

Topological surfaces in Bach's puzzle canons.

Tony Phillips

April 25, 2023

This talk is based on work with Eric Altschuler, published
in *Musical Times*, Winter 2015

A musical score has basically two dimensions:



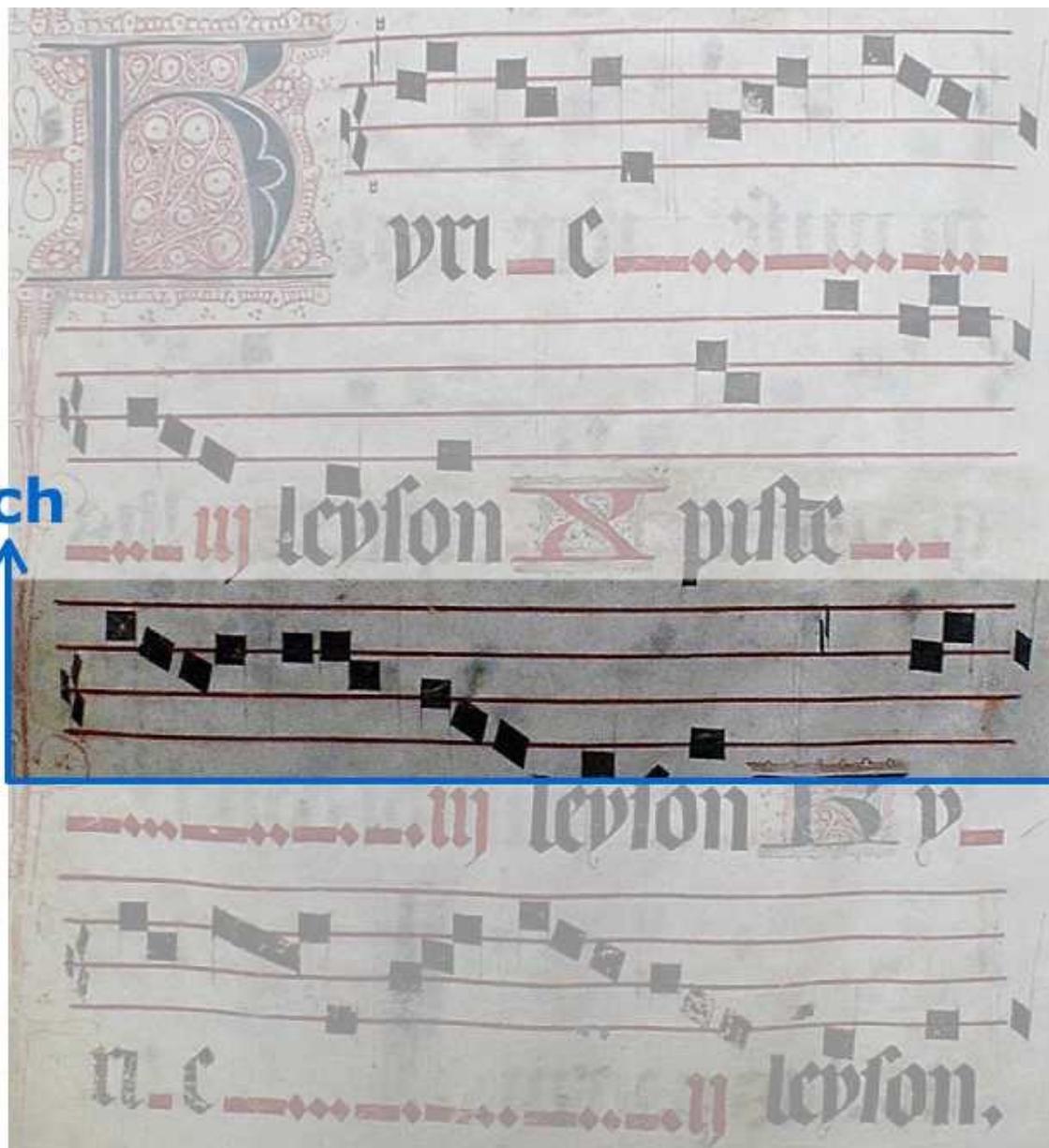
"Kyrie eleison"
(*Orbis factor*)
from a chantbook,
Miller Nichols Library
UMKC.
14th-15th century

A musical score has basically two dimensions:

pitch

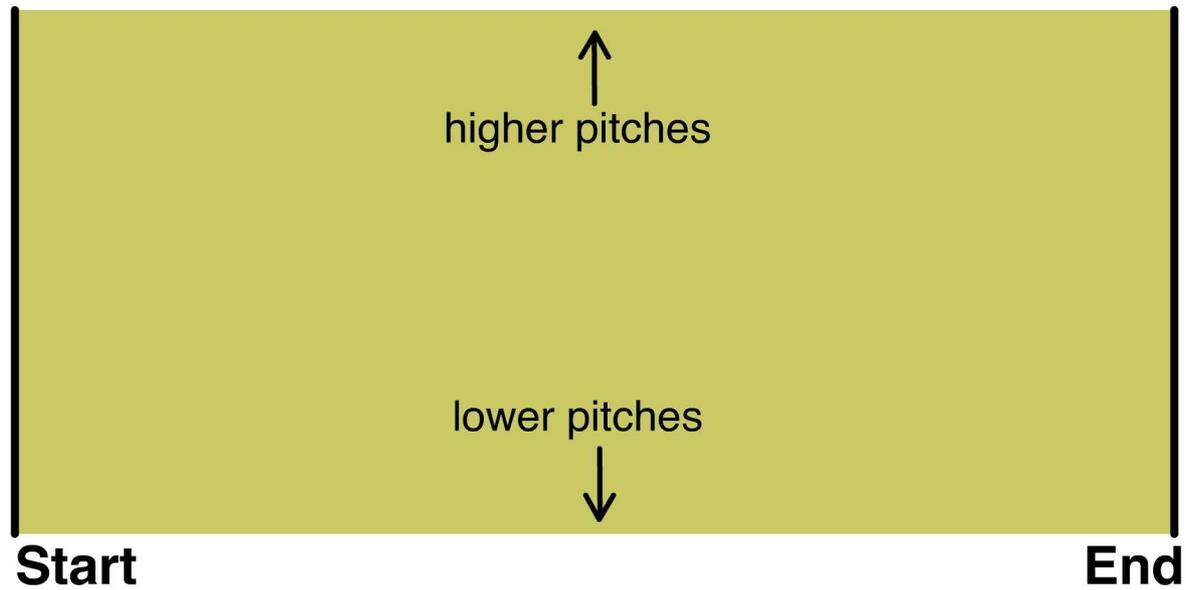


time



“Kyrie eleison”
(*Orbis factor*)
from a chantbook,
Miller Nichols Library
UMKC.
14th-15th century

So topologically a one-voice musical score is a 2-dimensional strip.



SCORES WITH SYMMETRY

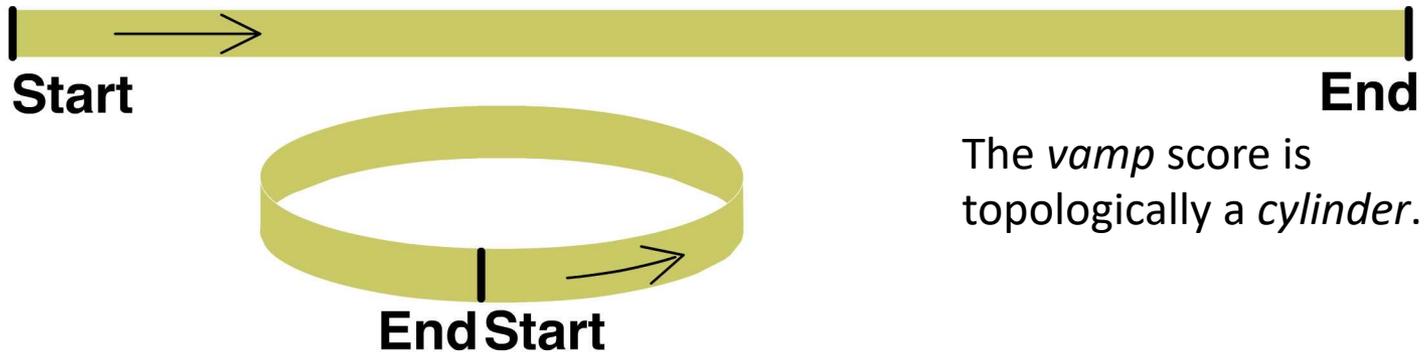
For example, suppose the score repeats. E.g. a *vamp*:



Repeat ad lib.

repeat marks

A musical staff in bass clef with a 4/4 time signature. It contains four quarter notes on the second line of the staff. The first and last notes are circled in red. A red line connects these two circles, with the text "repeat marks" written below it.



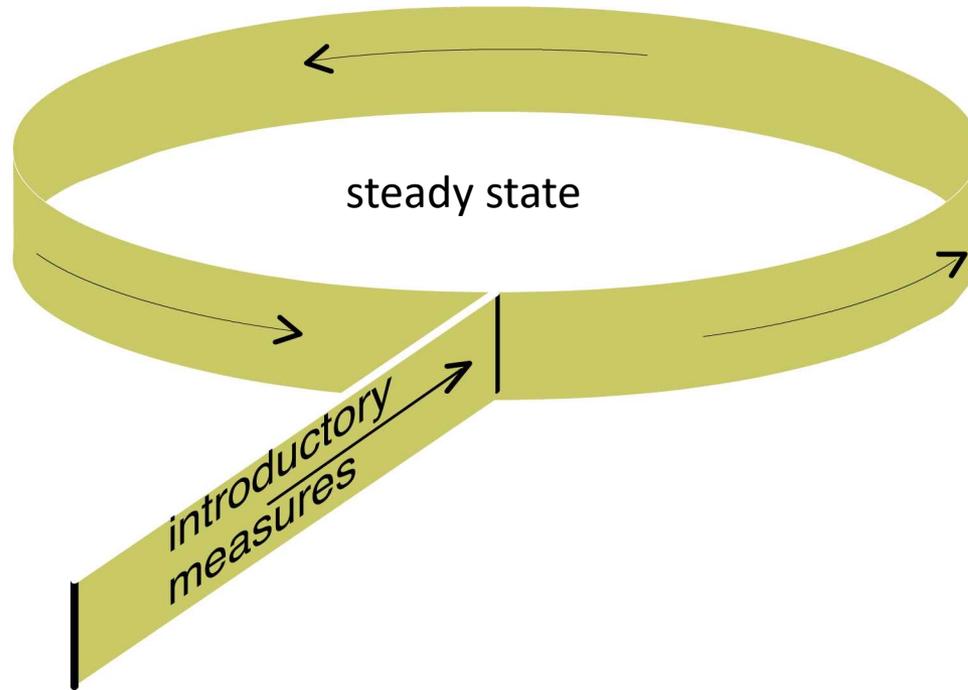
CANONS

A (2-part) **canon** is a score in which a second voice imitates the first voice after a delay. One of the best known is “Frère Jacques.”



A musical score for the canon 'Frère Jacques' in two parts. The score is written in G major (one sharp) and common time (C). It consists of three systems of two staves each. The first system shows the first voice starting with a half note G4, followed by a half note A4, and then a half note B4. The second system shows the first voice continuing with a half note C5, followed by a half note B4, and then a half note A4. The second voice enters in the second system with a half note G4, followed by a half note A4, and then a half note B4. The third system shows the first voice ending with a half note C5, followed by a half note B4, and then a half note A4. The second voice continues with a half note G4, followed by a half note A4, and then a half note B4. A blue box highlights the final two measures of the second voice, which harmonize with the final two measures of the first voice.

Note: The end of the second statement harmonizes with the beginning of the first, so the two voices dovetail to form a **steady state** that can be repeated *ad lib*.



Topologically, in two-part canons, after the introductory measures (the start of the lead voice) the score is a *cylinder*.

Canons were a speciality of J. S. Bach (1685-1750).

In this portrait, from 1746, he is holding the score of a canon:

one of the set of 14 canons, BWV 1087, on the bass line of the Goldberg Variations.



14 Canons on the Goldberg Ground.

This manuscript was discovered in 1974 in the back of Bach's own copy of the Goldberg Variations.

Definieren Canones libro III. notata a se fundamentalis
Noten nach eigener Art. von D. B. Bach.

1. Canon simplex $\frac{2}{2}$. al' roverscio.

3. 1. Canon varius Canones 3. 2. 3. 4. nota contraria & recta.

5. Canon duplex à 4.

6. Canon simplex. in byphat.

7. Canon simplex. in byphat.

8. Canon simplex. in byphat.

9. Canon in unione et scilicet. à 3.

10. Canon duplex. à 4.

11. Canon duplex. à 4.

12. Canon duplex. à 4.

13. Canon triplex.

14. Canon à 4. per Augustinum ad Divinum.

Fin.

