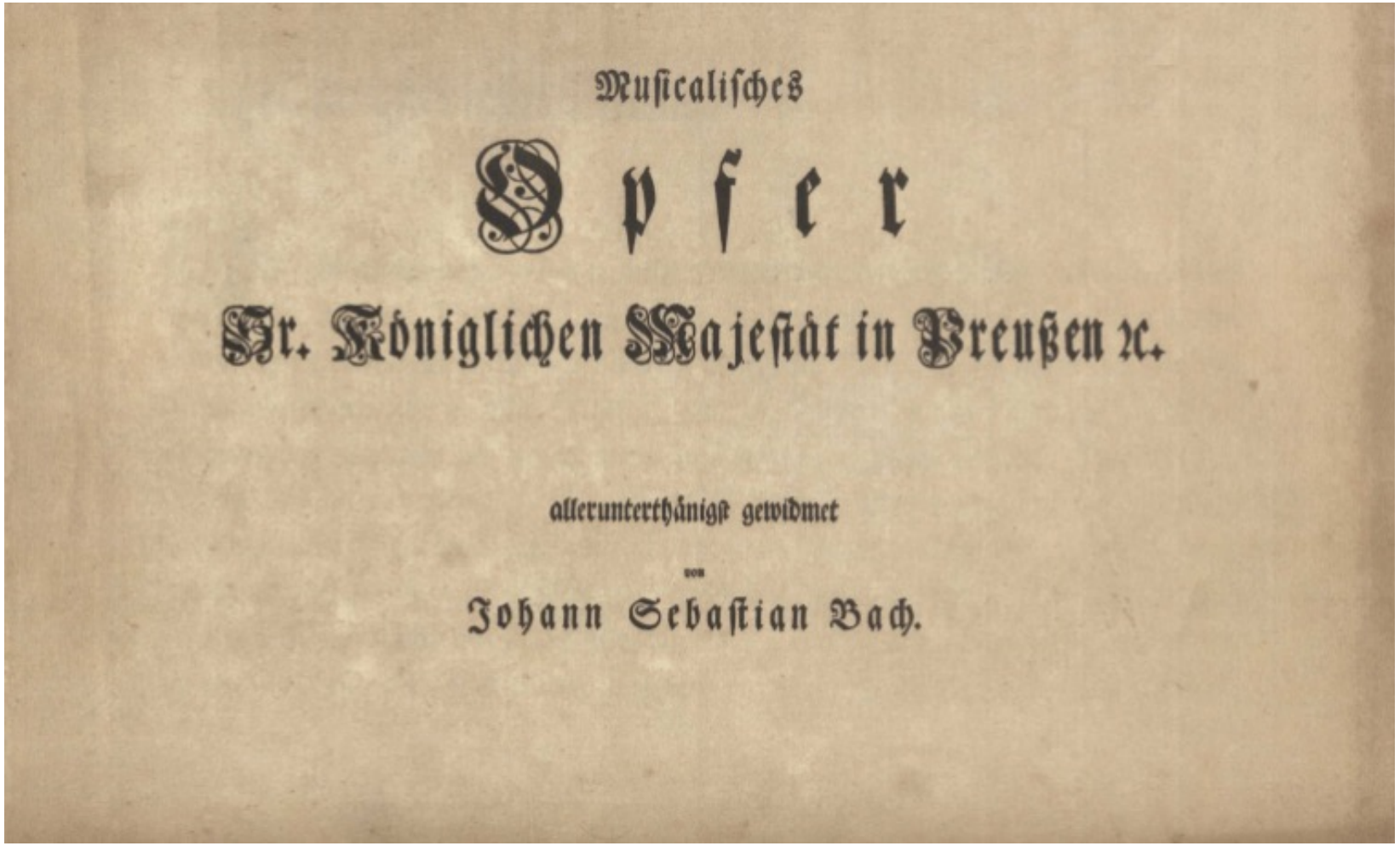
notes match exactly.

(tails are reversed).

Make your own: template available on my web page at Stony Brook.

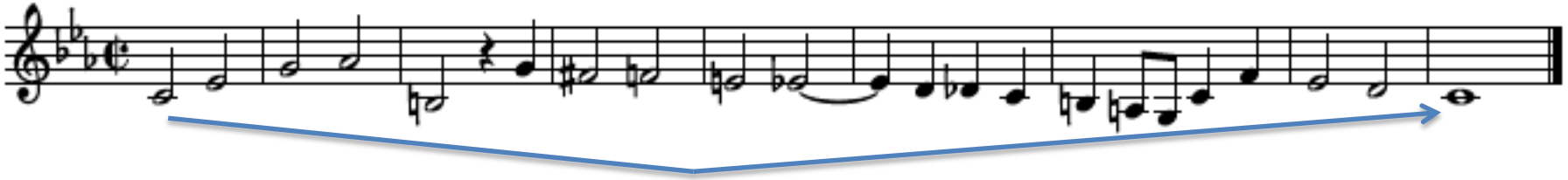


Another example: Canon 5 from
the Musical Offering, BWV 1079.



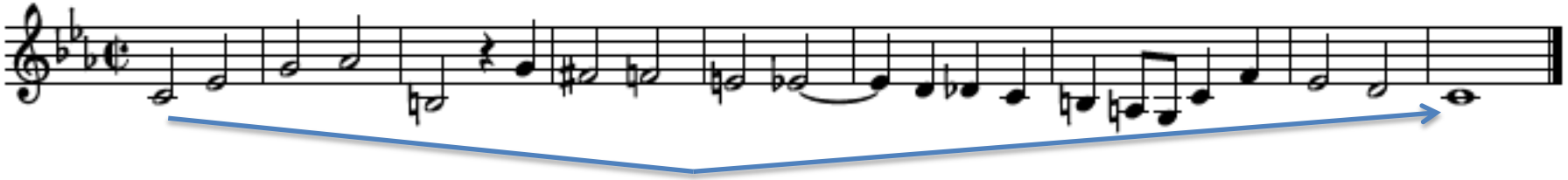
Title page of Musical Offering
“most humbly dedicated by Johann Sebastian Bach”

The elements of “The Musical Offering” are all based on a theme that was proposed to Bach by King Frederick II in 1747.



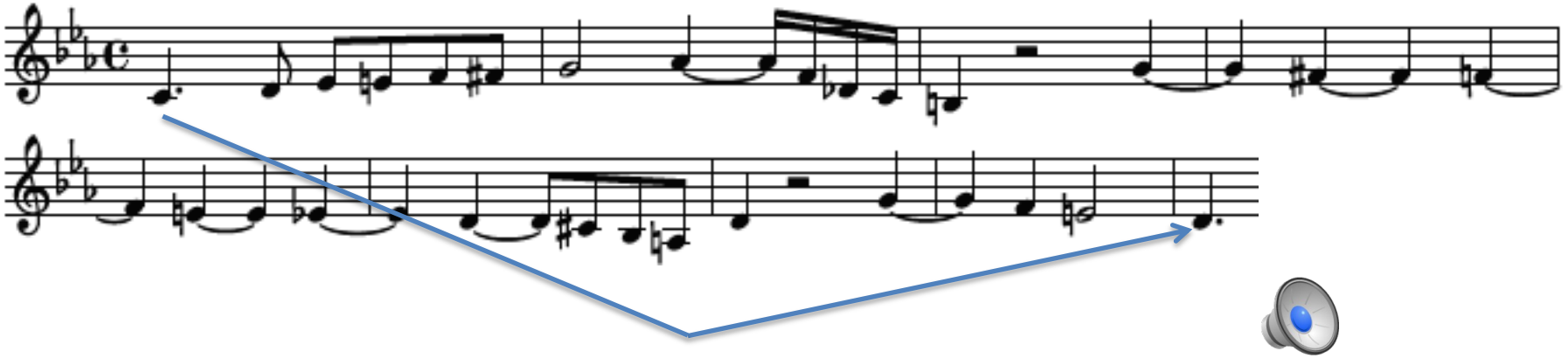
Starts and ends on the same note.

The elements of “The Musical Offering” are all based on a theme that was proposed to Bach by King Frederick II in 1747.



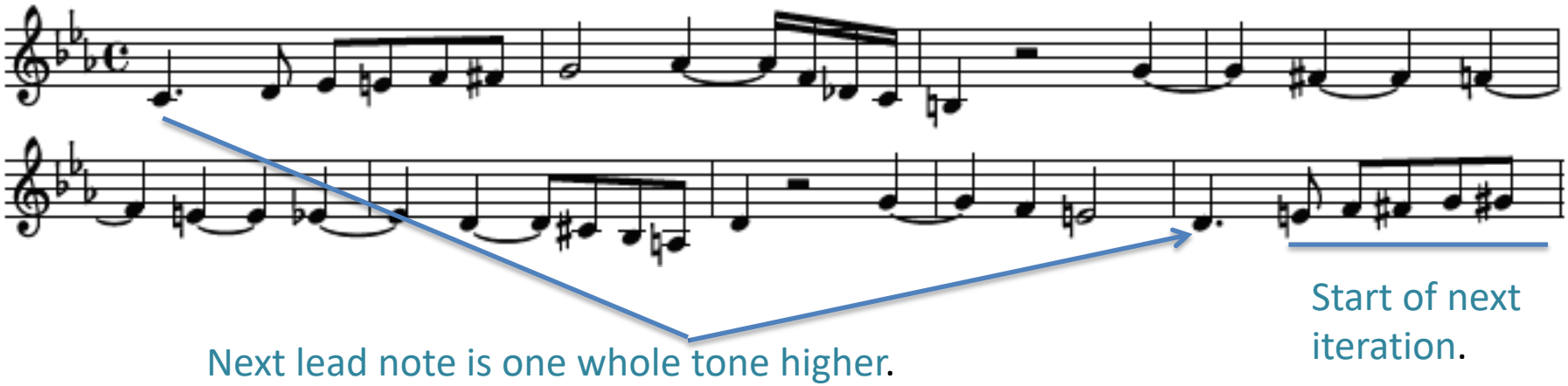
Starts and ends on the same note.

The version Bach uses in Canon 5:



Ends one whole tone higher.

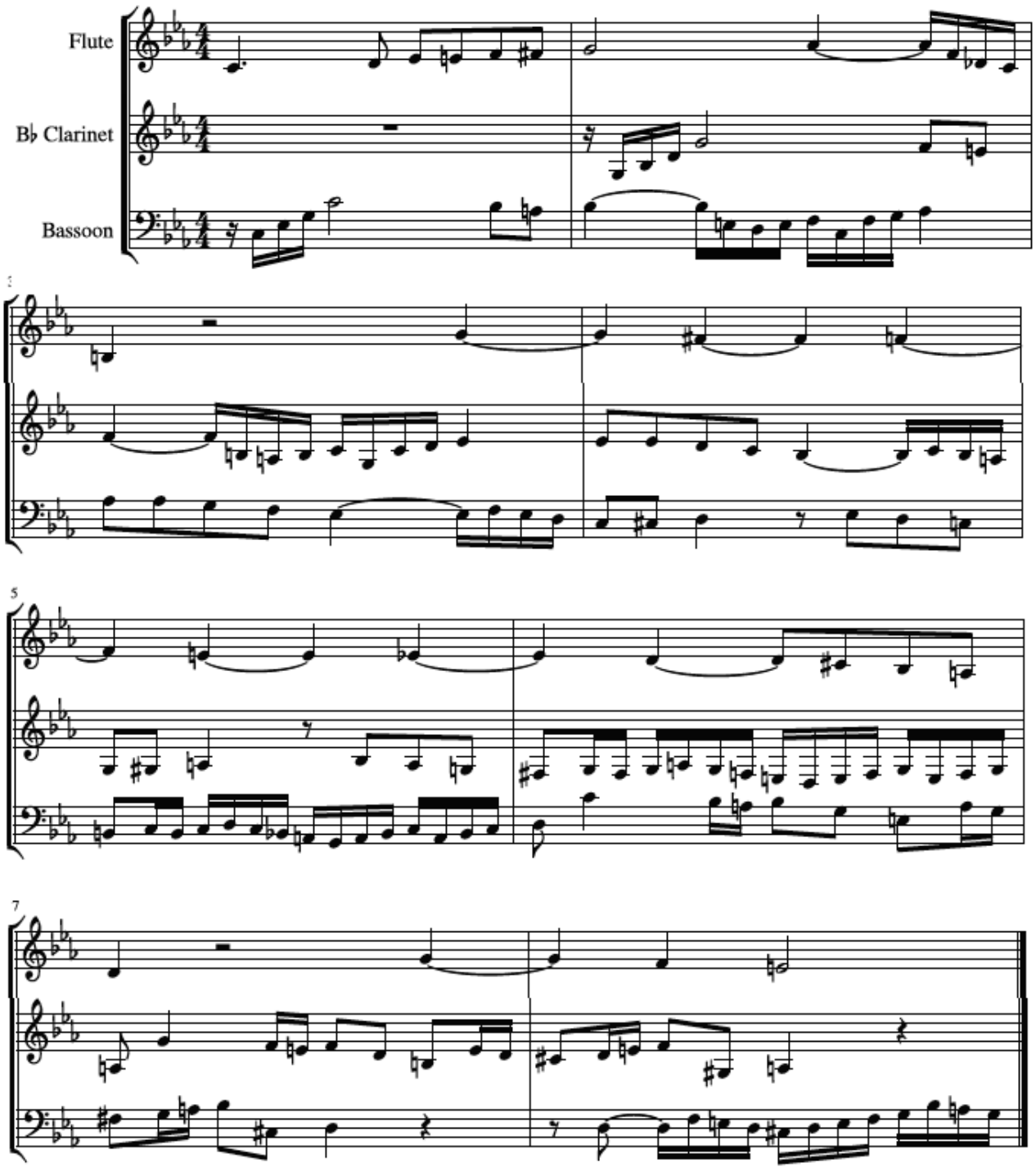
In Canon 5, Bach's version of the Royal Theme serves as a ground. It repeats as the highest voice; but each iteration starts one whole step higher than the one before.



A graphic representation of the melodic line.

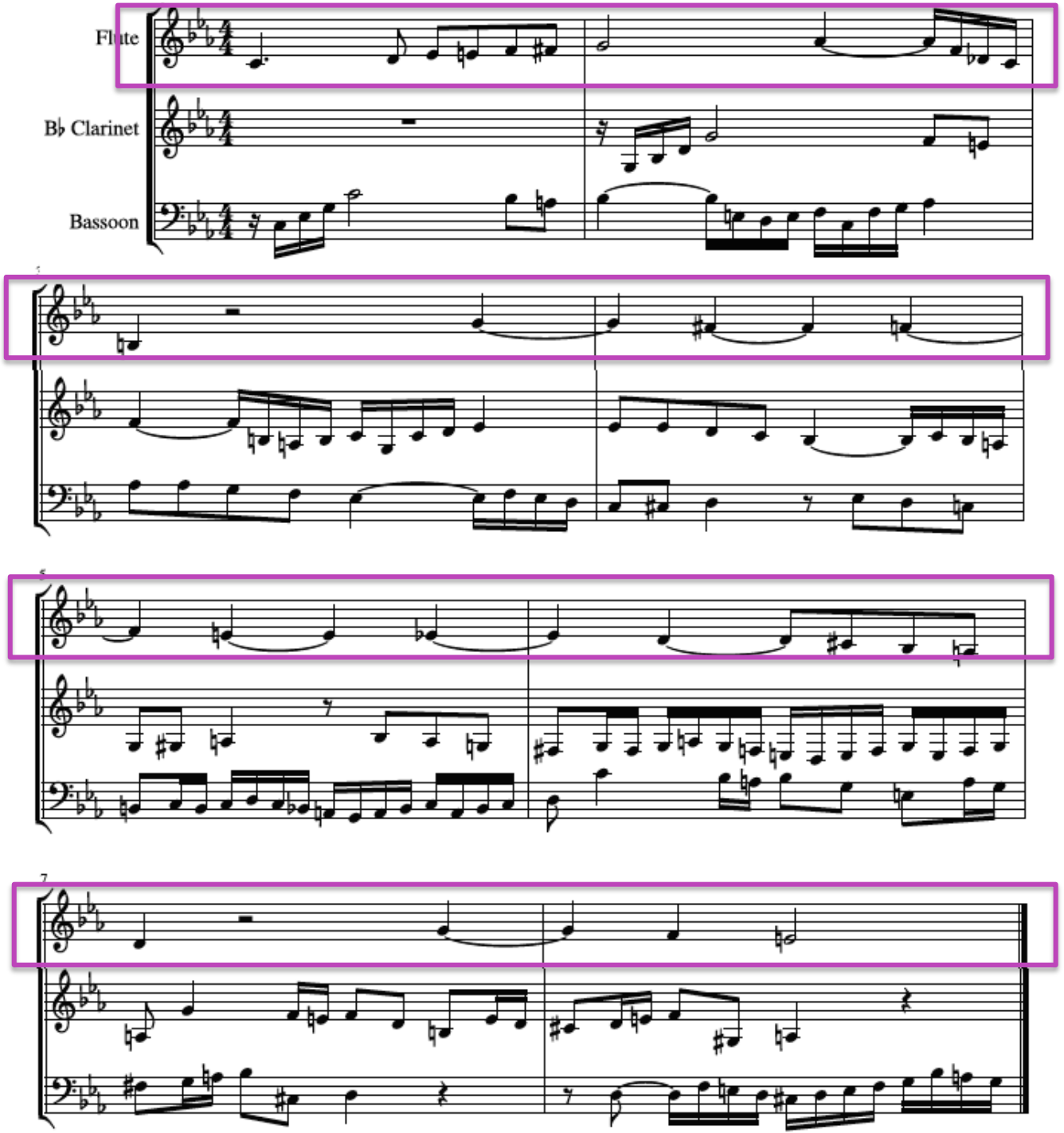
Canon 5 is a 2-part canon which repeats *with the same modulation* as the ground.

One iteration.



One iteration.

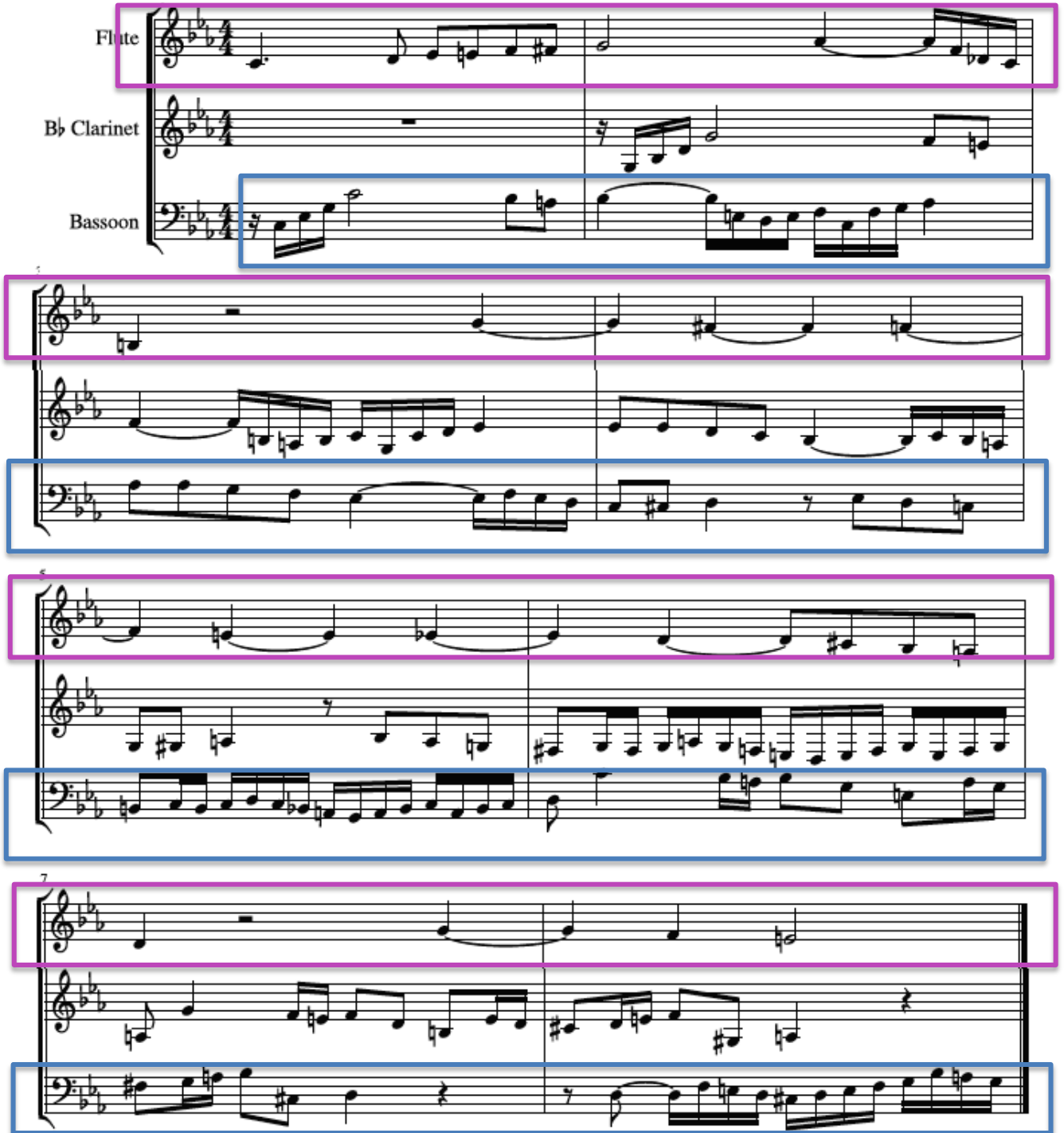
Top line:
Royal Theme



One iteration.

Top line:
Royal Theme

First canon voice

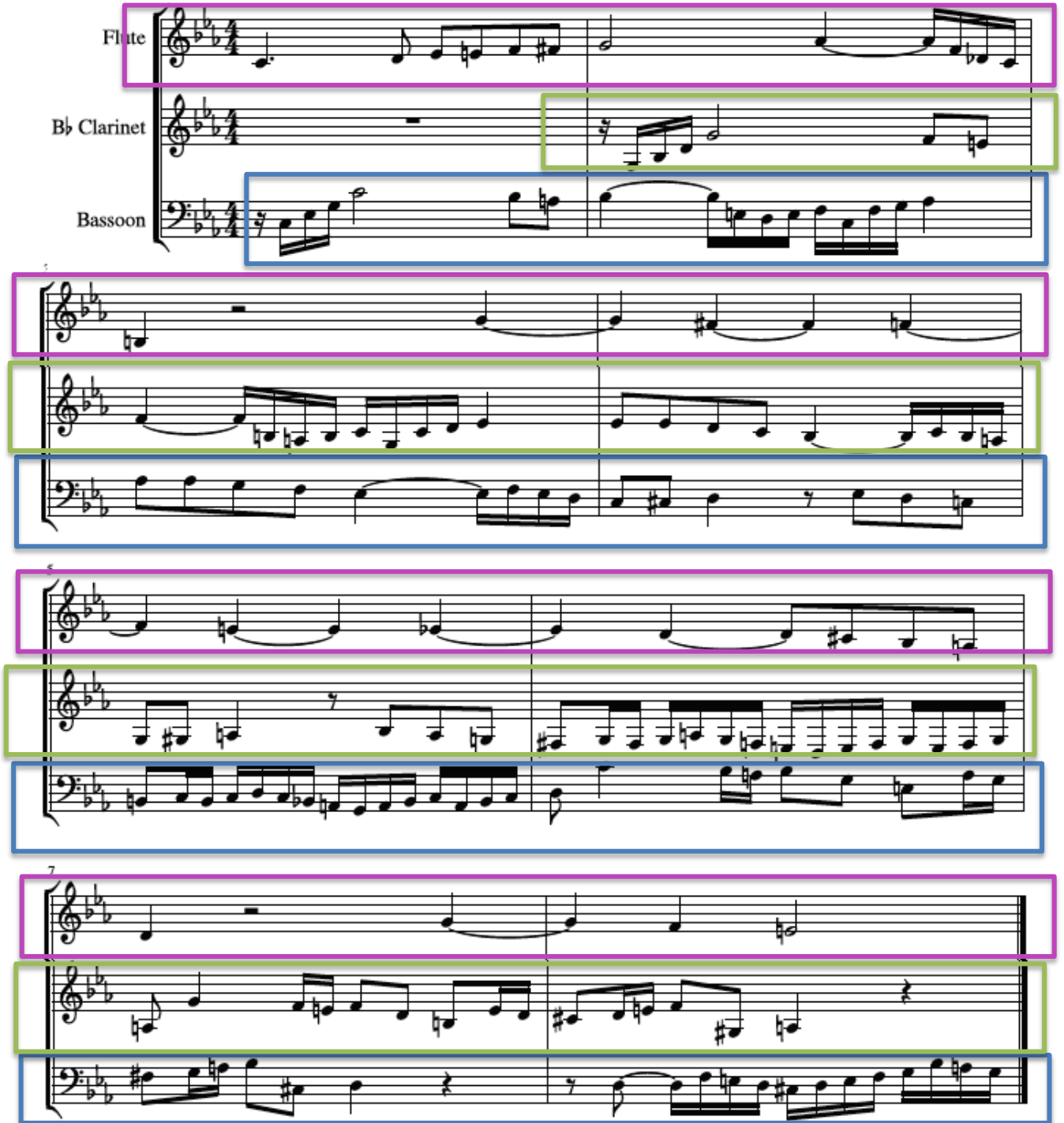


One iteration.

Top line:
Royal Theme

First canon voice

Second canon voice:
one measure later,
one fifth higher.



Here is the whole Canon, in a recording by Michael Monroe. (It lasts 3 minutes).
We will follow –graphically- the modulations of the Royal Theme.