Canon 13 (*triple*) –the canon in the painting– is one of the only two that were known before then.

14 Canons on the Goldberg Ground.

“Verschiedene Canones über die ersten acht fundamentalnoten vorheriger Arie.” von J. S. Bach.

(Various canons based on the first eight bass-line notes of the foregoing aria.)

Diversifizierte Canones über die ersten acht fundamentalnoten vorheriger Arie.
Noten nachfolgender Arie. von J. S. Bach.

1. Canon simplex 2. al roverscio.

3. Canon sopra Canones recto e capturno. 4. Nota contraria e recto.

5. Canon duplex à 4. 6. Canon simplex in blyght.

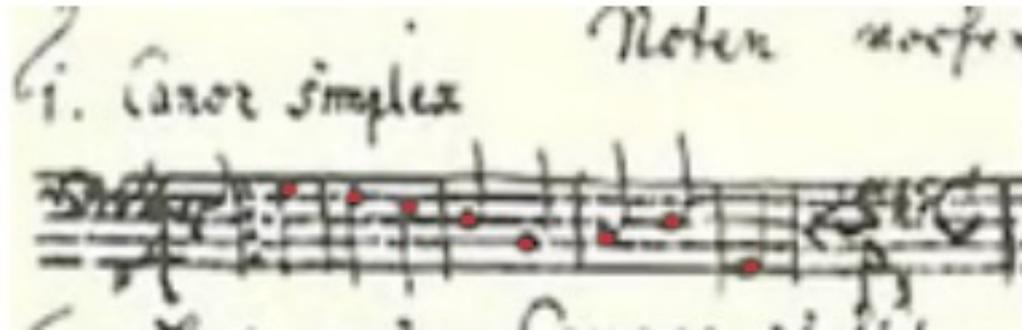
7. Canon simplex in g. 8. Canon simplex in g. 9. Canon in unisono et scissura. 10. Canon in unisono et scissura. 11. Canon in unisono et scissura. 12. Canon in unisono et scissura. 13. Canon in unisono et scissura. 14. Canon à 4. per augmentationem ad diminutionem.

Finis

ARIA.



The beginning of the Goldberg-variations Aria
with **the first eight notes of the bass-line** highlighted in red.



Canon 1 from the set of fourteen.

14 Canons on the Goldberg Ground.

Handwritten musical score for 14 canons on the Goldberg Ground. The title is "Deusfieriue Canones libro de organo aigl fundamentale Notu recto et contrario. von J.S. Bach." The score is written in C major and 4/1 time. The first canon is labeled "1. Canon simplex". The second is "2. Canon simplex al roverscio." The third is "3. Canon simplex Canones notu recto et contrario." The fourth is "4. Nota contraria et recto." The fifth is "5. Canon duplex a 4." The sixth is "6. Canon simplex libro bylight." The seventh is "7. Idea." The eighth is "8. Canon simplex il organo." The ninth is "9. Canon in unione col scambiano." The tenth is "10. Canon." The eleventh is "11. Canon." The twelfth is "12. Canon." The thirteenth is "13. Canon." The fourteenth is "14. Canon a 4. per augmentationem et diminutionem." The score is signed "Bach:" at the bottom right.

Canons 3 and 5 are the ones we will be considering.

Canons 3 and 5 are part of a set labeled "motu recto et contrario." They use both upright and contrary (upside-down) motion.

Canon 3.



Upside-down alto clef

Bass clef

"Segno" where second voice comes in.

Two staves of musical notation in 3/4 time, both in the key of D major (one sharp). The top staff begins with a bass clef and contains a sequence of notes: D2, E2, F2, G2, A2, B2, C3. A speaker icon is positioned below the first measure. The bottom staff begins with an upside-down alto clef and contains a sequence of notes: D3, E3, F3, G3, A3, B3, C4. A speaker icon is positioned above the final measure.

Canon 3.



Upside-down alto clef

Bass clef

"Segno" where second voice comes in.



Canon 5. This canon has 4 voices. The lowest two sing **Canon 3**.

Canon 3

Canon 5. This canon has 4 voices. The lowest two sing **Canon 3**.

Top 2 voices are also inverses.

Canon 3



Analysis of the top two voices in the steady state.

The image displays a musical score for two voices in G major, consisting of two staves. The key signature is one sharp (F#). The music is written in a steady state, featuring a consistent eighth-note rhythmic pattern. The upper voice begins with a melodic phrase, followed by a series of eighth-note runs. The lower voice provides a harmonic accompaniment with similar eighth-note patterns. The score concludes with a double bar line and repeat dots.

Analysis of the top two voices in the steady state.

The first system of music consists of two staves. Both staves are in treble clef and have a key signature of one sharp (F#). The music is divided into four measures. In the first measure, the upper staff has a long note (half note) on G4, while the lower staff has a half-note chord of F#4 and A4. The second measure shows the upper staff with eighth notes (G4, A4, B4, C5) and the lower staff with eighth-note chords (F#4-A4, G4-B4, A4-C5). The third measure continues with eighth notes in the upper staff (B4, C5, B4, A4) and eighth-note chords in the lower staff (G4-B4, F#4-A4, G4-B4). The fourth measure features a long note (half note) on G4 in the upper staff and eighth-note chords (F#4-A4, G4-B4, F#4-A4) in the lower staff.

The second system of music is a mirror image of the first system. It consists of two staves in treble clef with a key signature of one sharp (F#). The first measure has a long note (half note) on G4 in the lower staff and a half-note chord of F#4 and A4 in the upper staff. The second measure has eighth notes (G4, A4, B4, C5) in the lower staff and eighth-note chords (F#4-A4, G4-B4, A4-C5) in the upper staff. The third measure has eighth notes (B4, C5, B4, A4) in the lower staff and eighth-note chords (G4-B4, F#4-A4, G4-B4) in the upper staff. The fourth measure has a long note (half note) on G4 in the lower staff and eighth-note chords (F#4-A4, G4-B4, F#4-A4) in the upper staff.

Mirror image

Analysis of the top two voices in the steady state.

The image displays two systems of musical notation, each consisting of a grand staff with a treble and bass clef. The key signature is one sharp (F#). The first system shows a melodic line in the treble clef and a more rhythmic accompaniment in the bass clef. A blue box highlights the final two measures of the first system. A blue line extends from this box to a second blue box that highlights the first two measures of the second system. This second system shows the continuation of the melodic line in the treble clef and the accompaniment in the bass clef. The blue boxes and lines illustrate the relationship between the two systems, specifically focusing on the melodic line's movement.

“Optical” mirror image (inversion).

The last two measures are the upside-down mirror image of the first two.

Analysis of the top two voices in the steady state.

The image displays two systems of piano music, each consisting of two staves (treble and bass clef) with a key signature of one sharp (F#). The first system shows a melodic line in the treble clef and a supporting line in the bass clef. A green rectangular box highlights the first two measures of this system. The second system shows a similar melodic line in the treble clef and supporting line in the bass clef. A green rectangular box highlights the last two measures of this system. A green line connects the top-right corner of the first box to the bottom-left corner of the second box, illustrating an inversion.

“Optical” mirror image (inversion).

The last two measures are the upside-down mirror image of the first two.
(And, necessarily, vice-versa).

This symmetry is a special case of a *glide reflection*.

A periodic text in which the second half is a glide reflection of the first

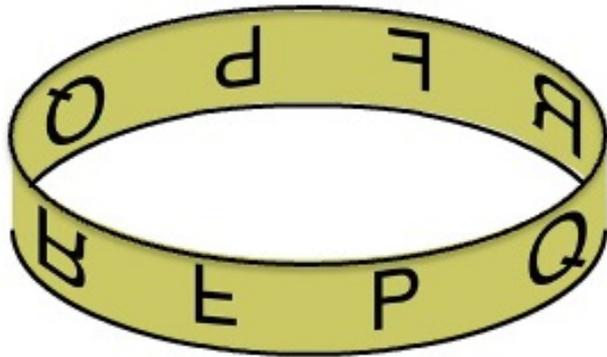


can be encoded in a Möbius strip.

A periodic text in which the second half is a glide reflection of the first



can be encoded in a Möbius strip.

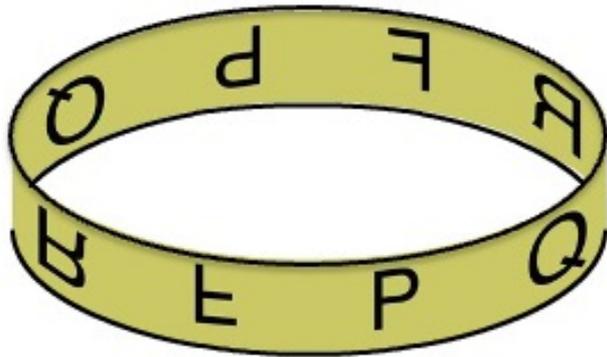


1. Periodicity means it can be read from a cylinder.

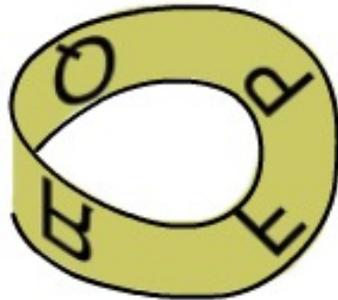
A periodic text in which the second half is a glide reflection of the first



can be encoded in a Möbius strip.



1. Periodicity means it can be read from a cylinder.

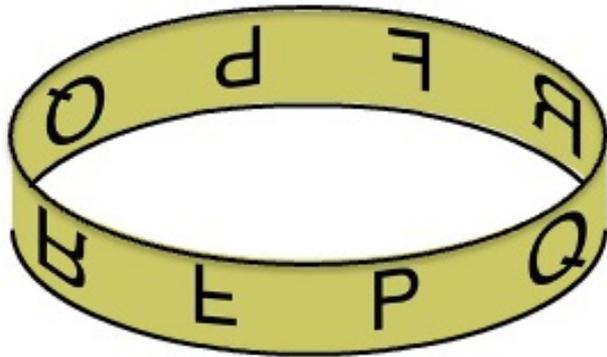


2. Half-and-half glide-reflection symmetry means the cylinder can be rolled up on itself, with a twist, to form a Möbius strip.

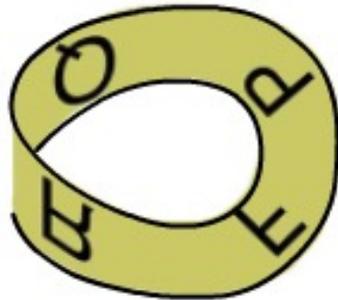
A periodic text in which the second half is a glide reflection of the first



can be encoded in a Möbius strip.

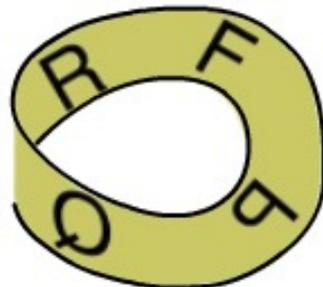


1. Periodicity means it can be read from a cylinder.



2. Half-and-half glide-reflection symmetry means the cylinder can be rolled up on itself, with a twist, to form a Möbius strip.

Same Möbius strip, flipped over.



Note that the text is *in the strip* (not on the surface) so it can be read from both sides.