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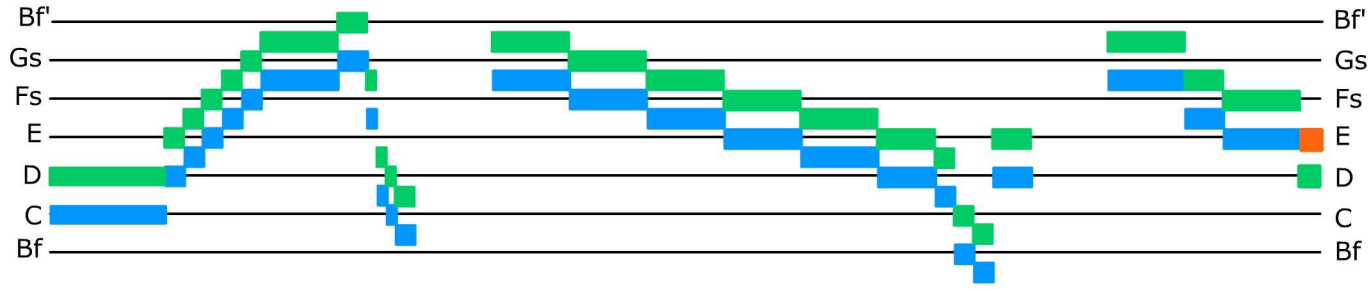
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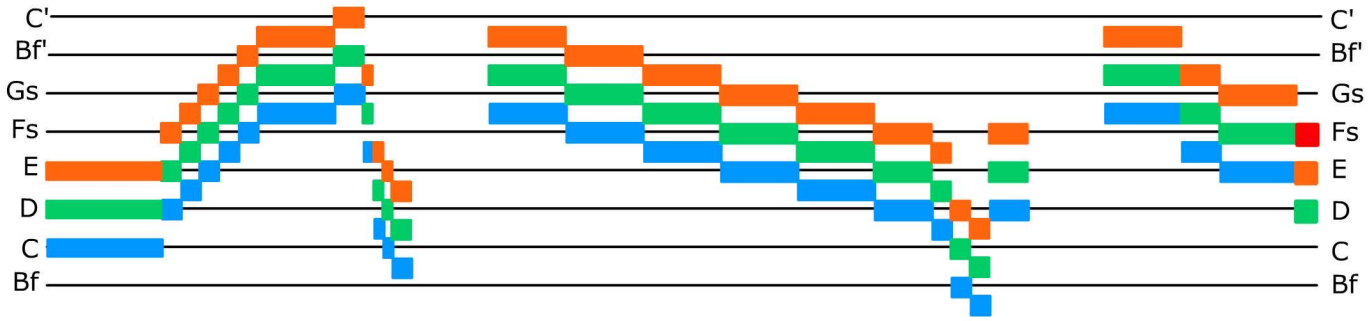
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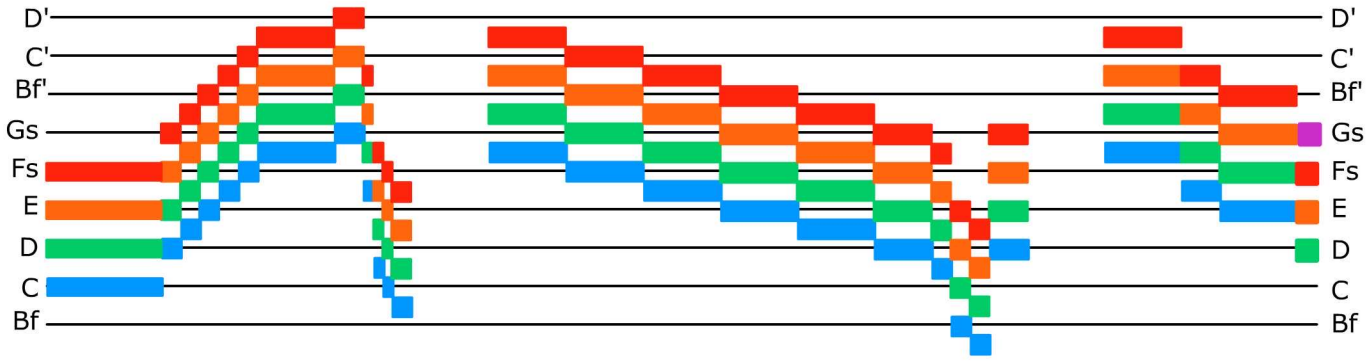
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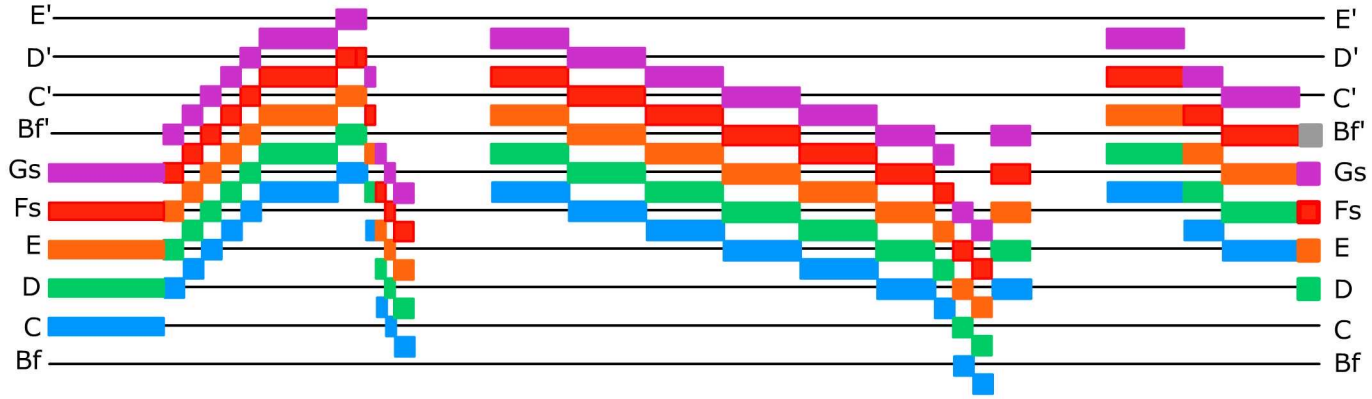
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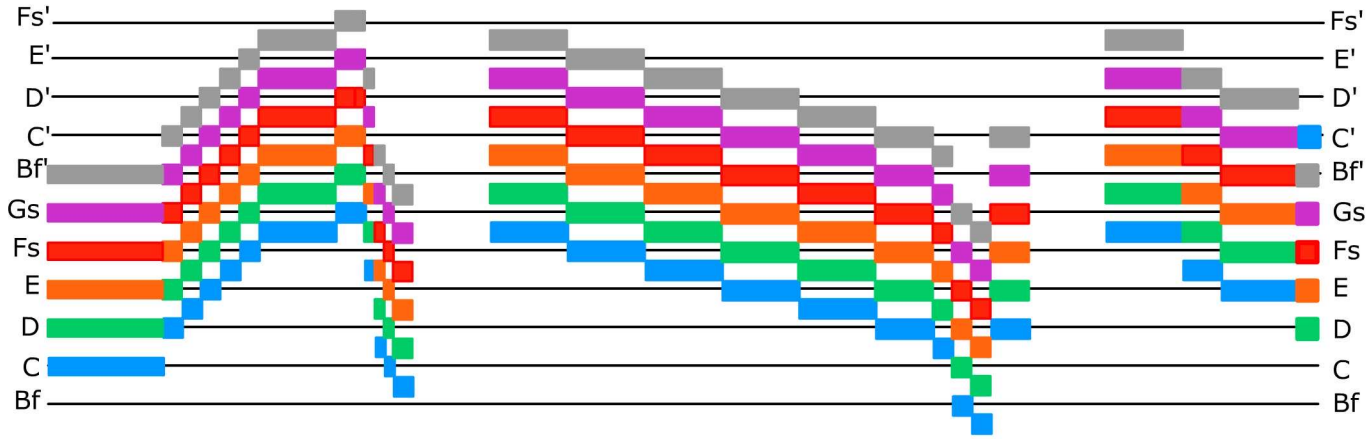
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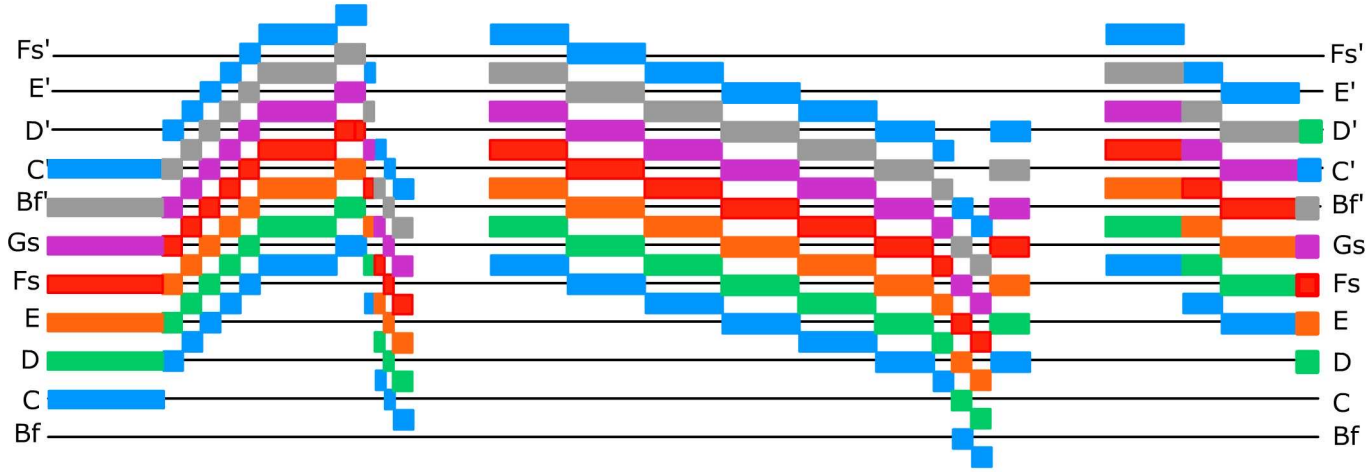
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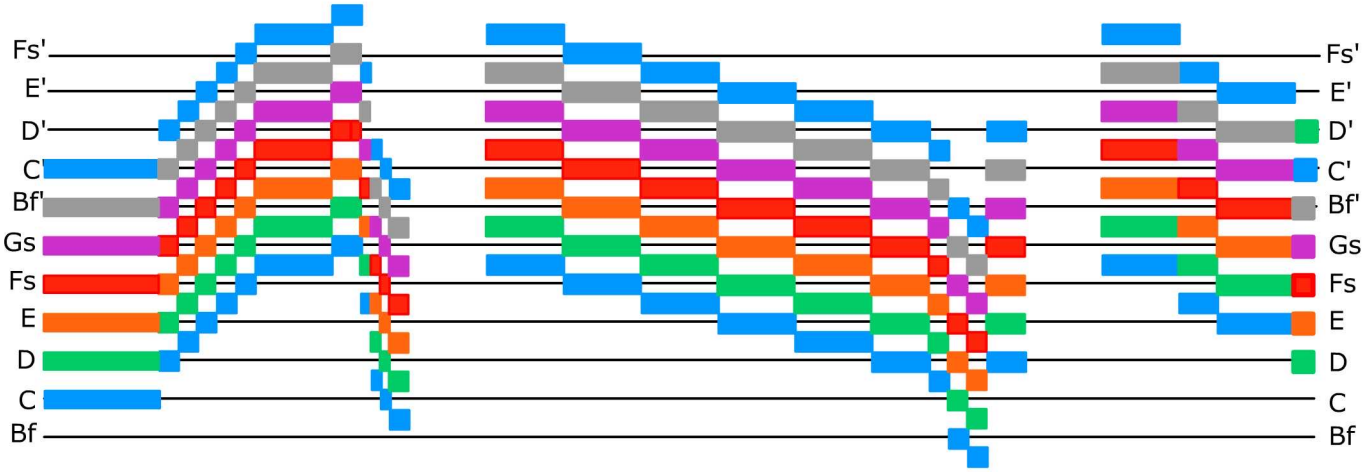
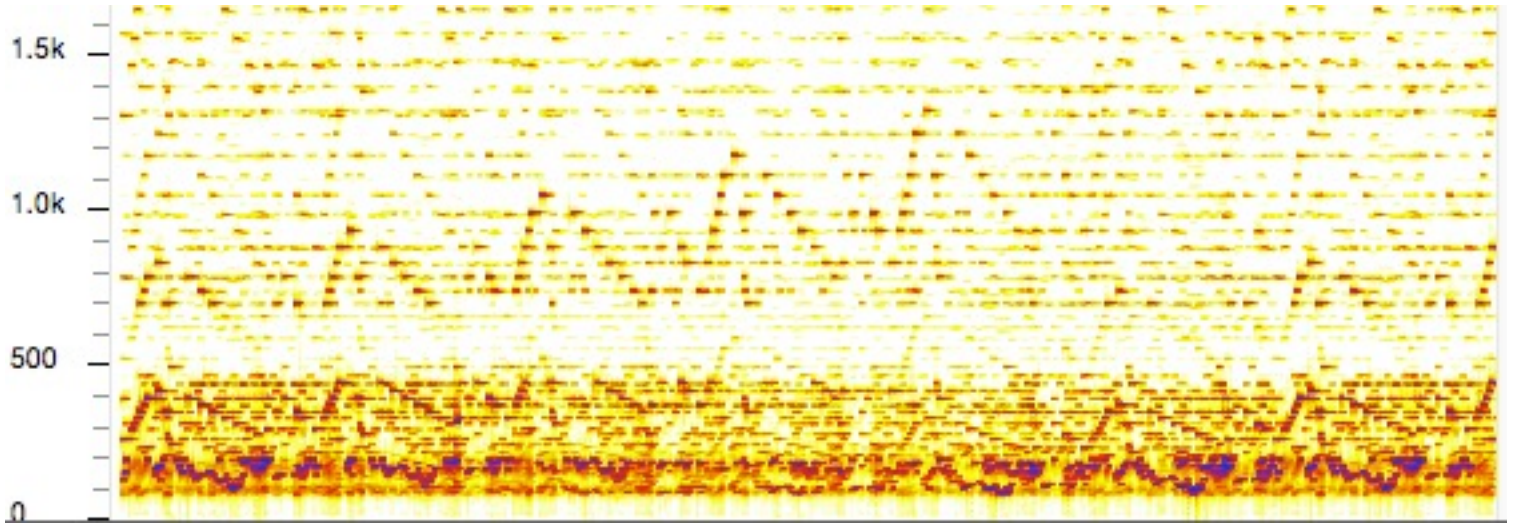
Bf

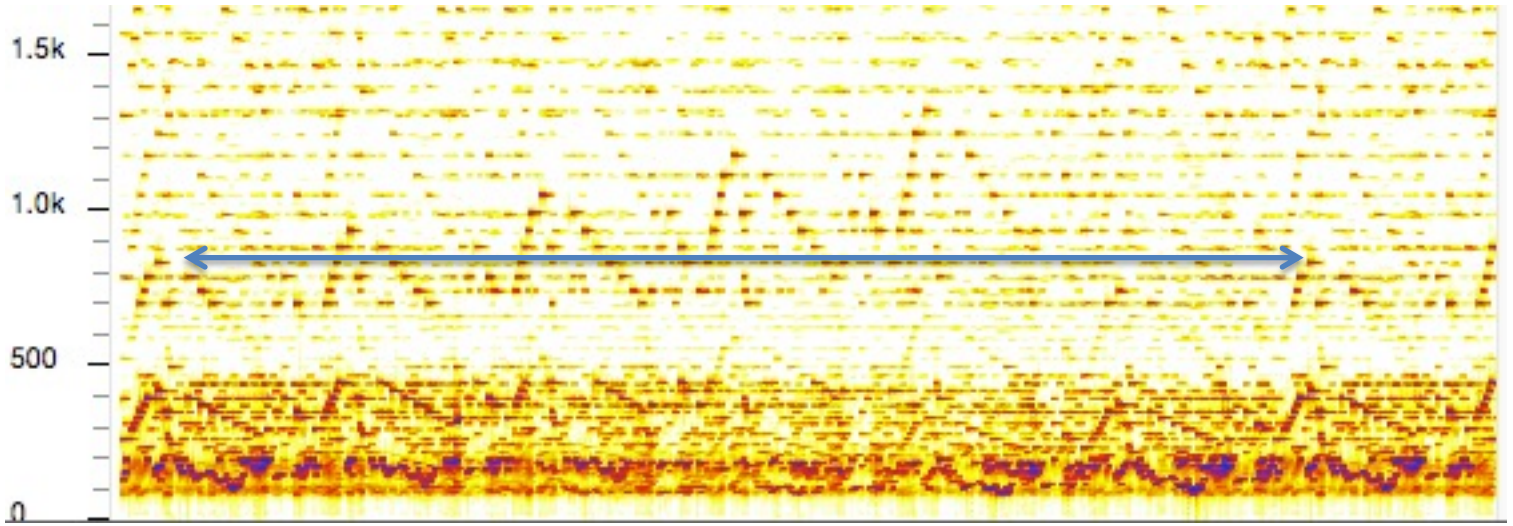
This is an illusion.