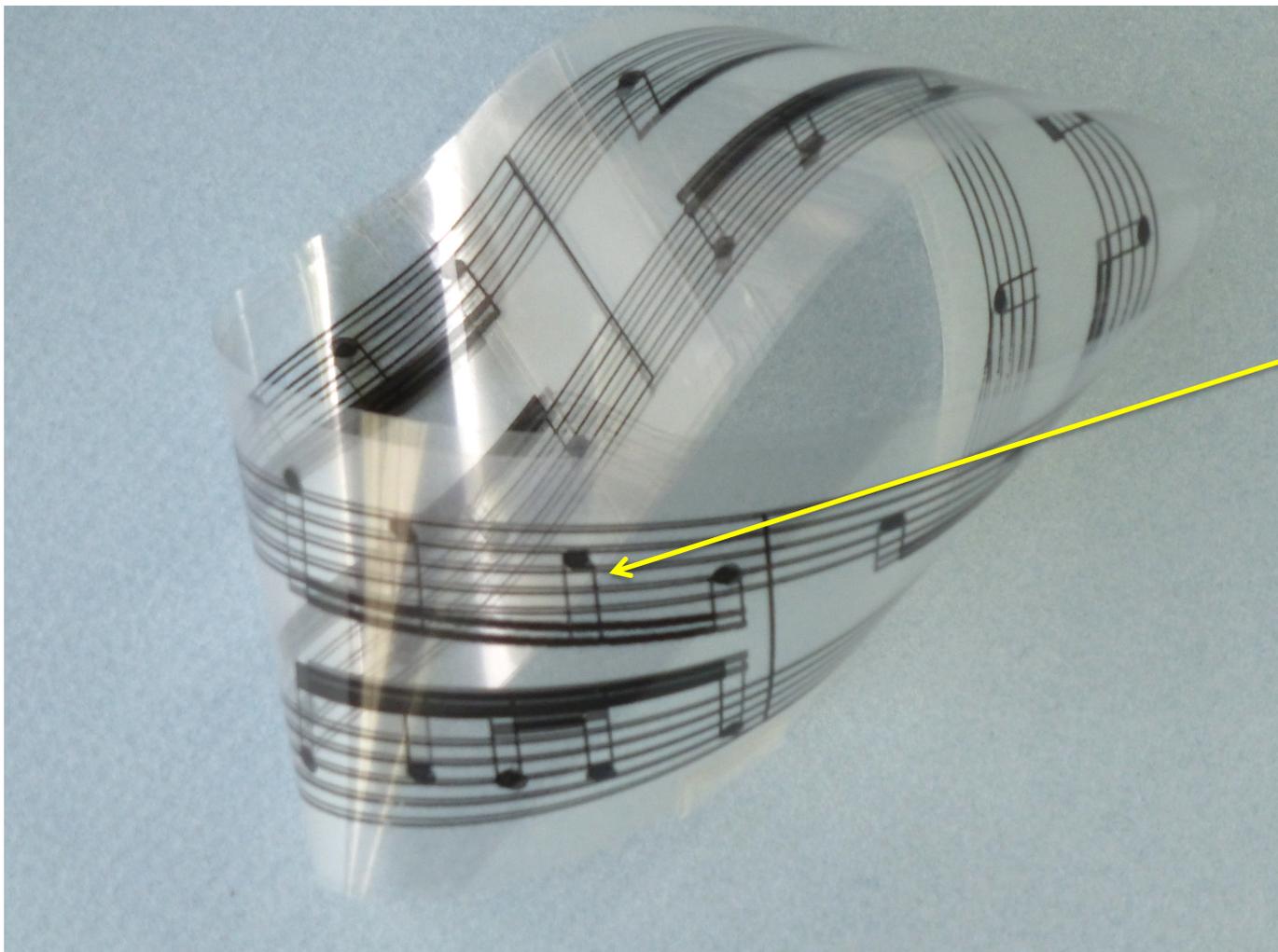
The score of the steady state of Canon 5
(upper two voices, without the clefs and time markings)
can be rolled into a Möbius strip so that the notes match exactly.



First as a cylinder (the notes are *in* the surface)



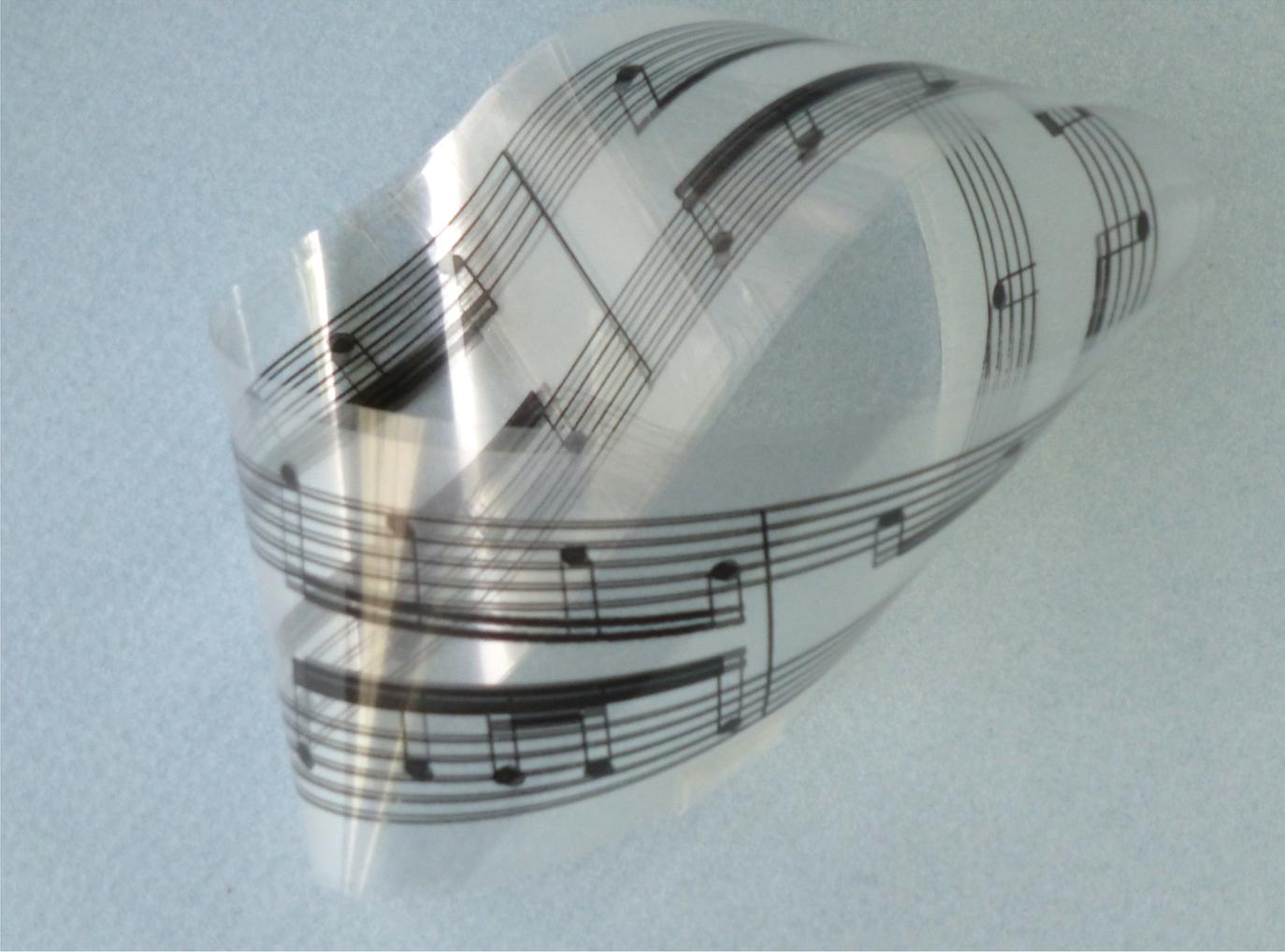
Then rolled into a Möbius strip.



notes
match
exactly.

(tails are
reversed).

Make your own: template available on my web page at Stony Brook.



Another example: Canon 5 from
the Musical Offering, BWV 1079.

Musicalisches

S p f e r

Mr. Königlichen Majestät in Preußen ꝛ.

allerunterthänigst gewidmet

von

Johann Sebastian Bach.

Title page of Musical Offering

“most humbly dedicated by Johann Sebastian Bach”

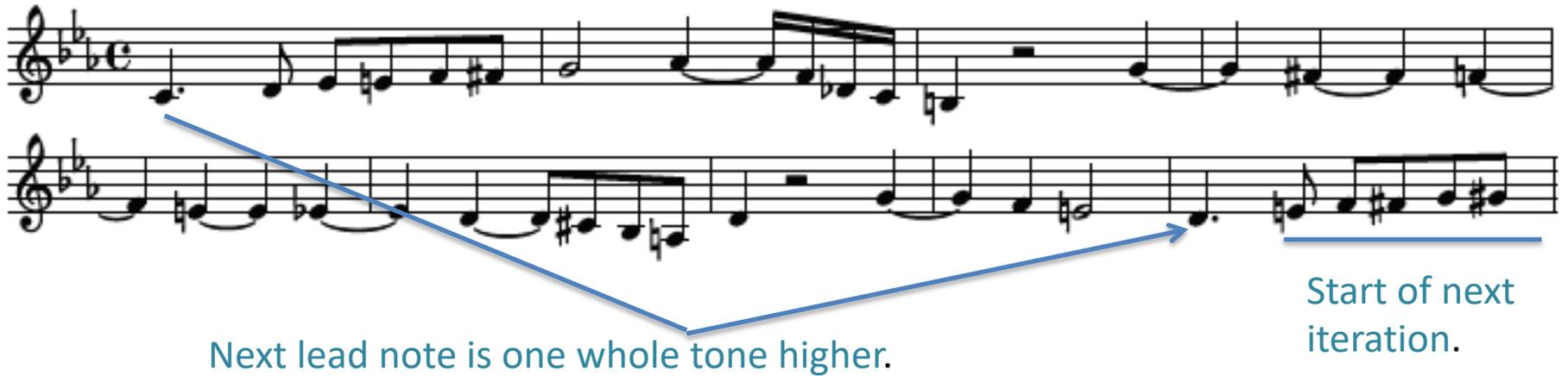
The elements of “The Musical Offering” are all based on a theme that was proposed to Bach by King Frederick II in 1747.



Starts and ends on the same note.

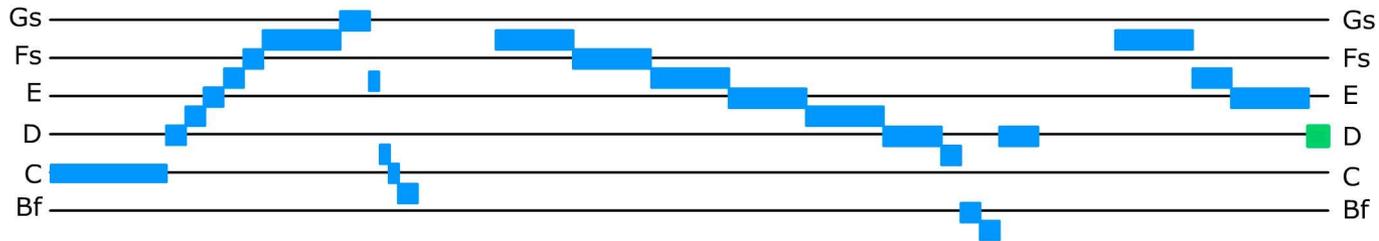


In Canon 5, Bach's version of the Royal Theme serves as a ground. It repeats as the highest voice; but each iteration starts one whole step higher than the one before.



Next lead note is one whole tone higher.

Start of next iteration.



A graphic representation of the melodic line.



Canon 5 is a 2-part canon which repeats *with the same modulation* as the ground.

One iteration.

Flute

B♭ Clarinet

Bassoon

The first system of the musical score covers measures 1 and 2. The Flute part (top staff) begins with a quarter note G4, followed by eighth notes A4, Bb4, and C5, then a quarter note D5. In measure 2, it has a half note E5 and a quarter note F5. The B♭ Clarinet part (middle staff) is silent in measure 1 and enters in measure 2 with a quarter rest, followed by eighth notes G4, A4, Bb4, and C5, then a quarter note D5. The Bassoon part (bottom staff) starts with a quarter note G3, followed by eighth notes A3, Bb3, and C4, then a quarter note D4. In measure 2, it has a half note E4 and a quarter note F4.

3

The second system covers measures 3 and 4. The Flute part (top staff) has a quarter note G4, followed by a half note A4, then a quarter note Bb4, and a quarter note C5. In measure 4, it has a half note D5 and a quarter note E5. The B♭ Clarinet part (middle staff) has a quarter note G4, followed by eighth notes A4, Bb4, and C5, then a quarter note D5. In measure 4, it has a half note E5 and a quarter note F5. The Bassoon part (bottom staff) has a quarter note G3, followed by eighth notes A3, Bb3, and C4, then a quarter note D4. In measure 4, it has a half note E4 and a quarter note F4.

5

The third system covers measures 5 and 6. The Flute part (top staff) has a quarter note G4, followed by a half note A4, then a quarter note Bb4, and a quarter note C5. In measure 6, it has a half note D5 and a quarter note E5. The B♭ Clarinet part (middle staff) has a quarter note G4, followed by eighth notes A4, Bb4, and C5, then a quarter note D5. In measure 6, it has a half note E5 and a quarter note F5. The Bassoon part (bottom staff) has a quarter note G3, followed by eighth notes A3, Bb3, and C4, then a quarter note D4. In measure 6, it has a half note E4 and a quarter note F4.

7

The fourth system covers measures 7 and 8. The Flute part (top staff) has a quarter note G4, followed by a half note A4, then a quarter note Bb4, and a quarter note C5. In measure 8, it has a half note D5 and a quarter note E5. The B♭ Clarinet part (middle staff) has a quarter note G4, followed by eighth notes A4, Bb4, and C5, then a quarter note D5. In measure 8, it has a half note E5 and a quarter note F5. The Bassoon part (bottom staff) has a quarter note G3, followed by eighth notes A3, Bb3, and C4, then a quarter note D4. In measure 8, it has a half note E4 and a quarter note F4.

One iteration.

Top line:
Royal Theme

Flute

B♭ Clarinet

Bassoon

This system shows the first two measures of the score. The Flute part (top line) is highlighted with a purple box. It begins with a quarter rest in measure 1, followed by a quarter note G4, and then a half note G4-A4-Bb4 in measure 2. The B♭ Clarinet part has a whole rest in measure 1 and enters in measure 2 with a quarter note G4, followed by eighth notes A4-Bb4-A4-G4. The Bassoon part starts in measure 1 with a quarter note G2, followed by eighth notes A2-Bb2-A2-G2, and then a half note G2 in measure 2.

This system shows measures 3 and 4. The Flute part (top line) is highlighted with a purple box. It has a quarter rest in measure 3, followed by a half note G4 in measure 4. The B♭ Clarinet part continues with eighth notes A4-Bb4-A4-G4 in measure 3, followed by a half note G4 in measure 4. The Bassoon part continues with eighth notes A2-Bb2-A2-G2 in measure 3, followed by a half note G2 in measure 4.

This system shows measures 5 and 6. The Flute part (top line) is highlighted with a purple box. It has a quarter rest in measure 5, followed by a half note G4 in measure 6. The B♭ Clarinet part has a quarter rest in measure 5, followed by a half note G4 in measure 6. The Bassoon part continues with eighth notes A2-Bb2-A2-G2 in measure 5, followed by a half note G2 in measure 6.

This system shows measures 7 and 8. The Flute part (top line) is highlighted with a purple box. It has a quarter rest in measure 7, followed by a half note G4 in measure 8. The B♭ Clarinet part has a quarter rest in measure 7, followed by a half note G4 in measure 8. The Bassoon part continues with eighth notes A2-Bb2-A2-G2 in measure 7, followed by a half note G2 in measure 8.

One iteration.

Top line:
Royal Theme

First canon voice

The image displays a musical score for three instruments: Flute, B♭ Clarinet, and Bassoon. The score is organized into four systems, each containing three staves. The key signature is B-flat major (two flats) and the time signature is 4/4. The first system is labeled 'One iteration.' and the top line is identified as 'Royal Theme'. The 'First canon voice' is indicated by a blue box around the Bassoon staff in the first system. The score shows the progression of the canon through the instruments across the four systems, with measures 1 through 7 indicated by small numbers at the beginning of each system.

One iteration.

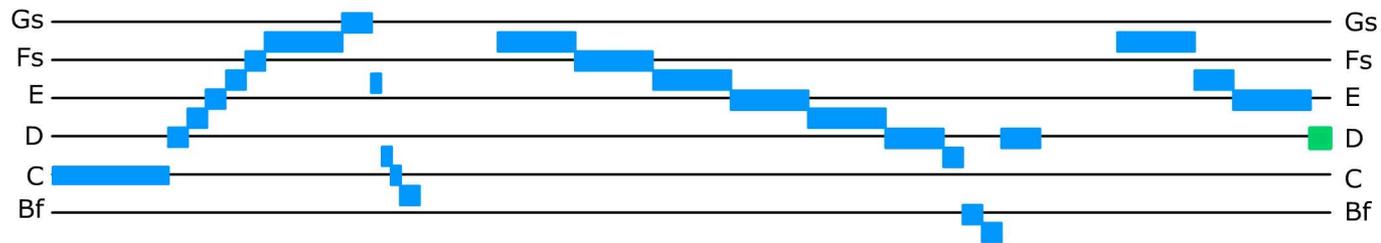
Top line:
Royal Theme

First canon voice

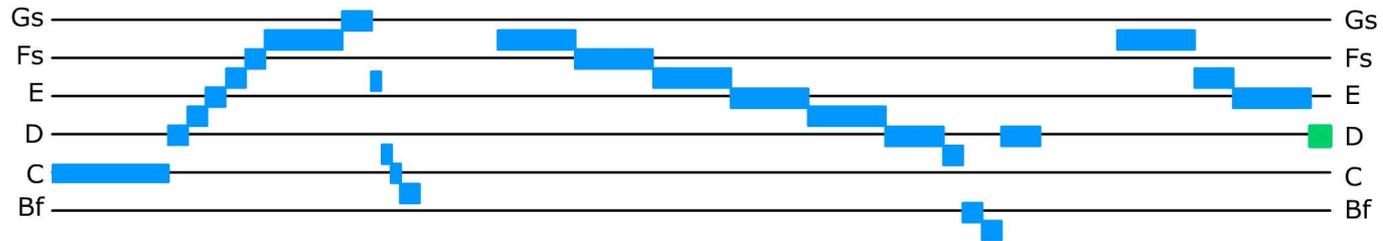
Second canon voice:
one measure later,
one fifth higher.

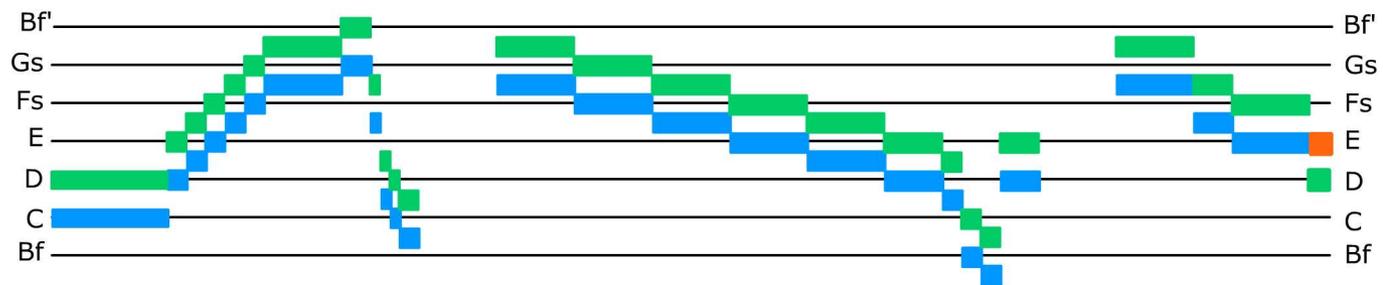
The image displays a musical score for three instruments: Flute, B♭ Clarinet, and Bassoon. The score is organized into three systems, each representing an iteration of a canon. The key signature is B-flat major (two flats) and the time signature is 4/4. The Flute part is consistently highlighted with a purple box, the B♭ Clarinet part with a green box, and the Bassoon part with a blue box. The first system (measures 1-4) shows the initial entry of the 'Royal Theme' in the Flute. The second system (measures 5-8) shows the first canon voice, where the Flute part is repeated one measure later and one fifth higher. The third system (measures 9-12) shows the second canon voice, where the Flute part is repeated again, one measure later and one fifth higher than the previous iteration.

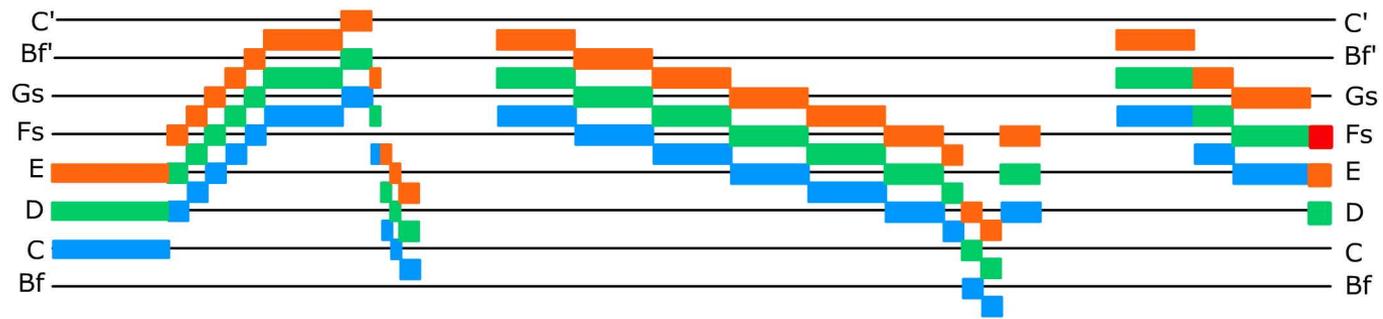
Here is the whole Canon, in a recording by Michael Monroe. (It lasts 3 minutes). We will follow –graphically- the modulations of the Royal Theme.

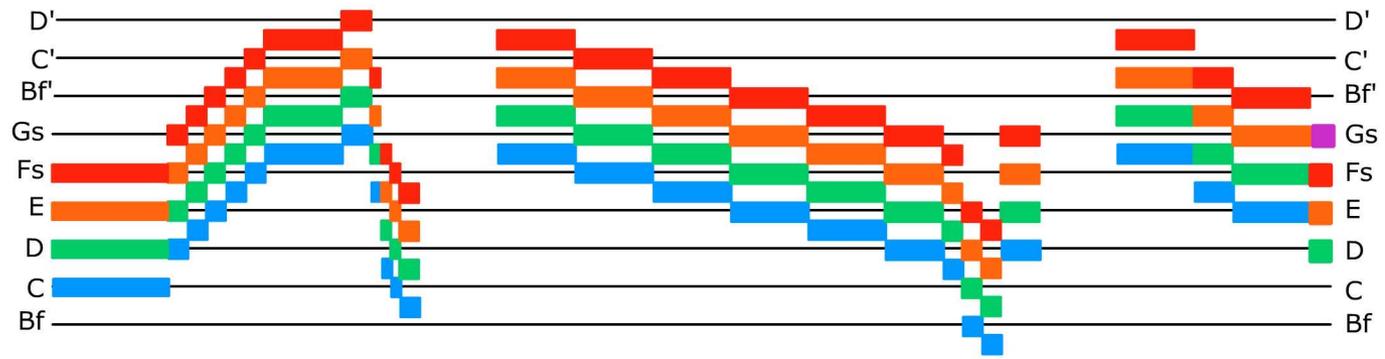


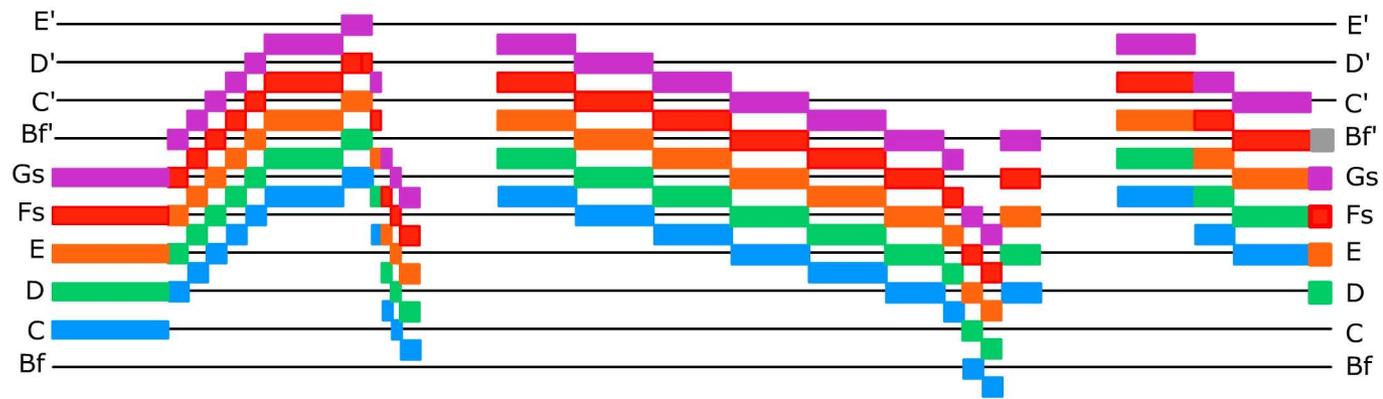
Here is the whole Canon, in a recording by Michael Monroe.
We will follow the modulations of the Royal Theme.