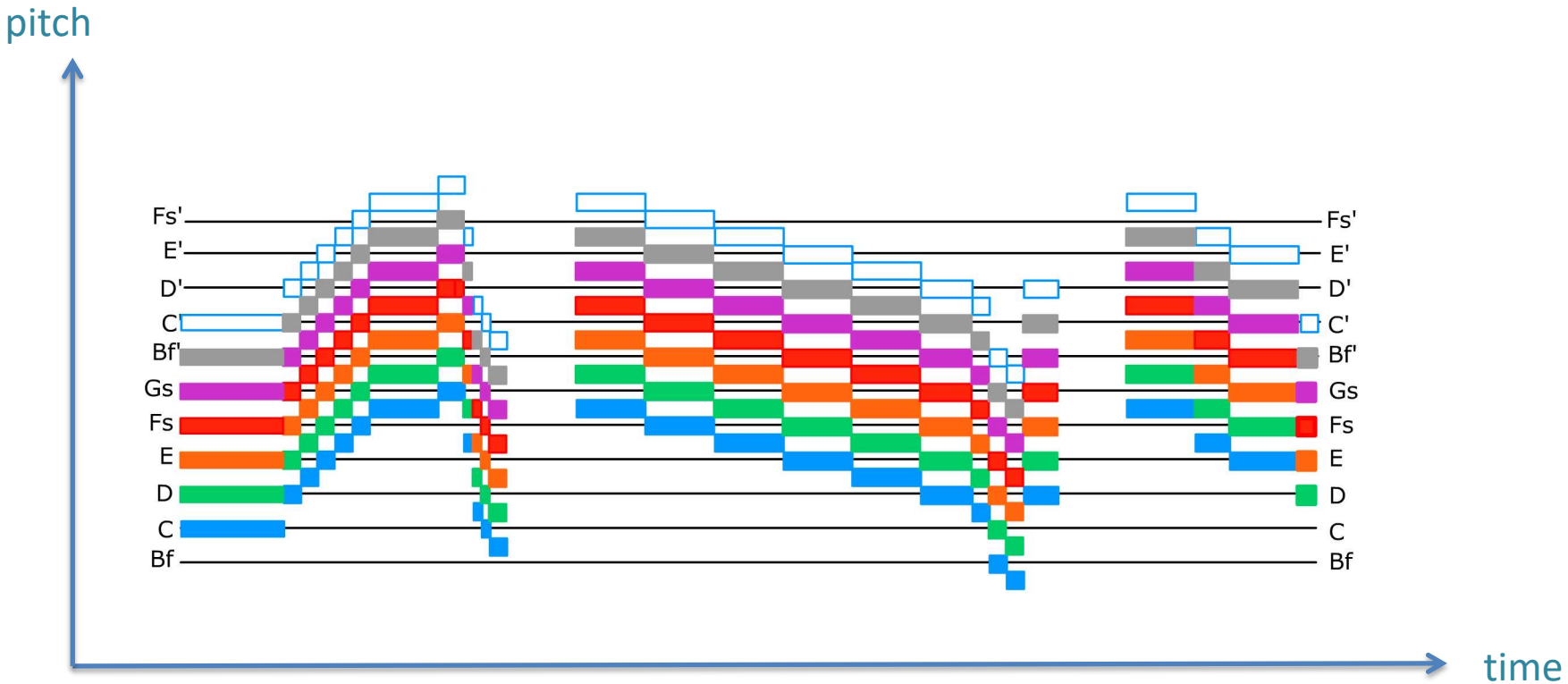
A sonogram of Michael Monroe’s recording of Canon 5.

The seventh iteration is *at the same pitch* as the first.



A sonogram of Michael Monroe's recording of Canon 5.

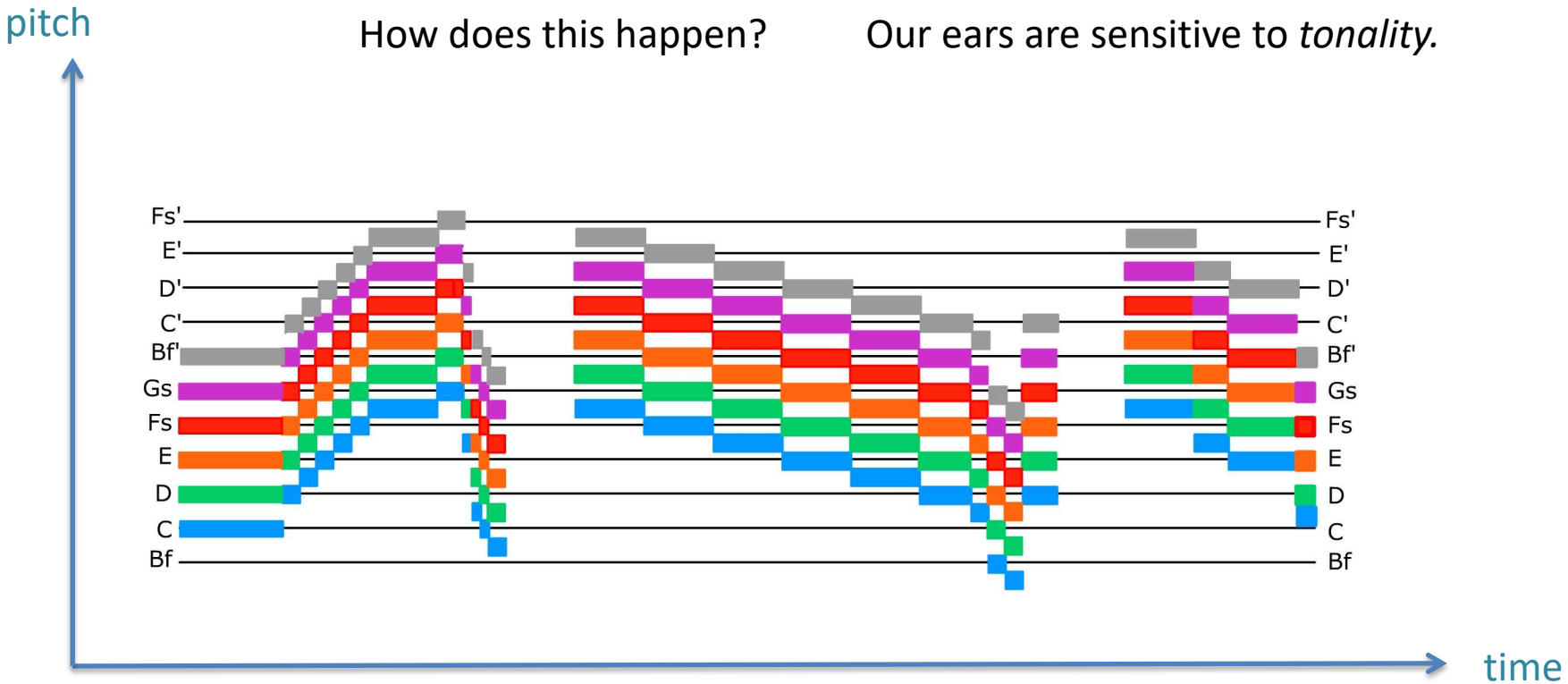
We perceive the pitch as rising.

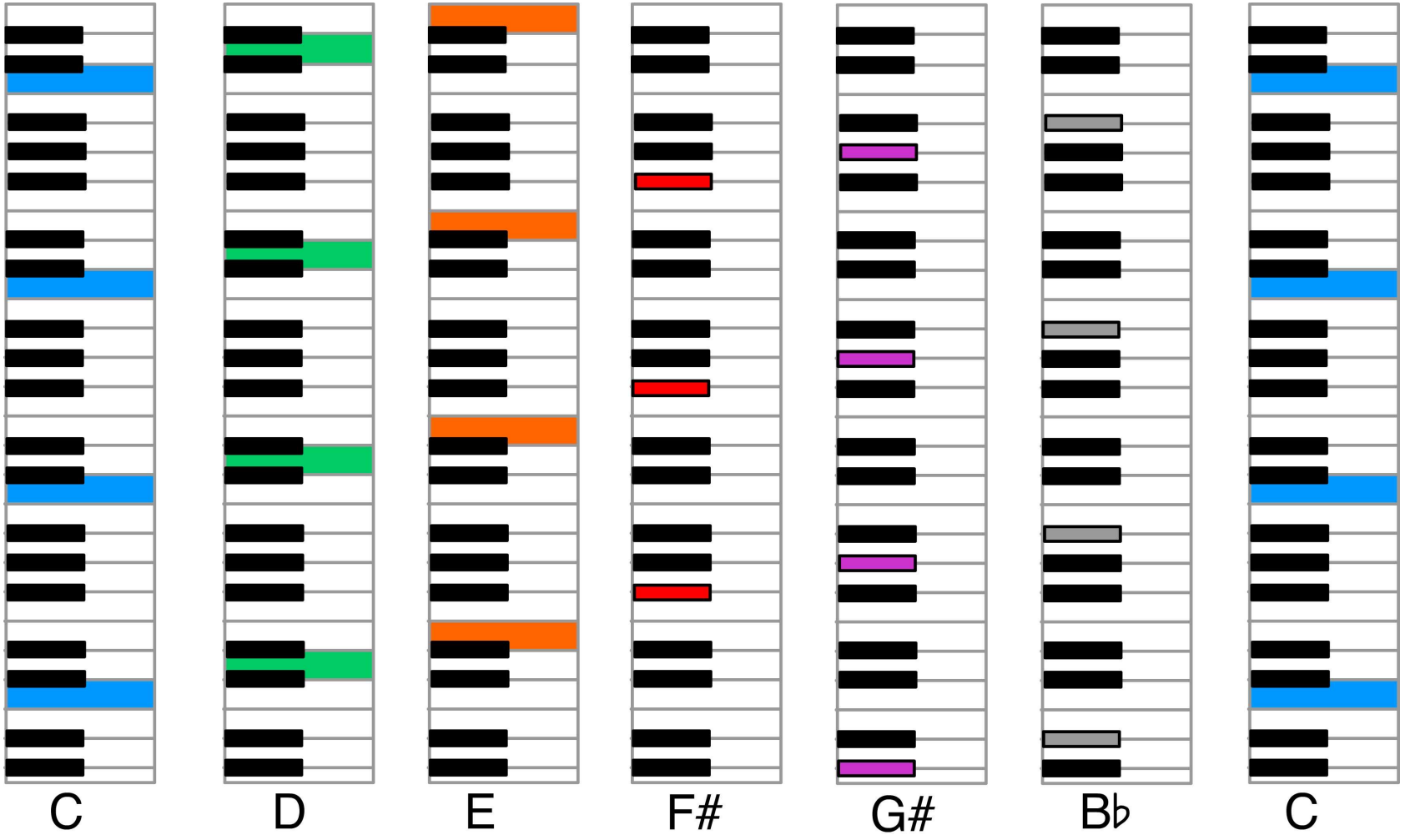


We perceive the pitch as rising.

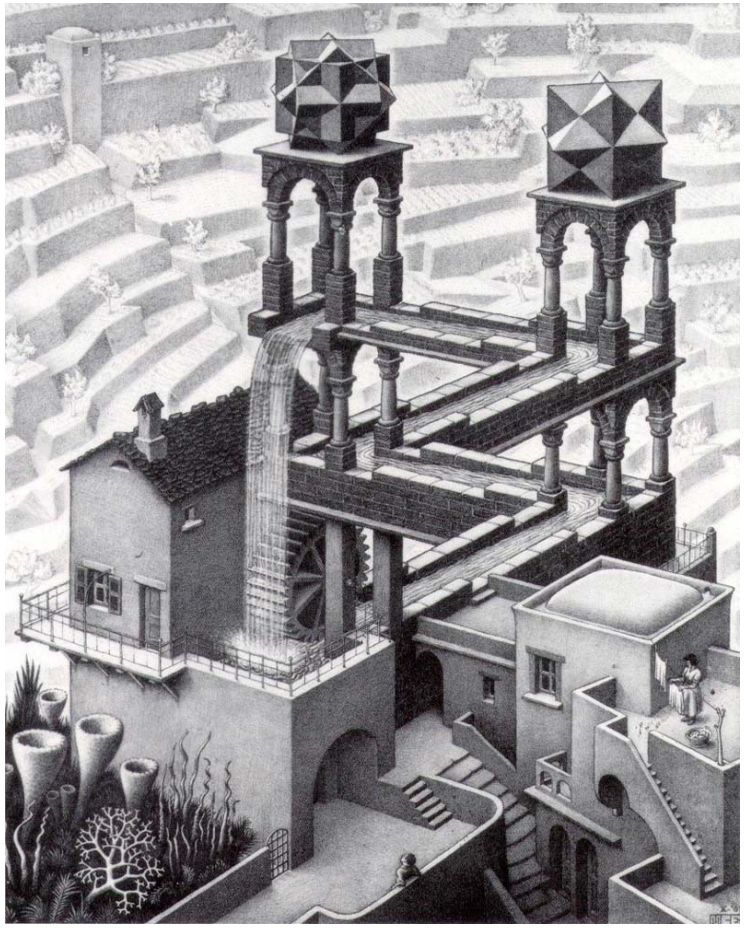
But it has actually gone back to its original frequency.

How does this happen? Our ears are sensitive to *tonality.*

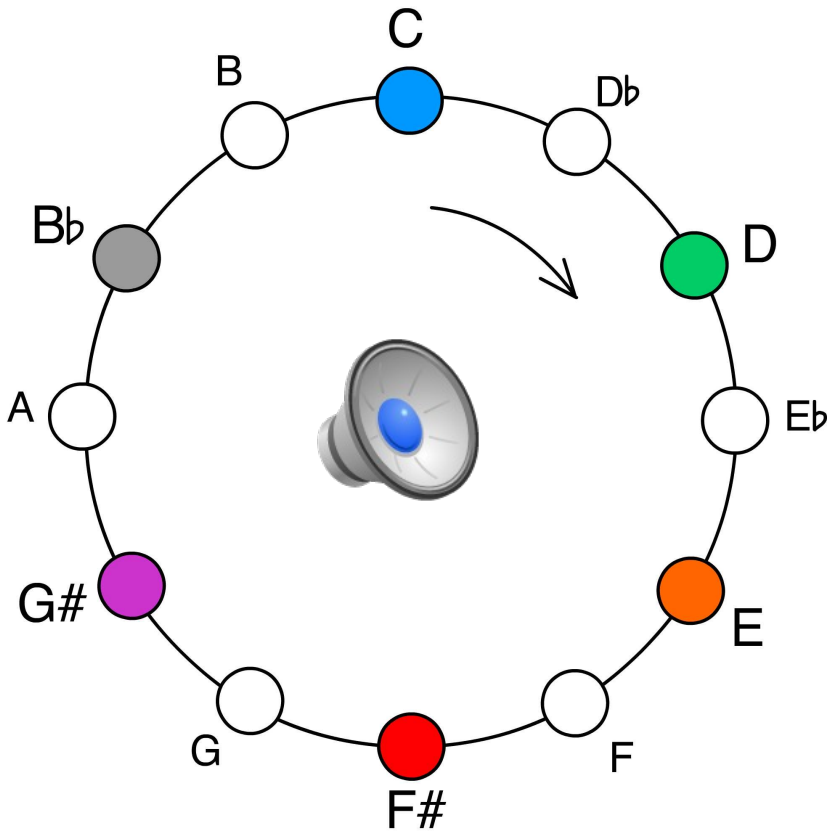




This is an example of a Shepard Scale: the tone seems to be always ascending, but ends up in the same place.