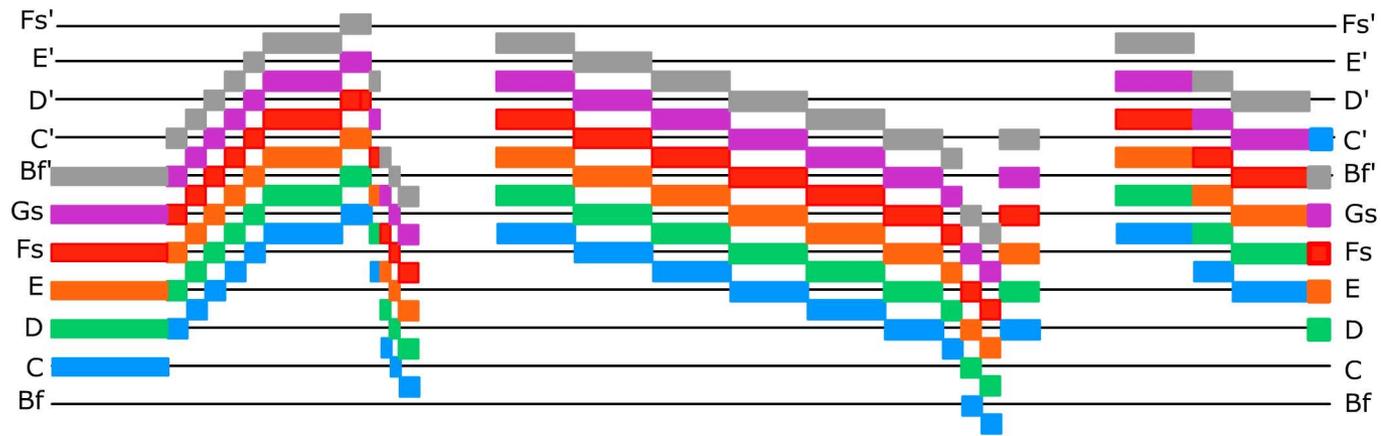


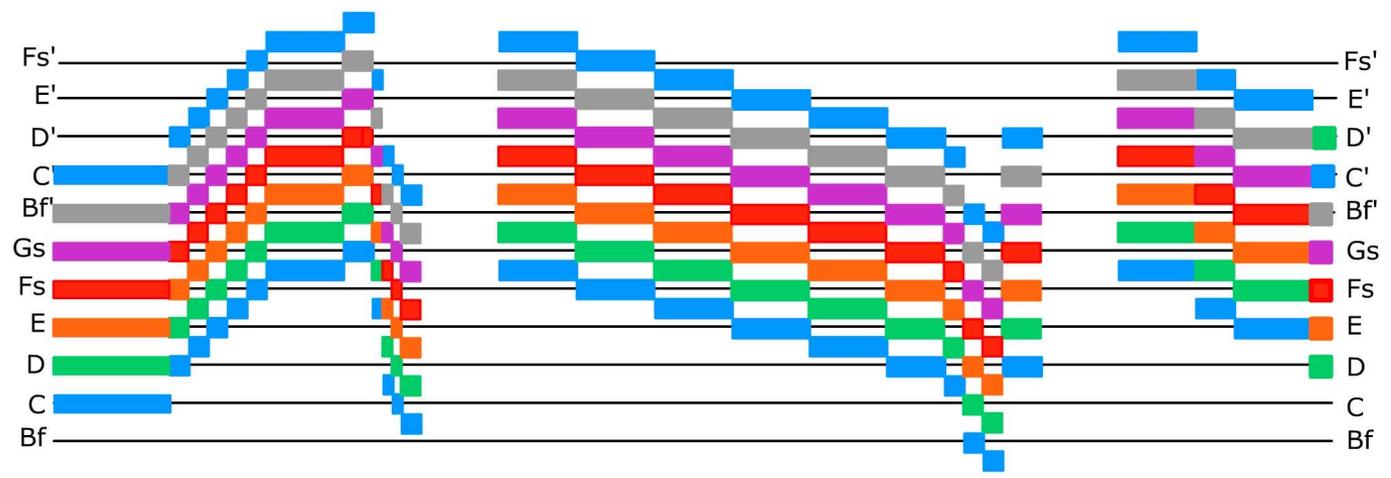




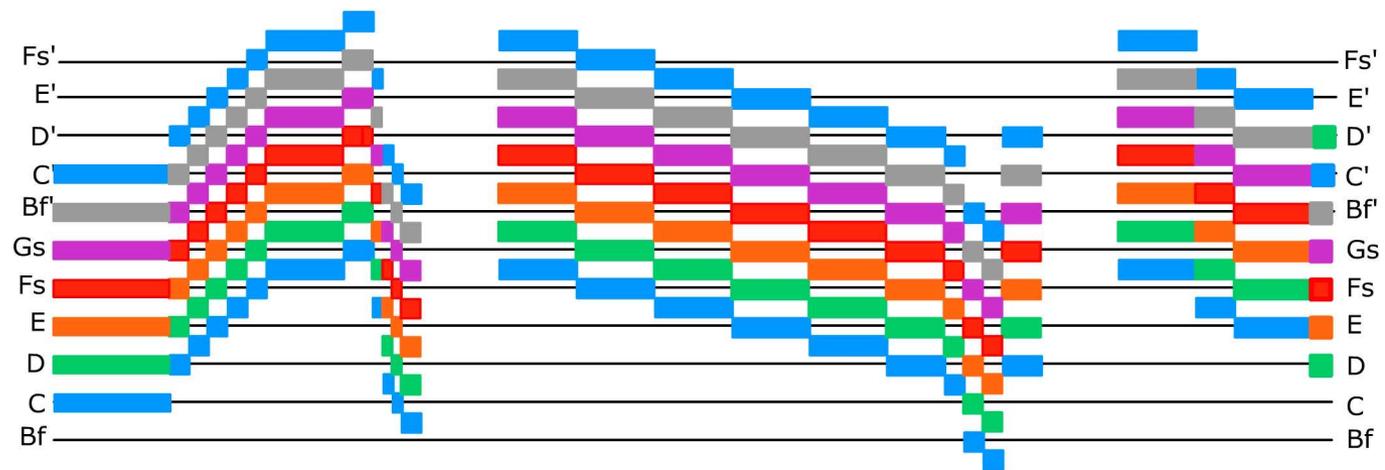


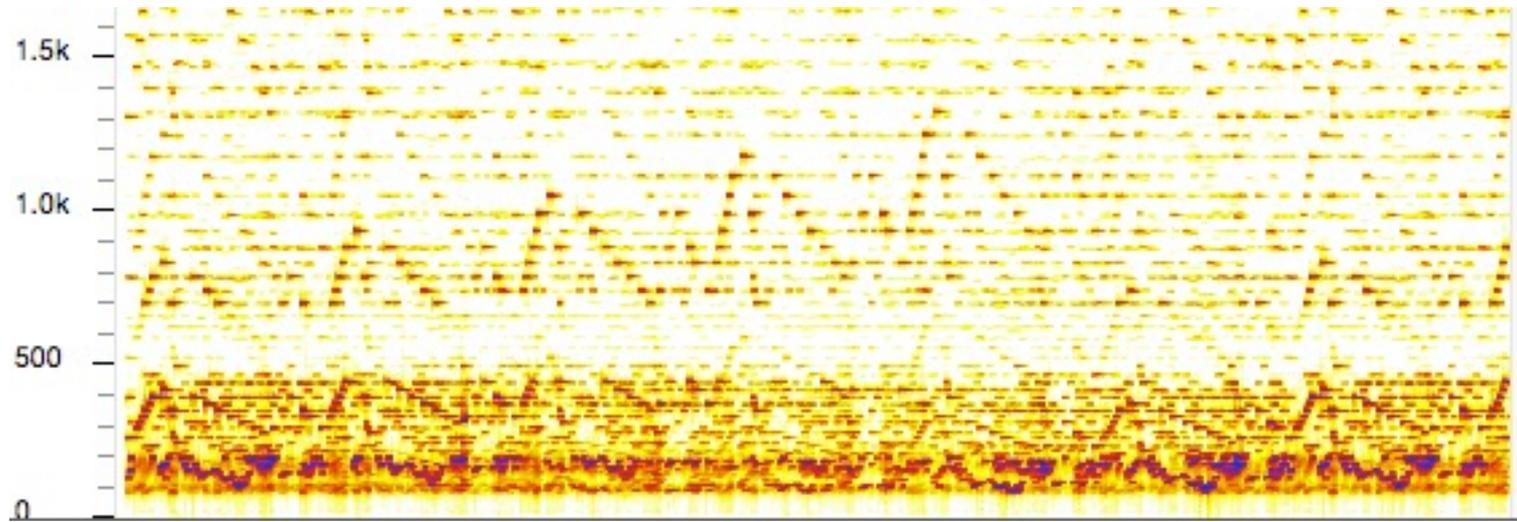






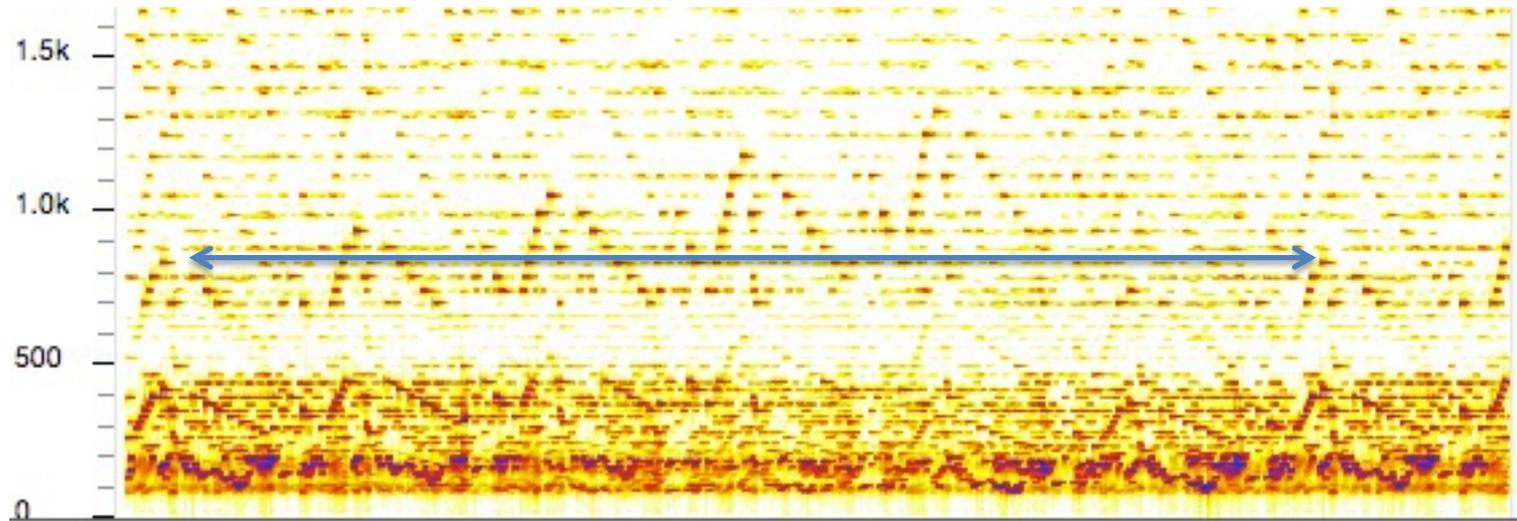
This is an illusion.





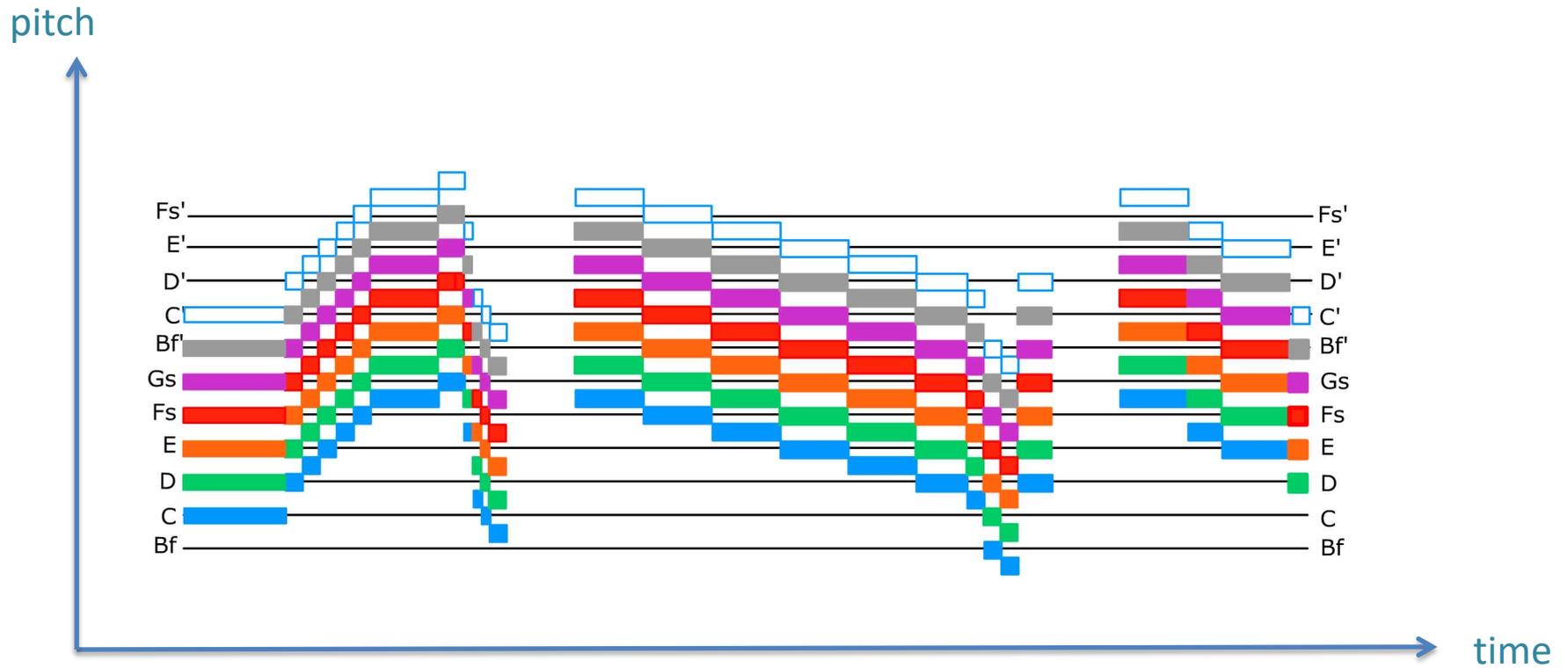
A sonogram of Michael Monroe's recording of Canon 5.