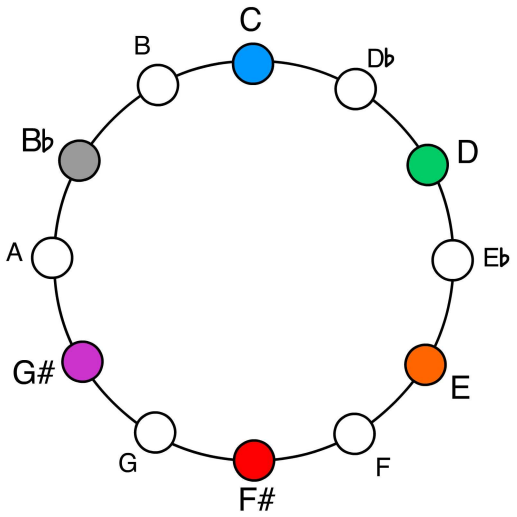


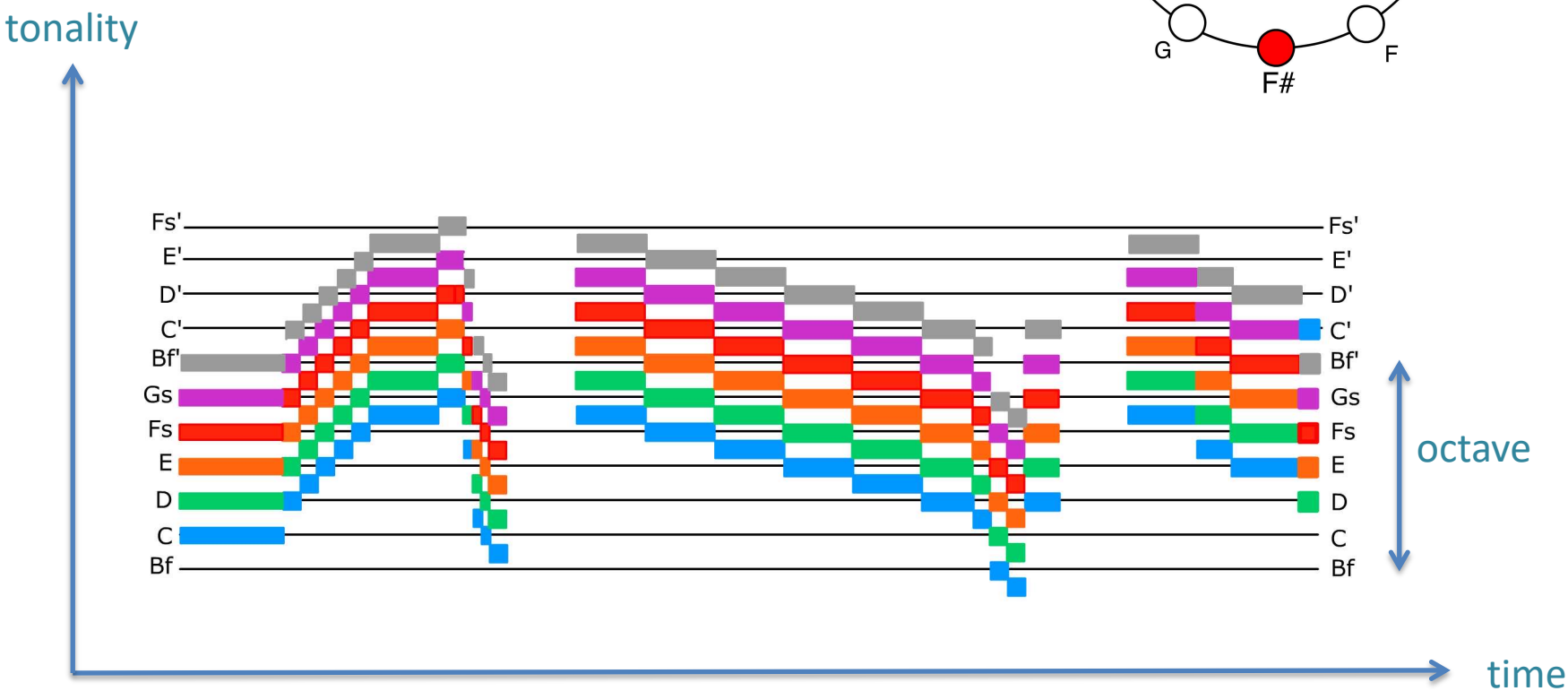
M. C. Escher, *Waterfall*, 1961



It’s an acoustic analogue of Escher’s impossible waterfall.

In TONALITY space, pitches an octave apart are equivalent.





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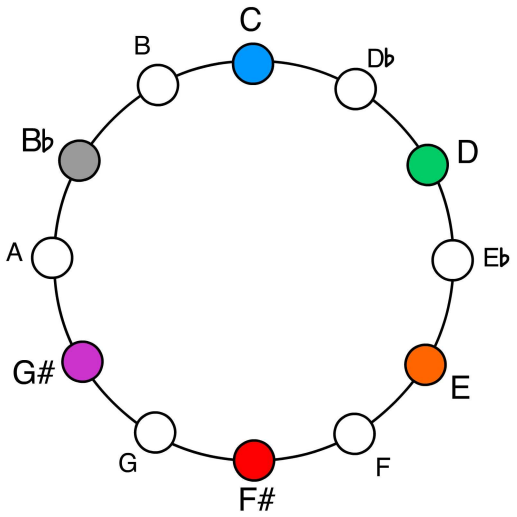

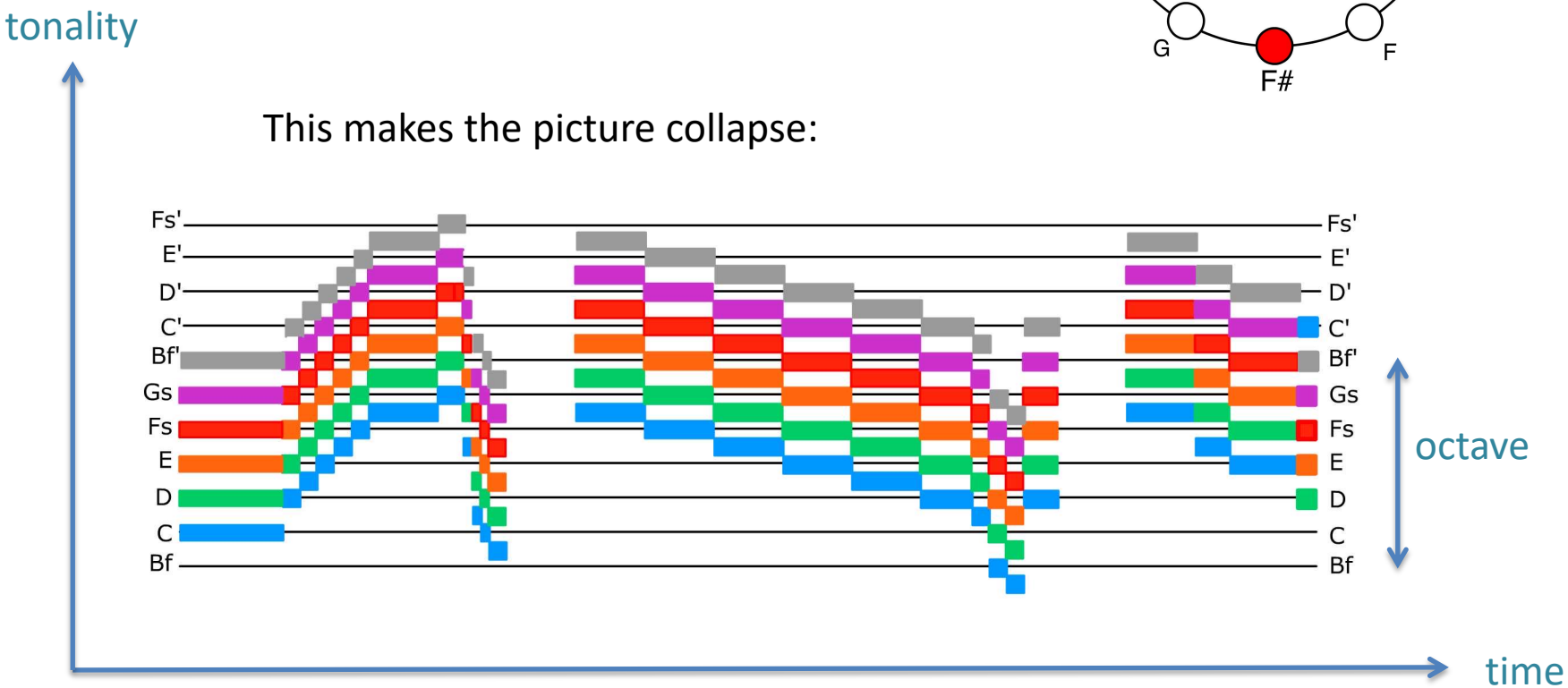

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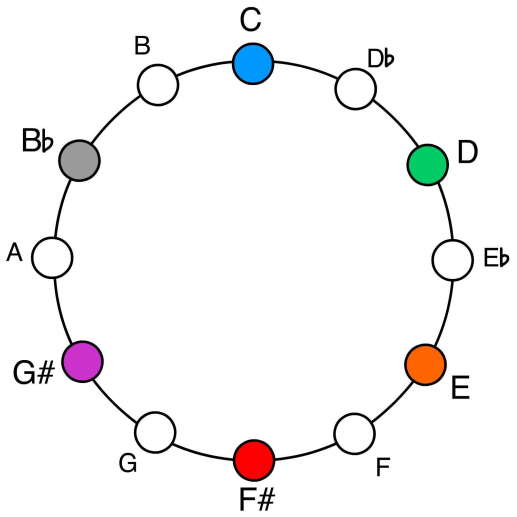
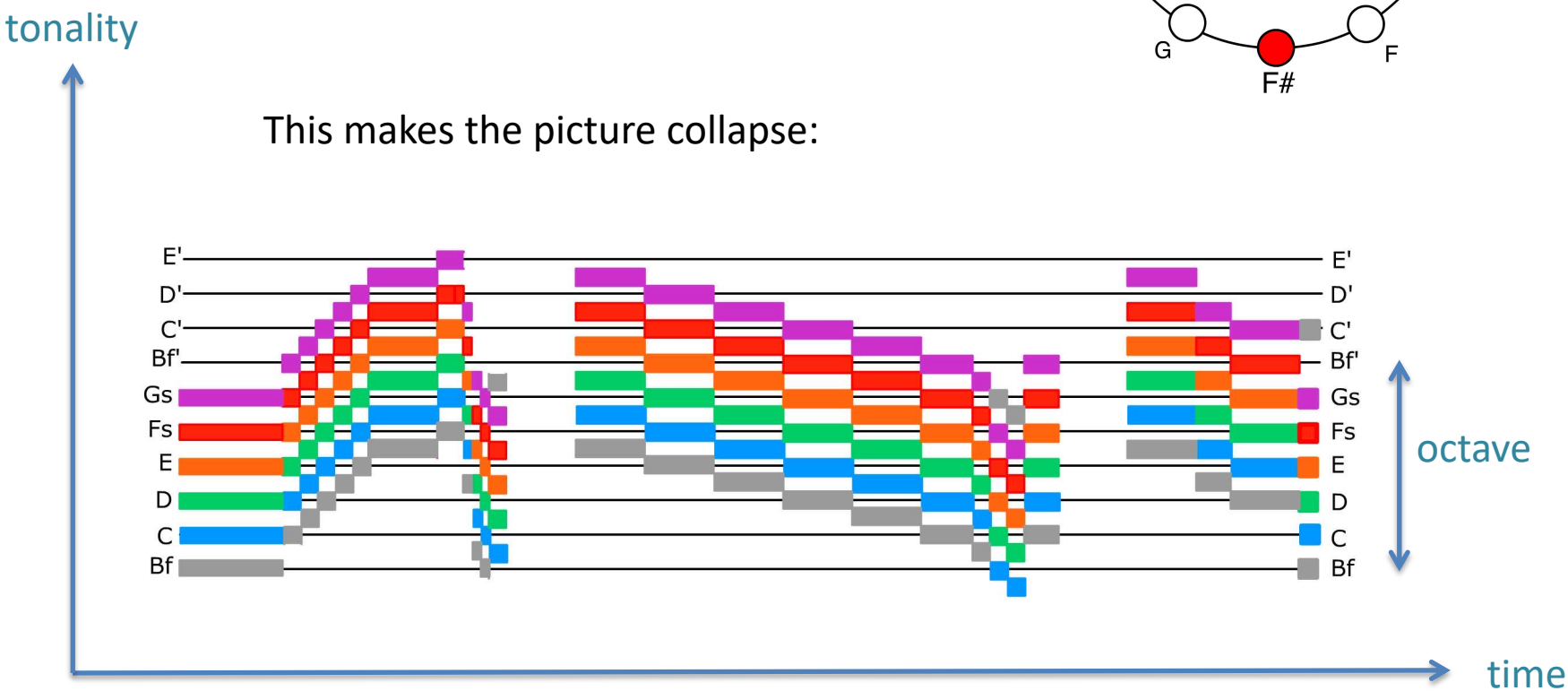


This makes the picture collapse:



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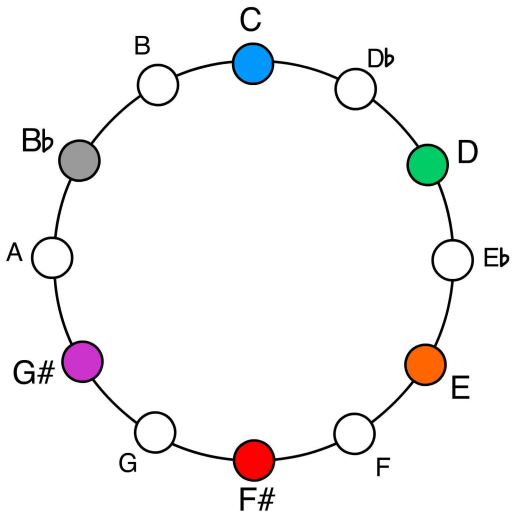

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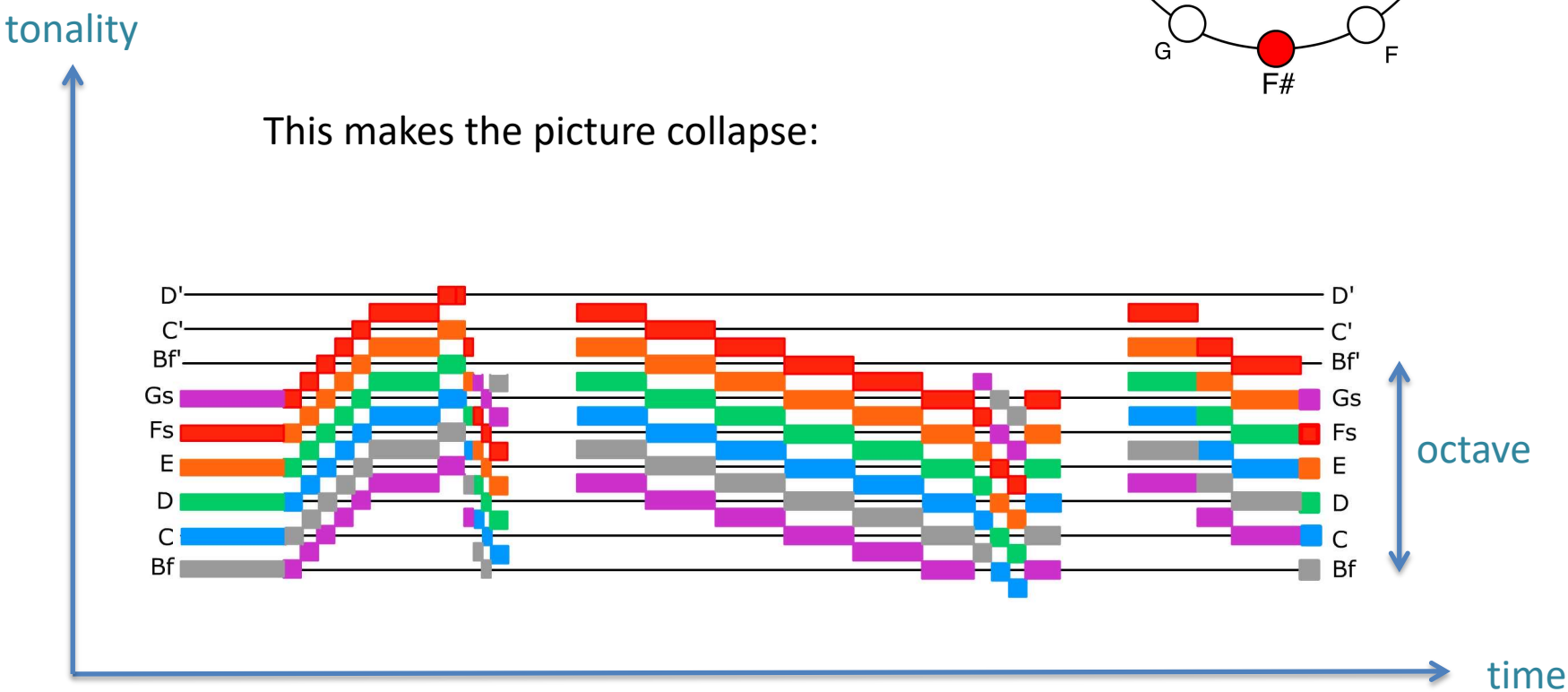

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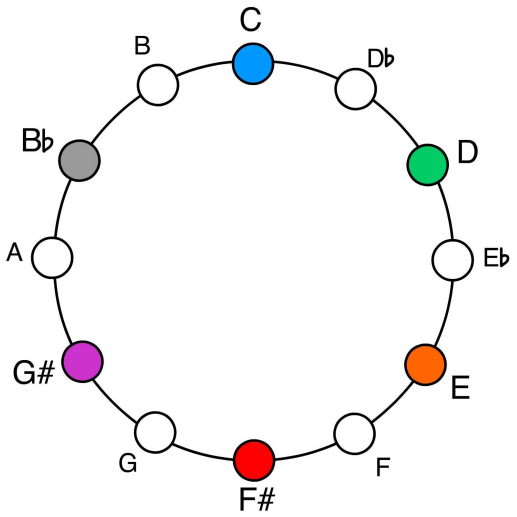

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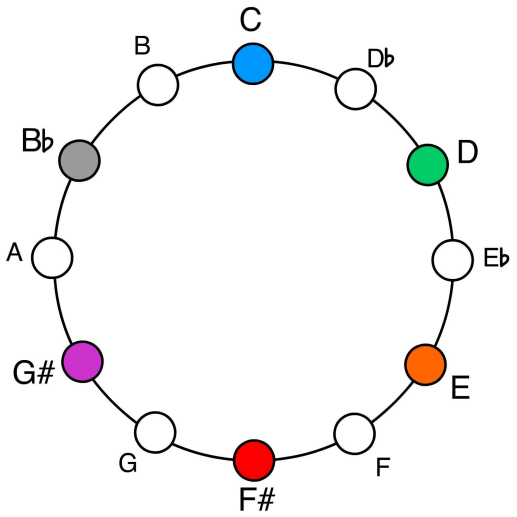

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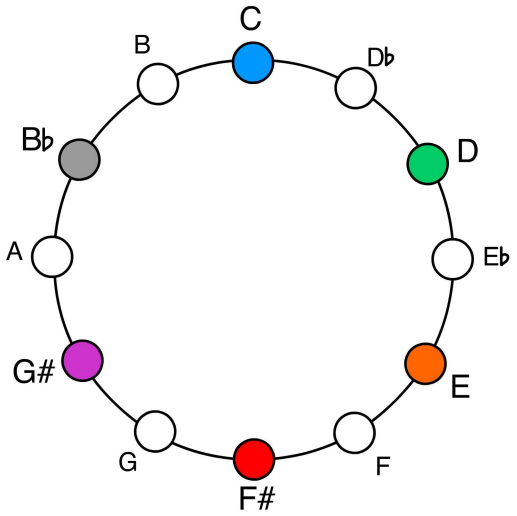

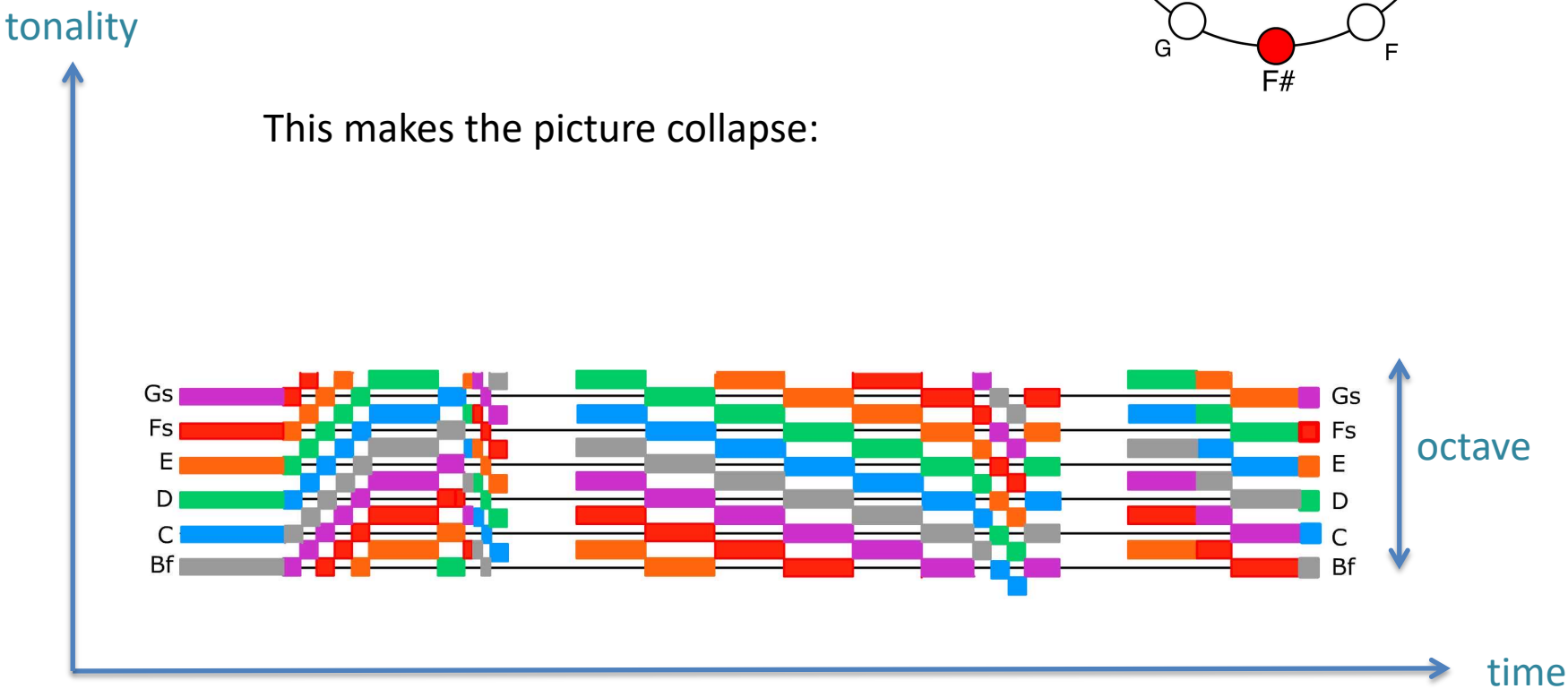

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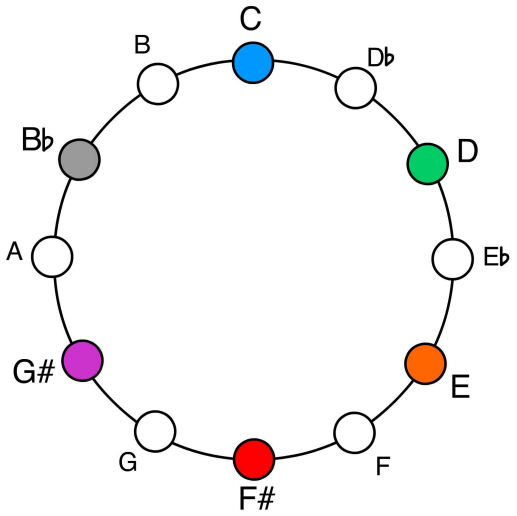