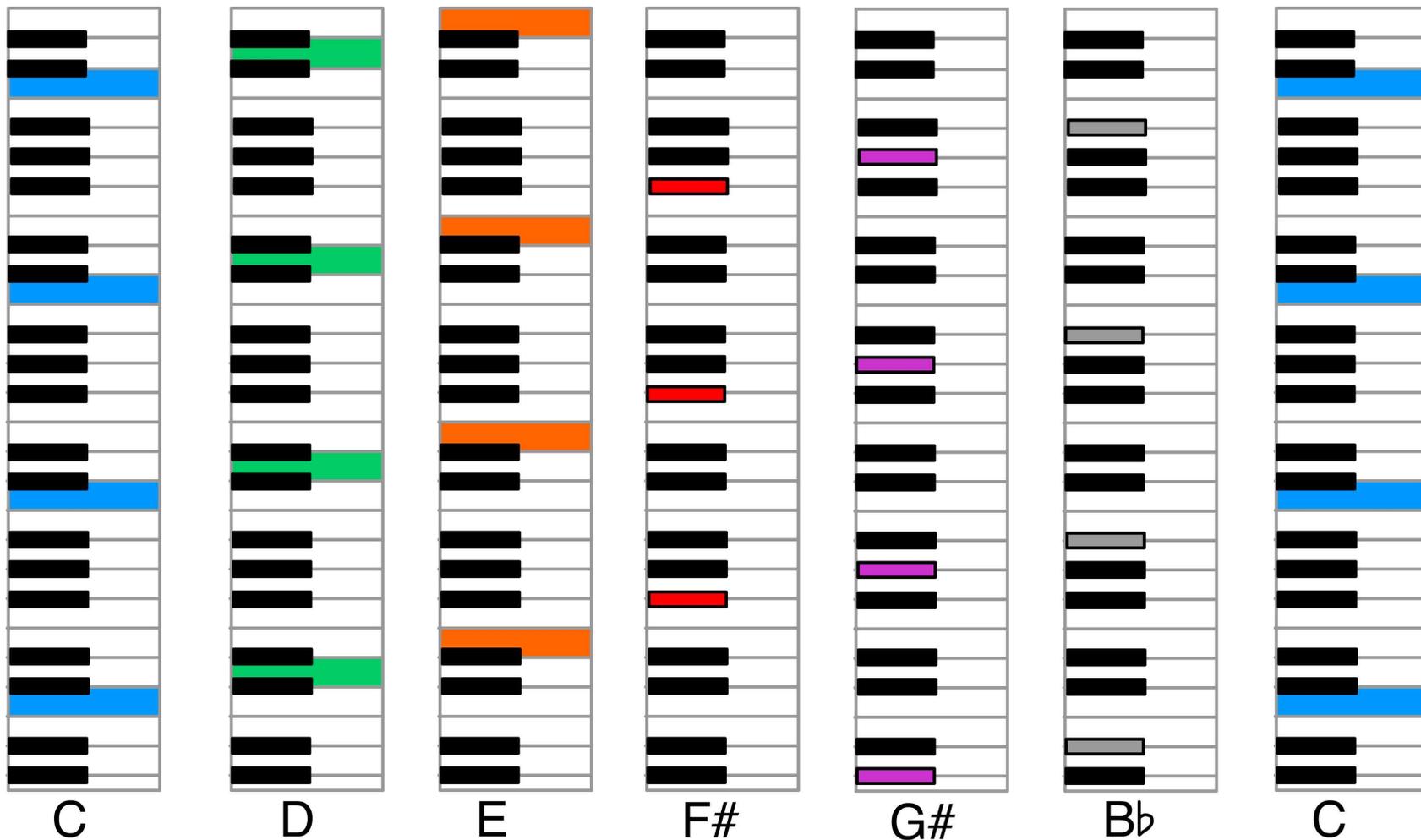
The seventh iteration is *at the same pitch* as the first.



A sonogram of Michael Monroe's recording of Canon 5.

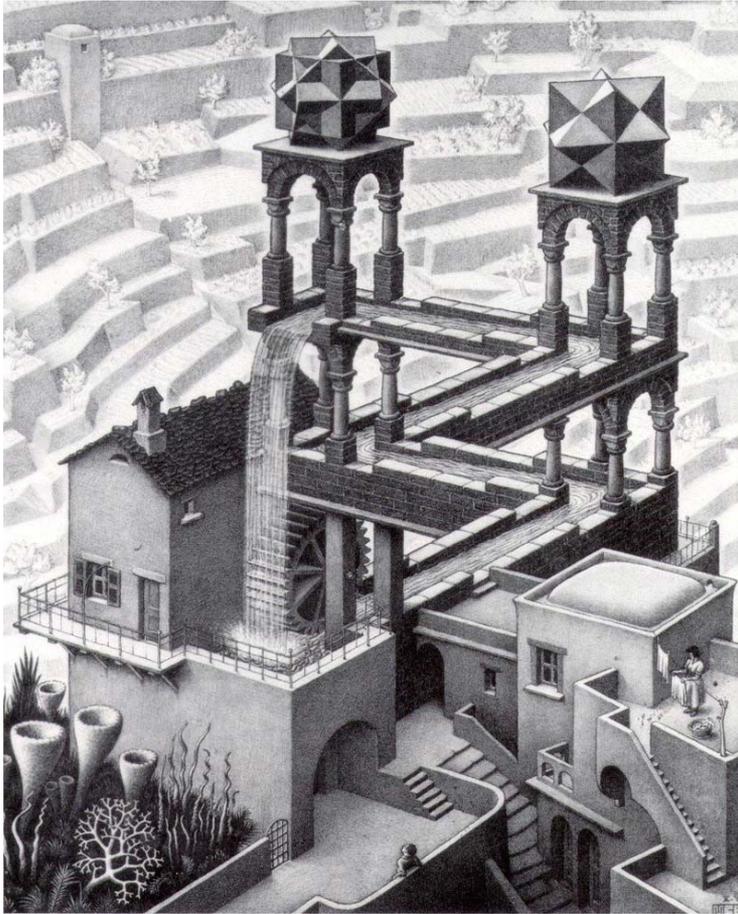
We perceive the pitch as rising.



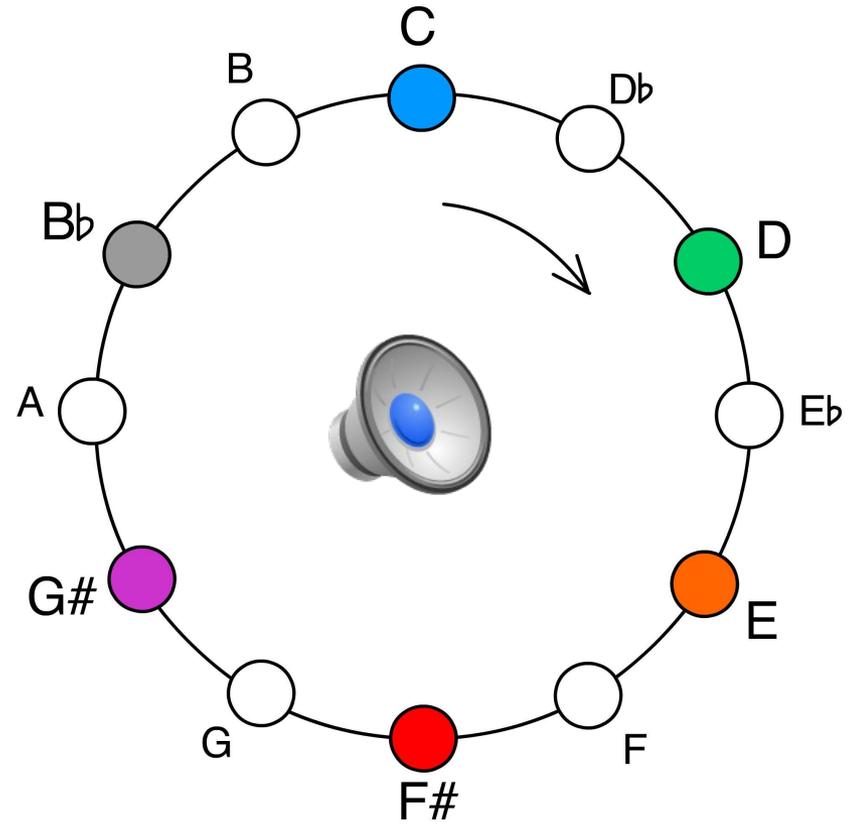


This is an example of a Shepard Scale: the tone seems to be always ascending, but ends up in the same place.



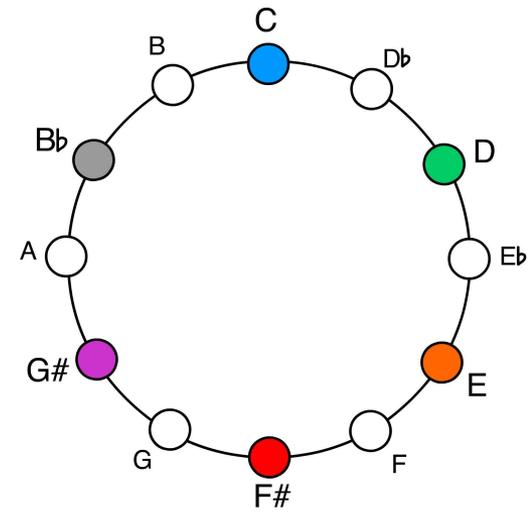


M. C. Escher, *Waterfall*, 1961

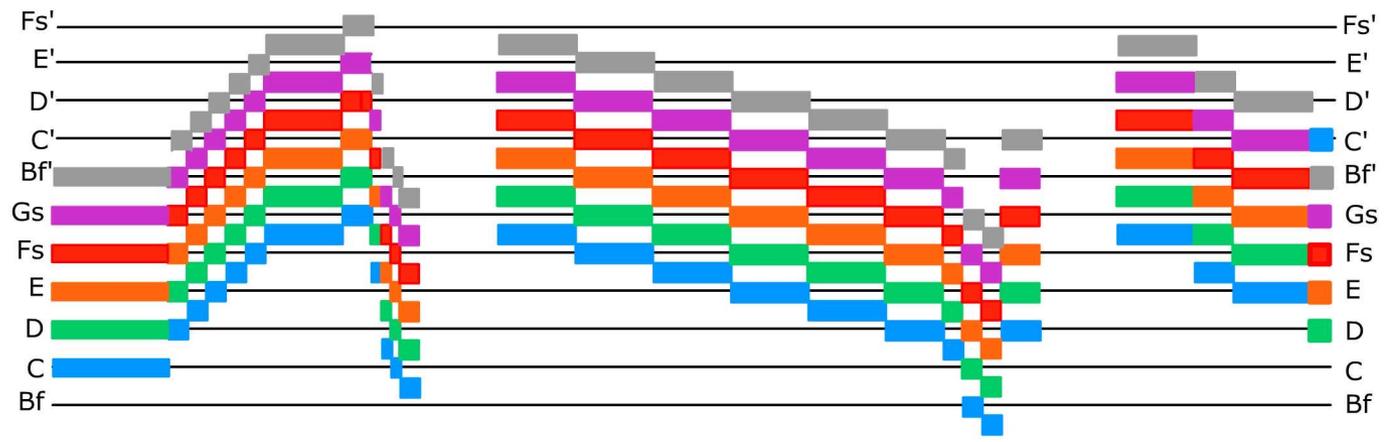


It's an acoustic analogue of Escher's impossible waterfall.

In TONALITY space, pitches an octave apart are equivalent.