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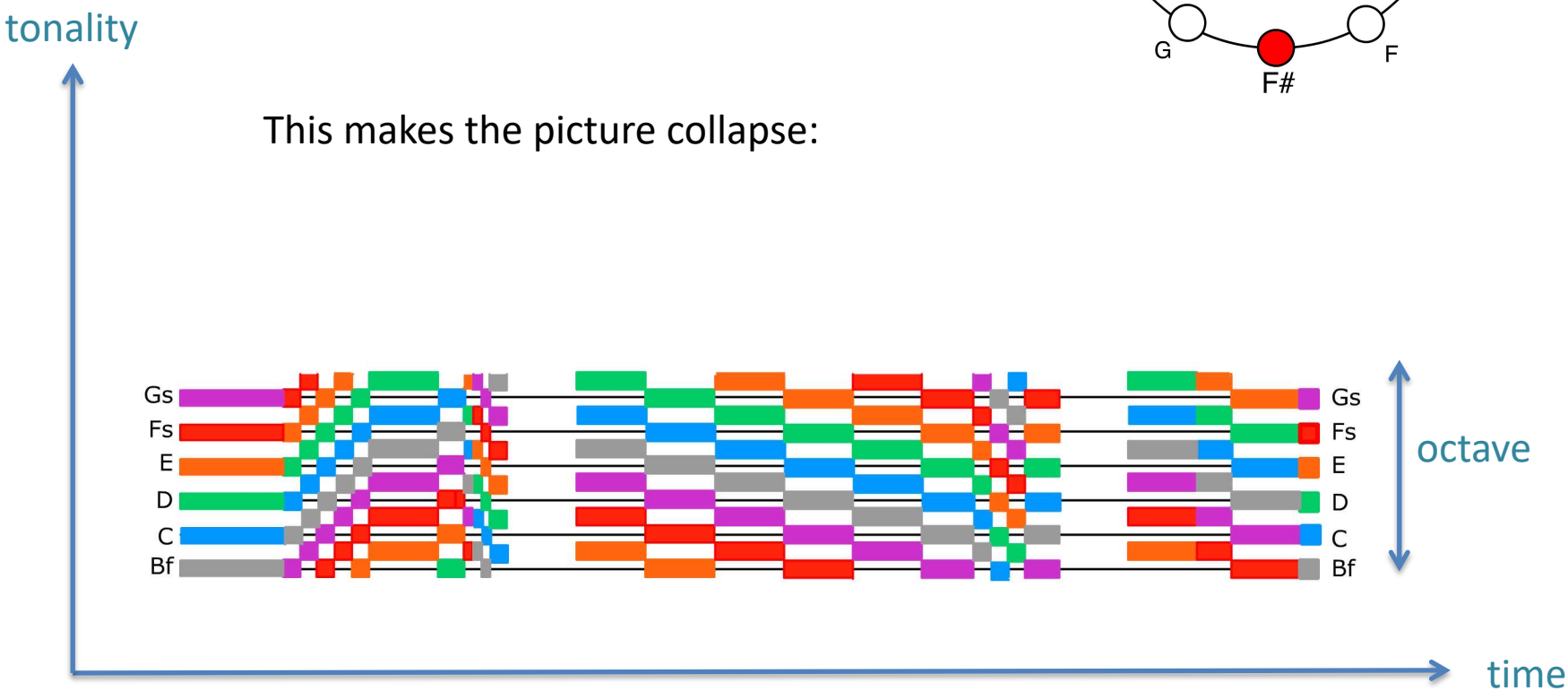
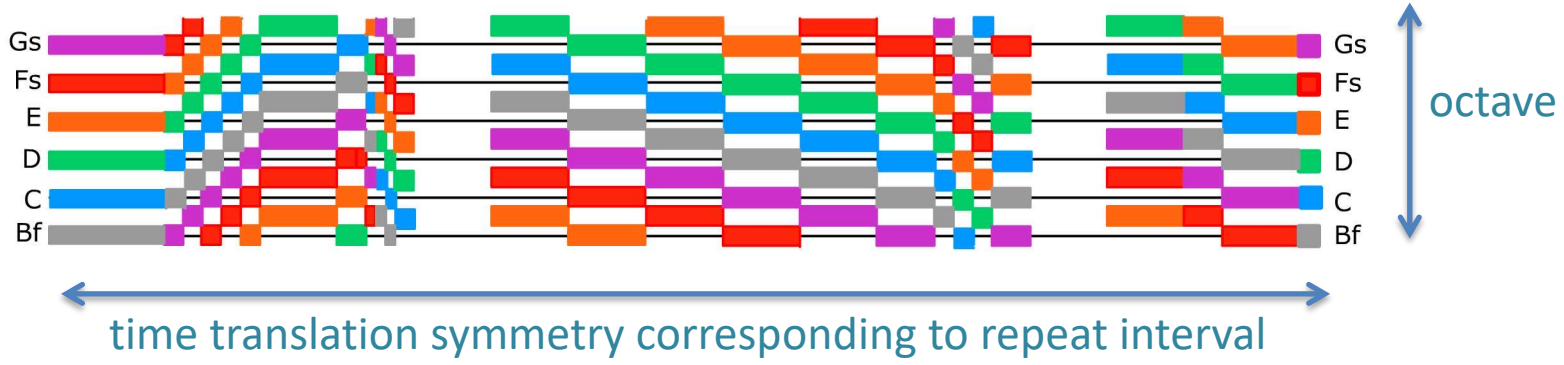
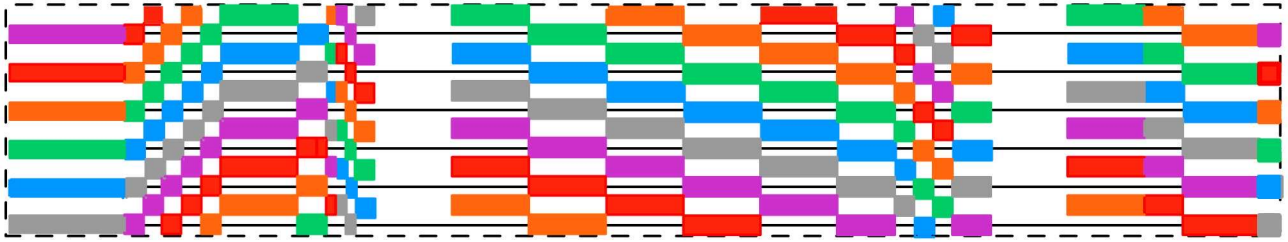
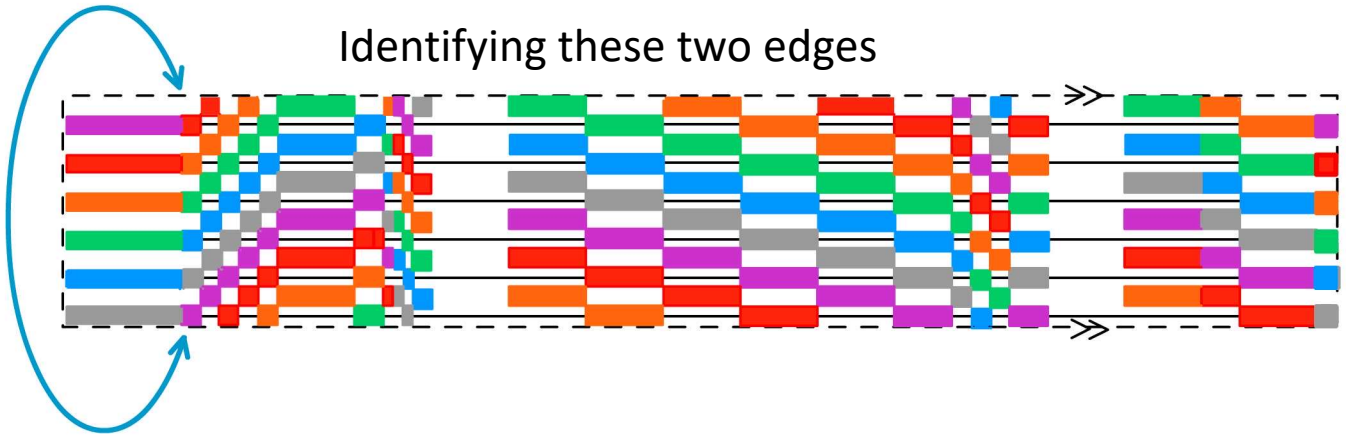
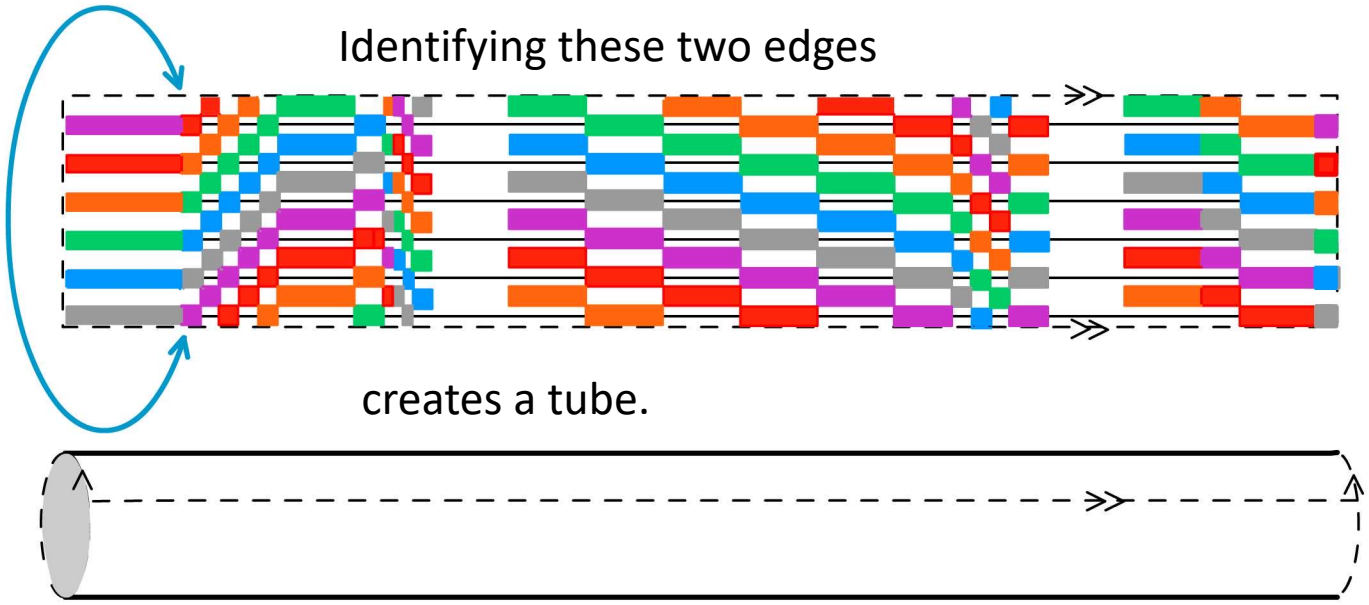


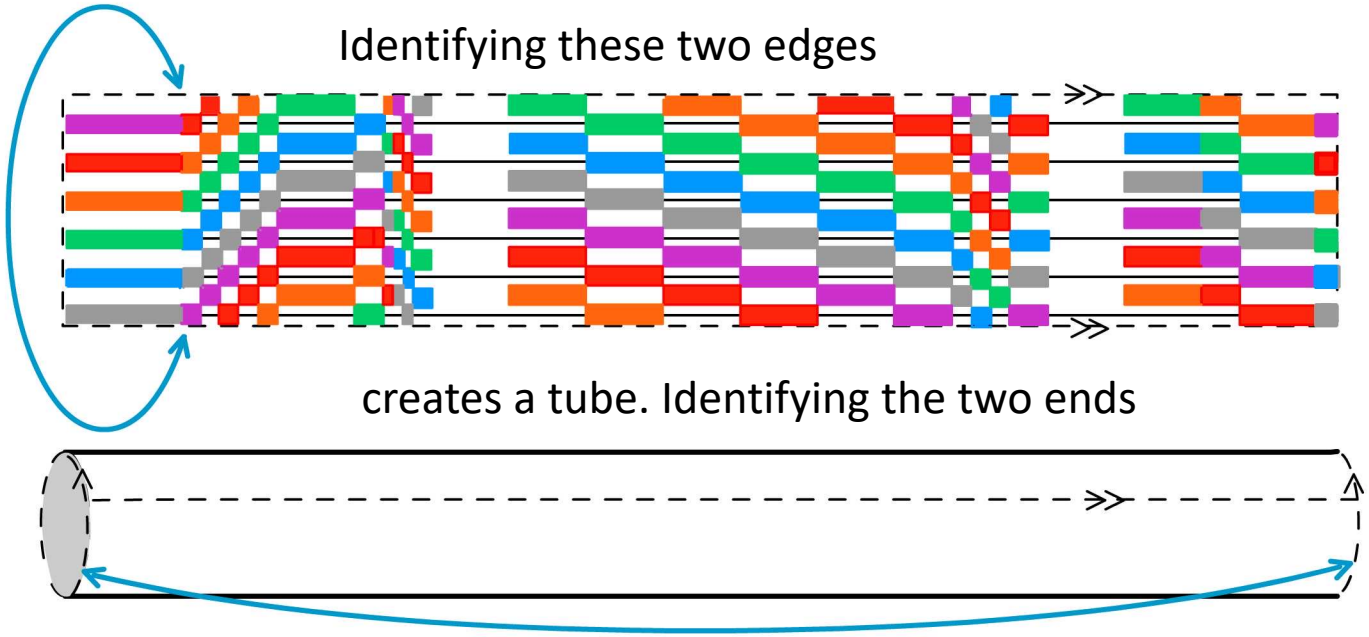
Together with the horizontal translation symmetry corresponding to the repeat, this gives the score of Canon 5 the topology of a ***torus***.






Identifying these two edges


Identifying these two edges
creates a tube.


Identifying these two edges
creates a tube. Identifying the two ends

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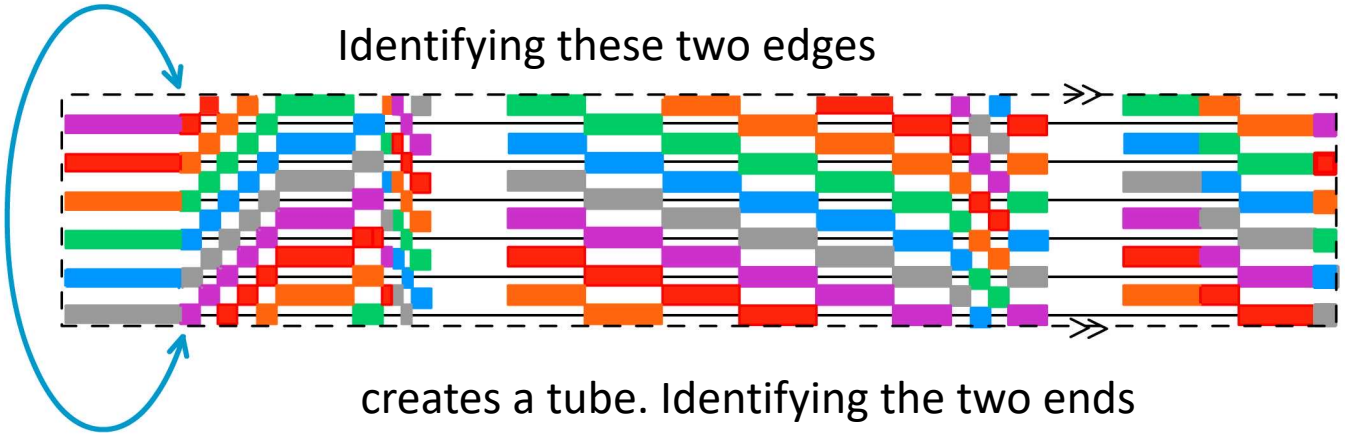
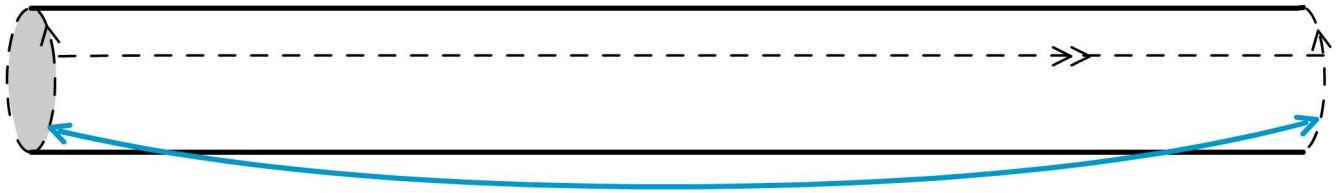
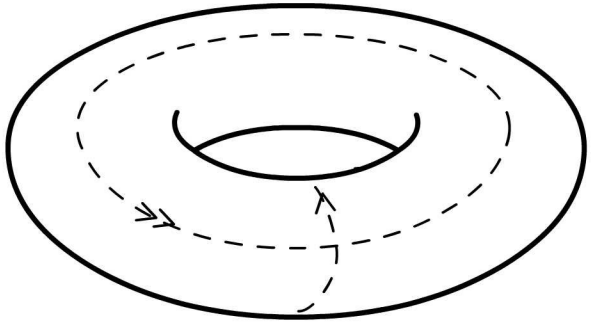


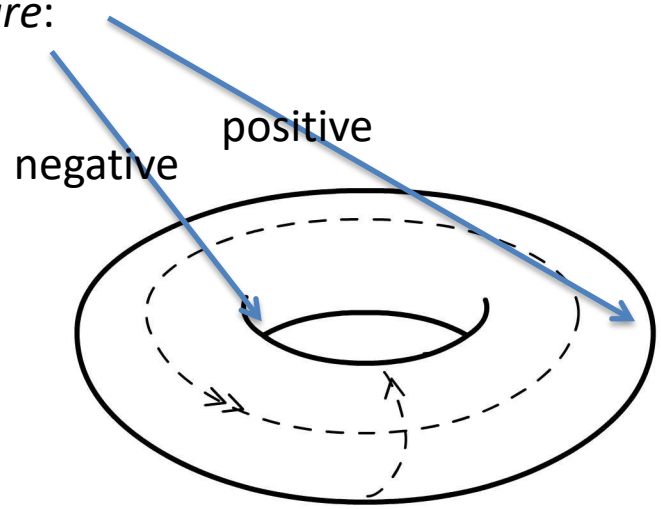
creates a tube. Identifying the two ends



creates a torus.