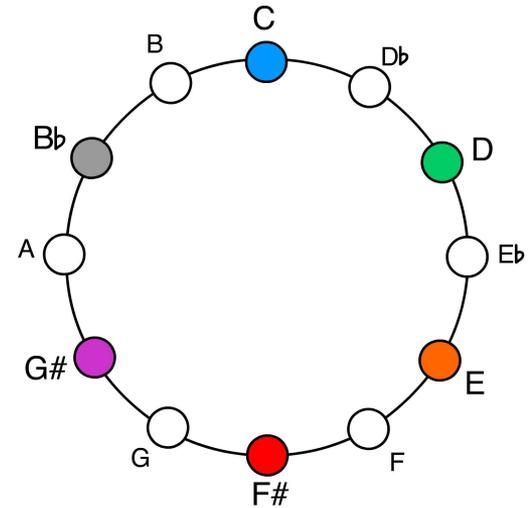
tonality



octave

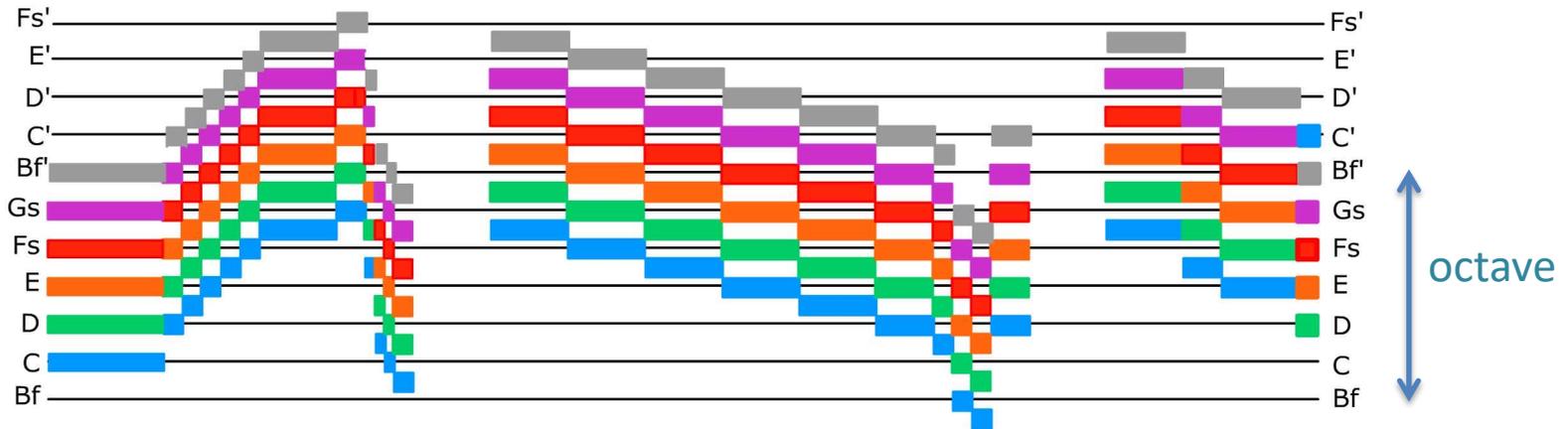
time

In TONALITY space, pitches an octave apart are equivalent.



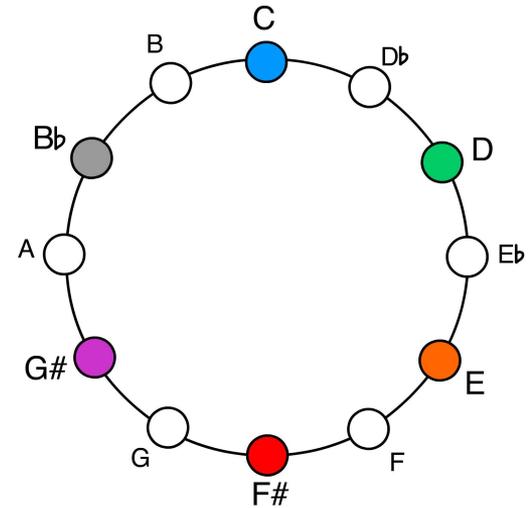
tonality

This makes the picture collapse:



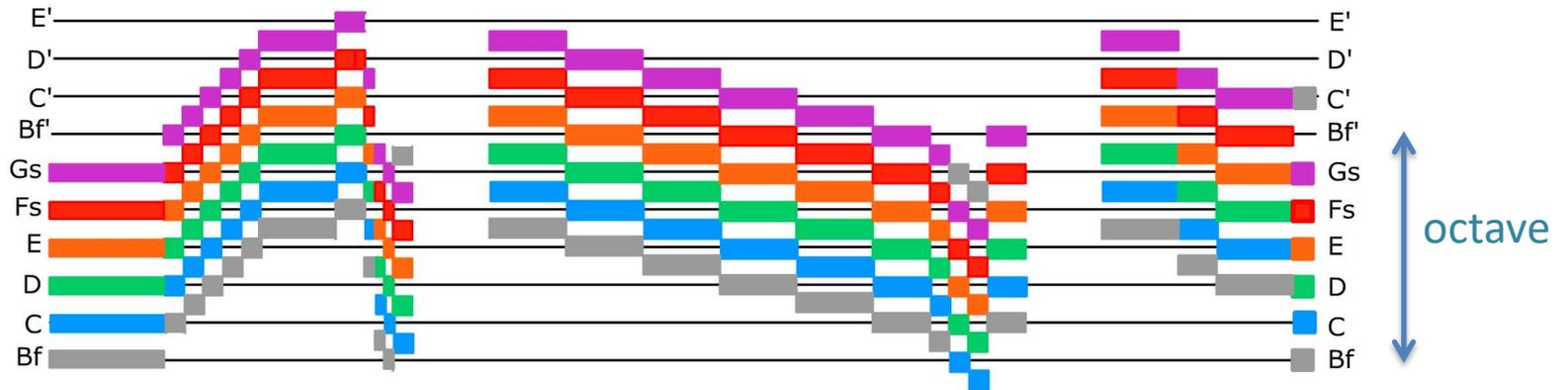
time

In TONALITY space, pitches an octave apart are equivalent.



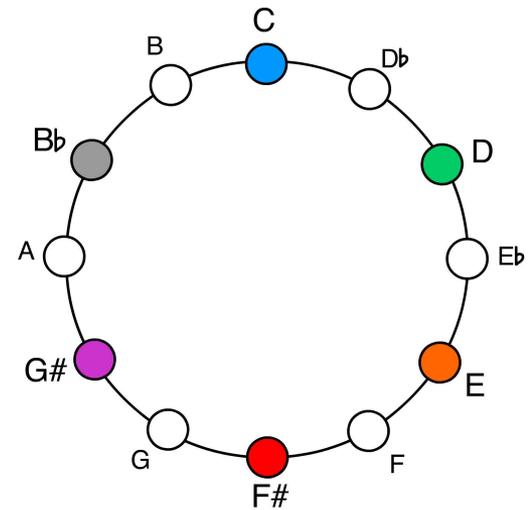
tonality

This makes the picture collapse:



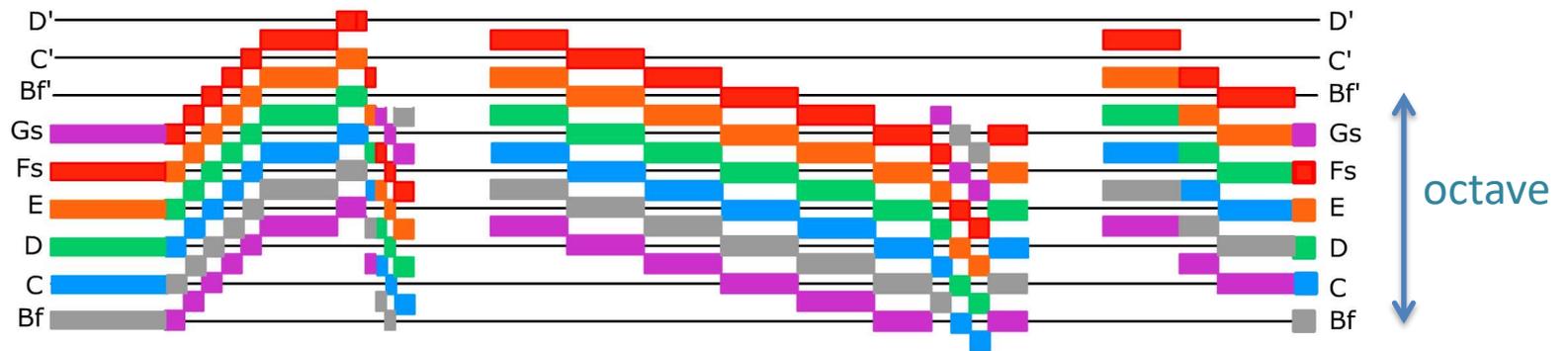
time

In TONALITY space, pitches an octave apart are equivalent.



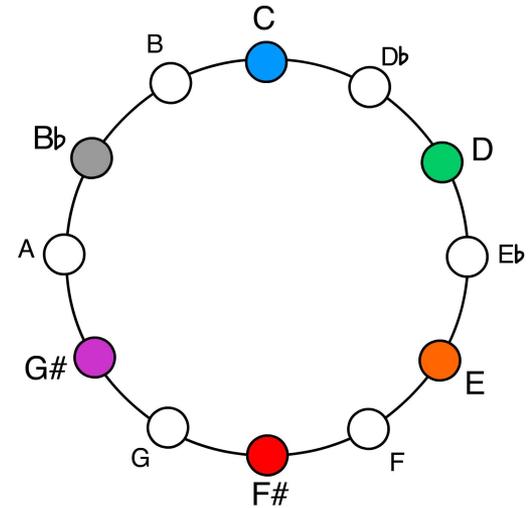
tonality

This makes the picture collapse:



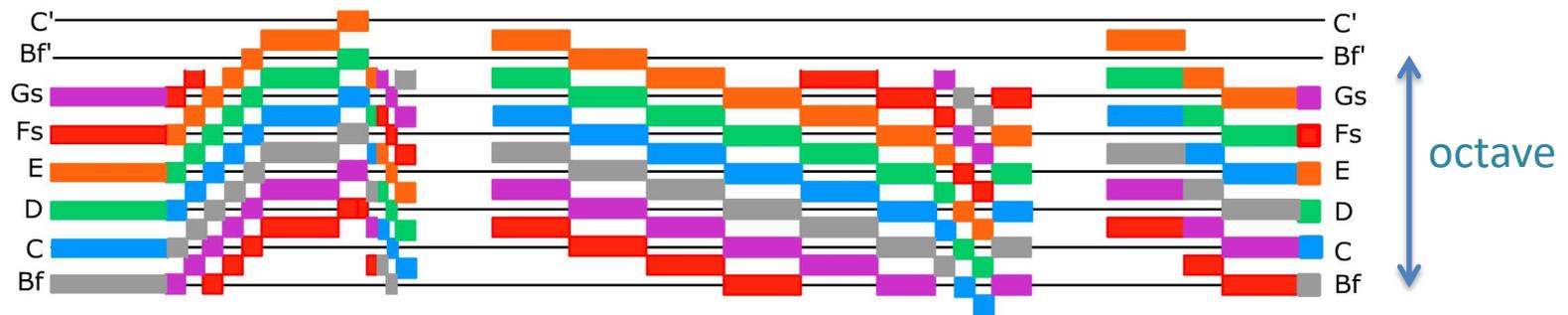
time

In TONALITY space, pitches an octave apart are equivalent.



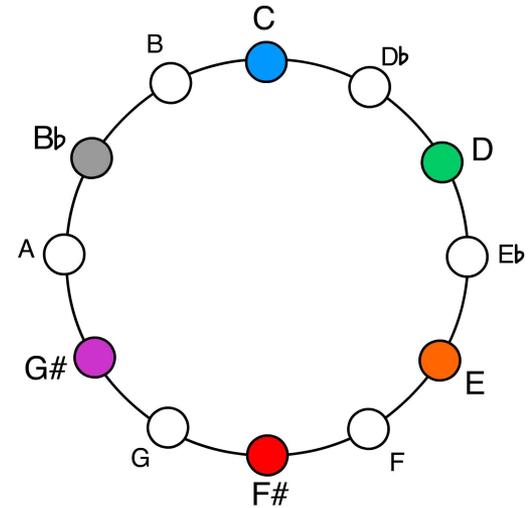
tonality

This makes the picture collapse:



time

In TONALITY space, pitches an octave apart are equivalent.