Note that our geometric picture of a smooth torus in 3-space has *curvature*:



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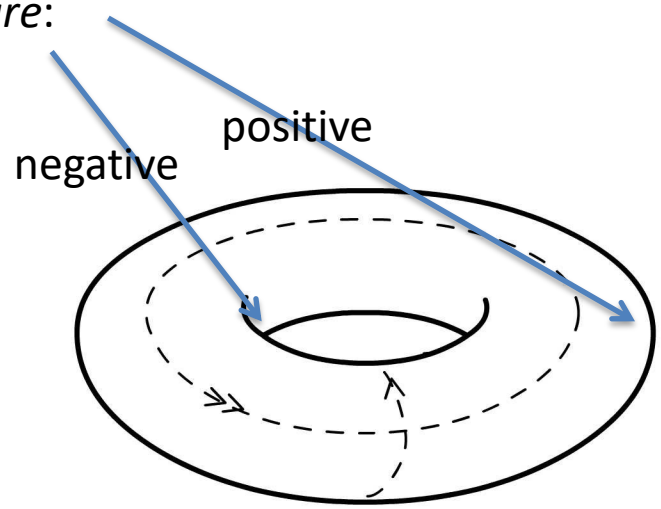
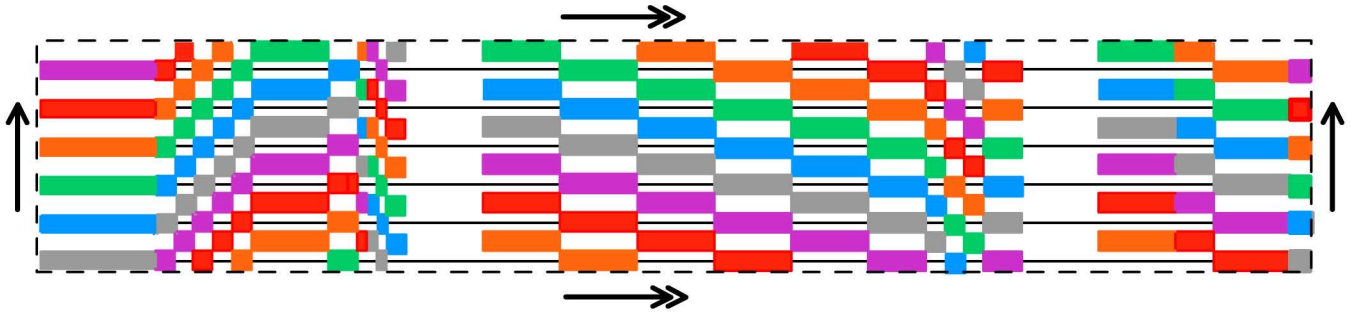
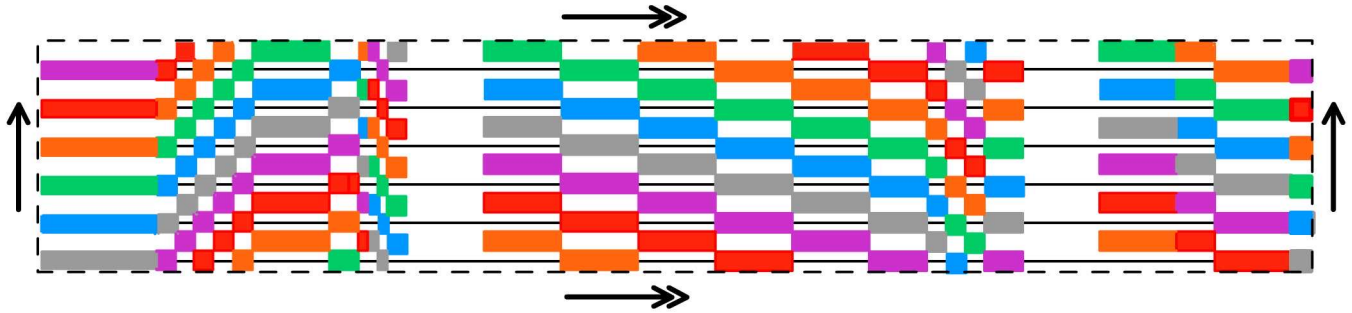


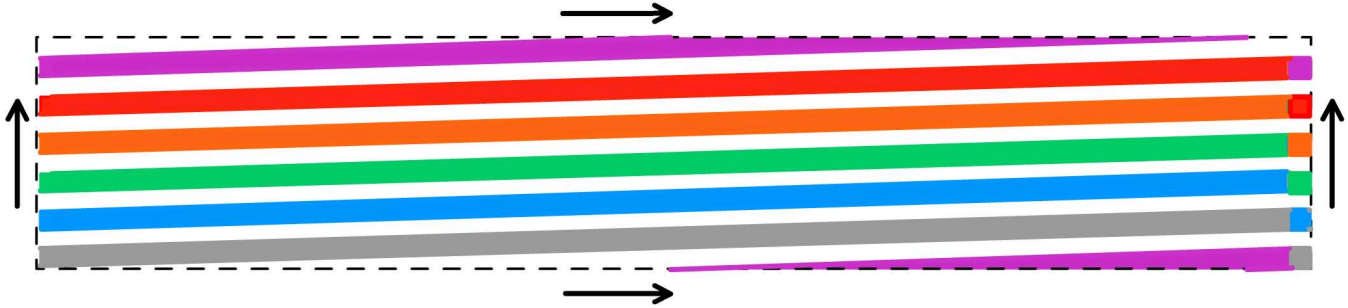
This does not make sense for a musical score.



The score of Canon 5 is a *flat* torus (no curvature), obtained by identifying opposite edges of a rectangle. It is out there, but it cannot be smoothly exhibited in 3-dimensional space.



On this ideal torus the melodic line of the canon winds around six times in one direction, and once in the other.



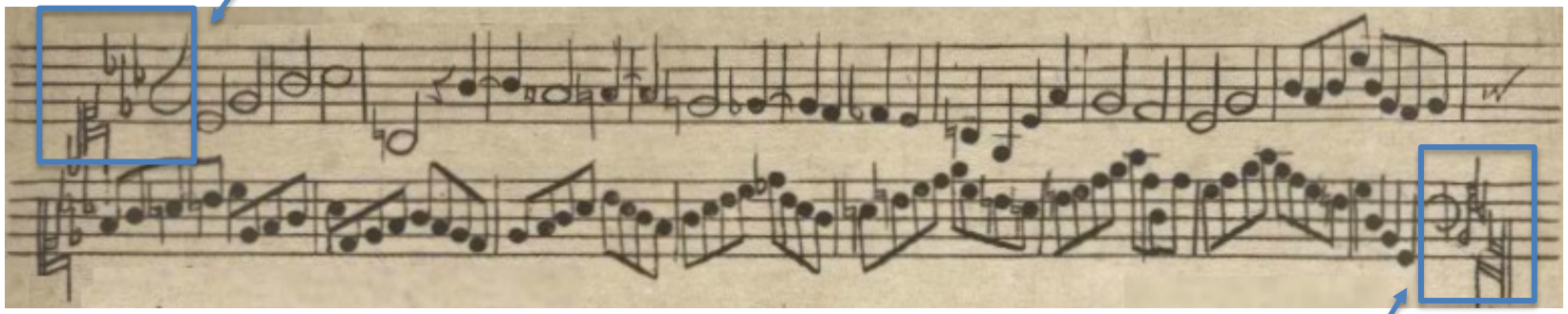
A beautiful video has been posted on YouTube

Explaining how the “Crab Canon” can be written on a Möbius strip.

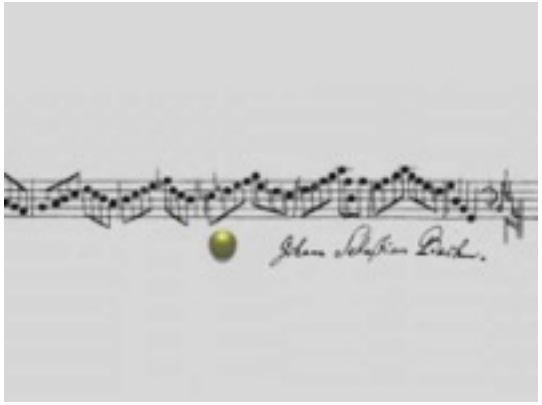
The Crab Canon is also derived from the Royal Theme.

The two voices play the score starting from opposite ends.

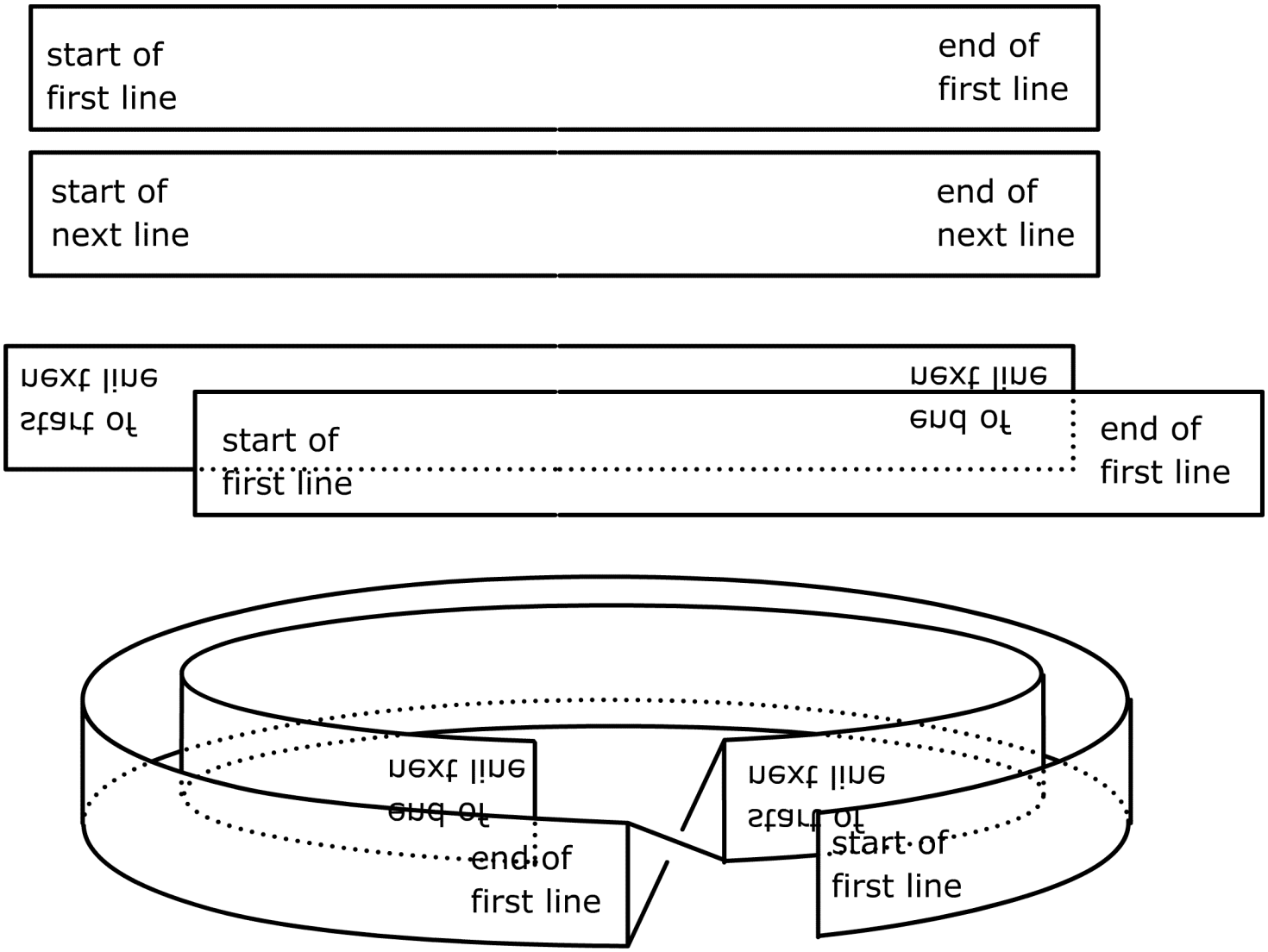
Alto clef, time and key signatures

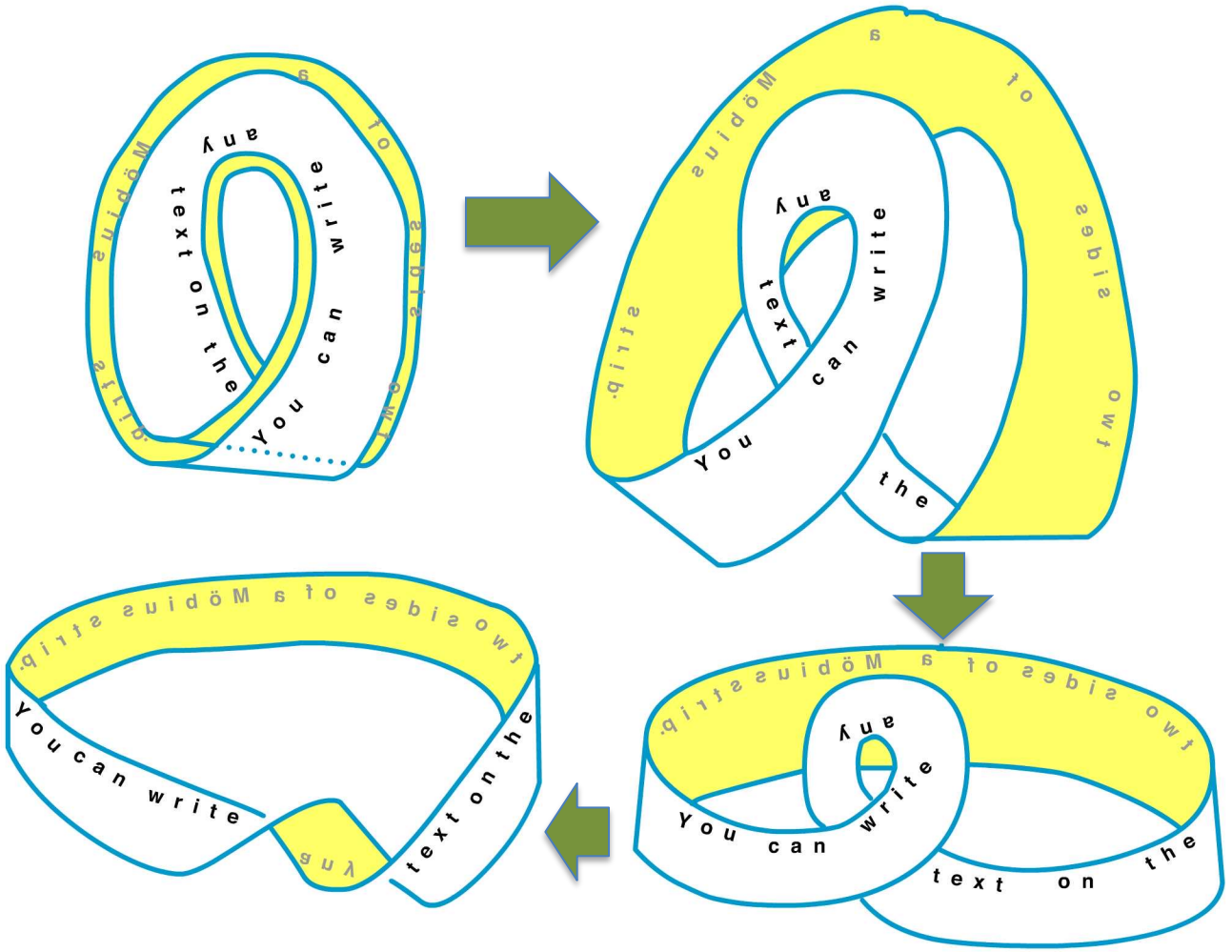


Backwards alto clef, time and key signatures



The problem is that this procedure works with *any* one-line text.


start of
first line
end of
first line
start of
next line
end of
next line
start of
first line
end of
first line
start of
first line
end of
first line



(You can write any text on the two sides of a Möbius strip.)


That's all Folks!