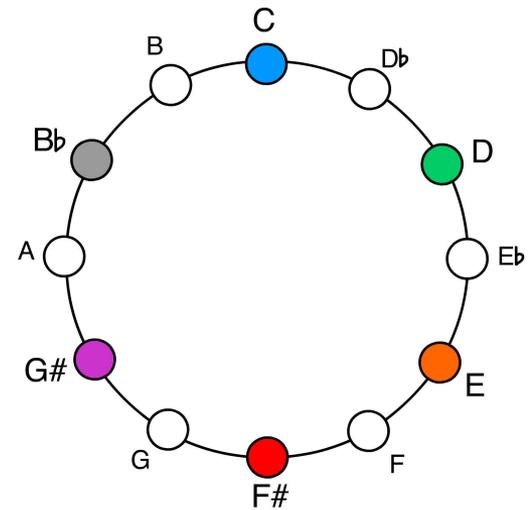
tonality

This makes the picture collapse:



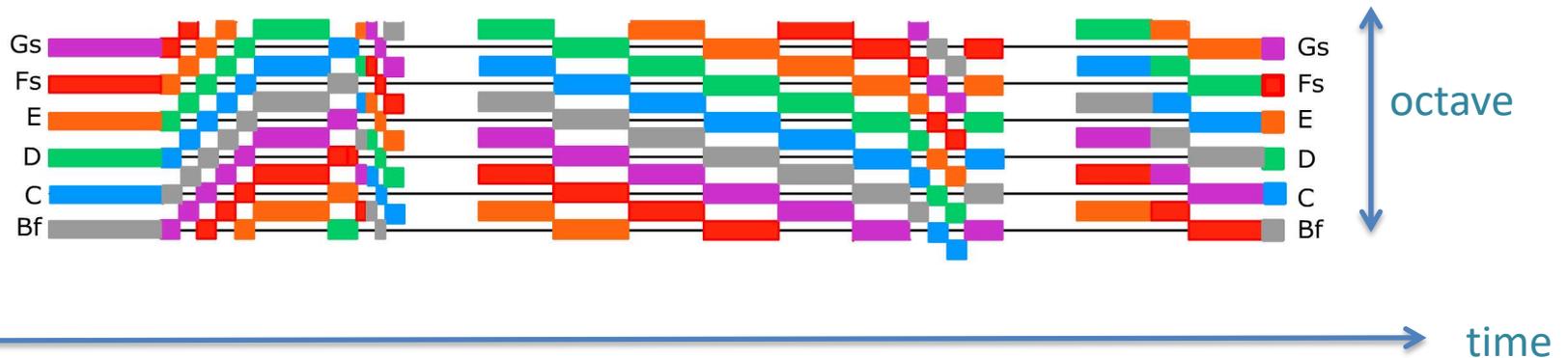
time

In TONALITY space, pitches an octave apart are equivalent.

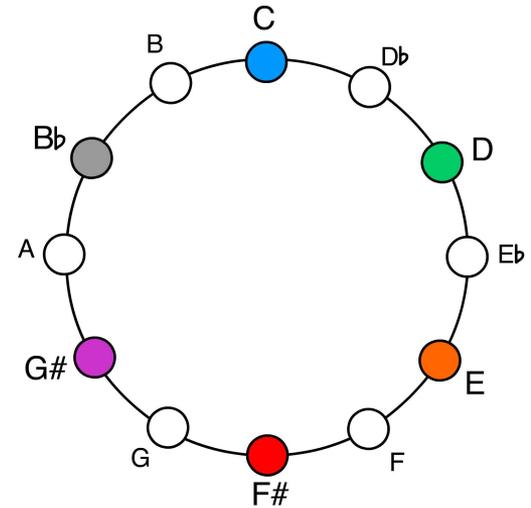


tonality

This makes the picture collapse:

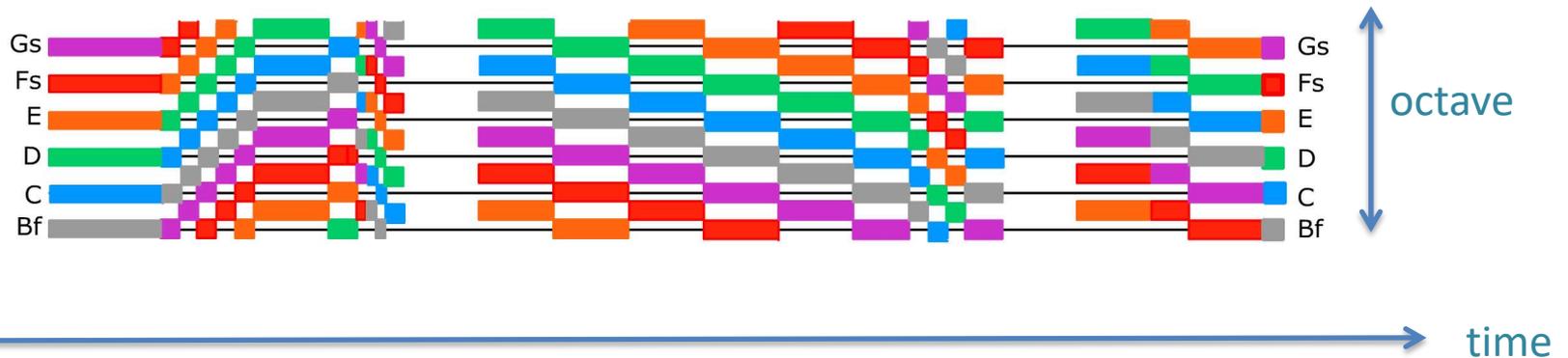


In TONALITY space, pitches an octave apart are equivalent.

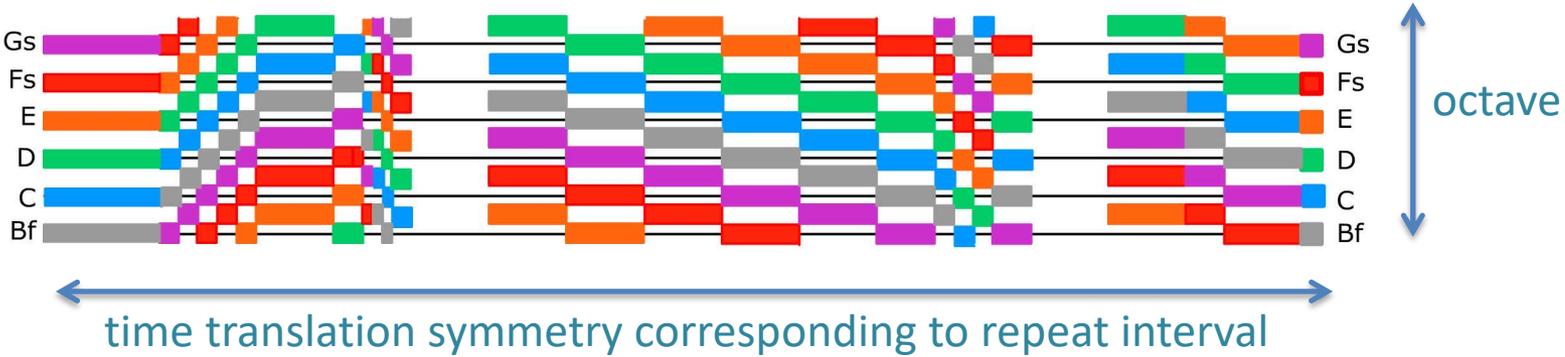


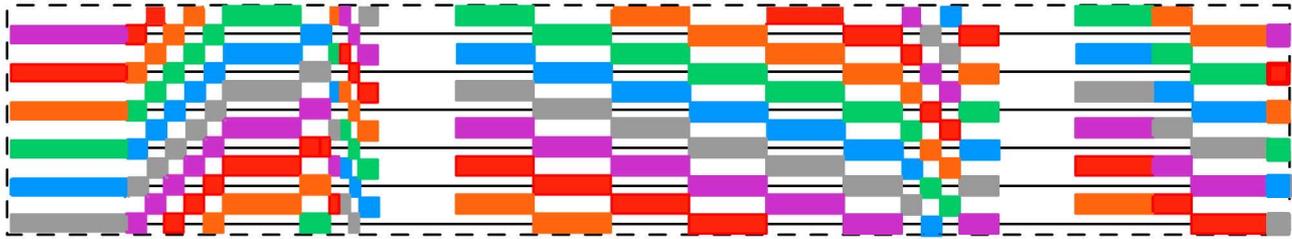
tonality

This makes the picture collapse:

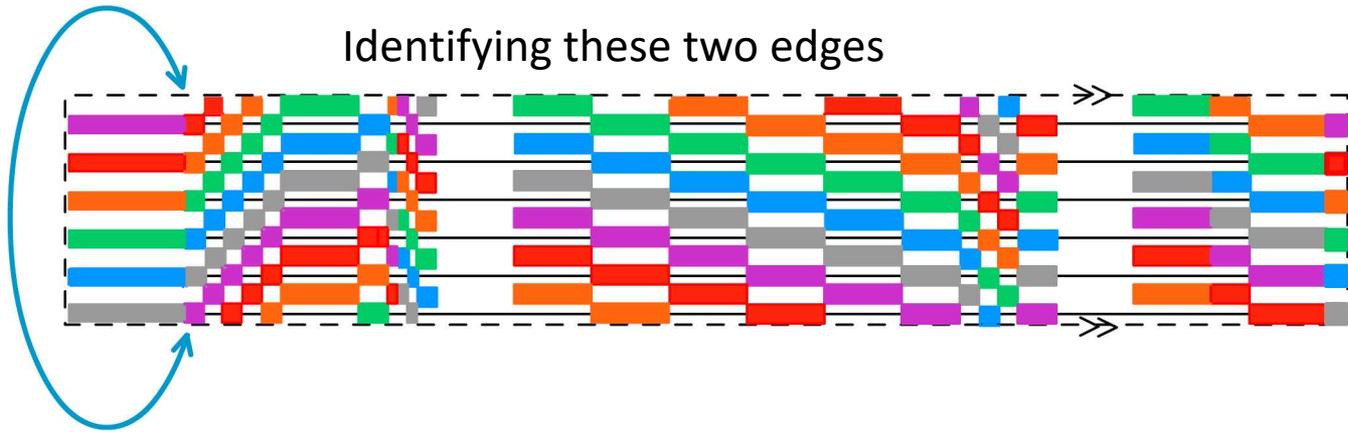


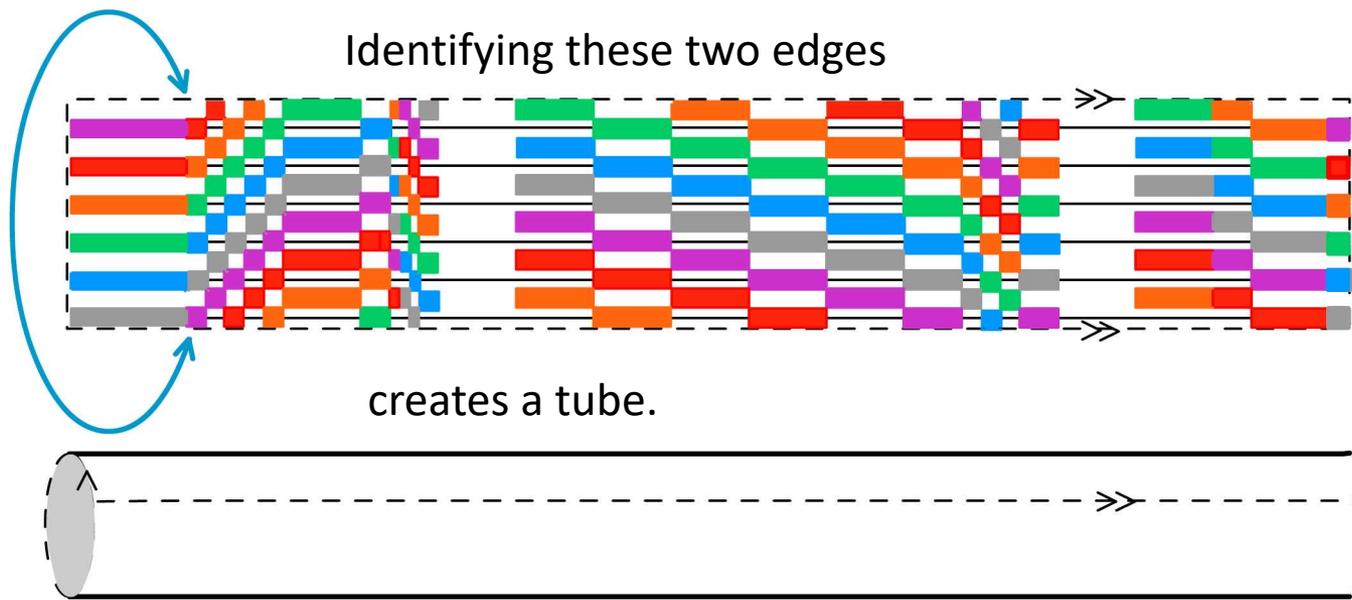
Together with the horizontal translation symmetry corresponding to the repeat, this gives the score of Canon 5 the topology of a **torus**.

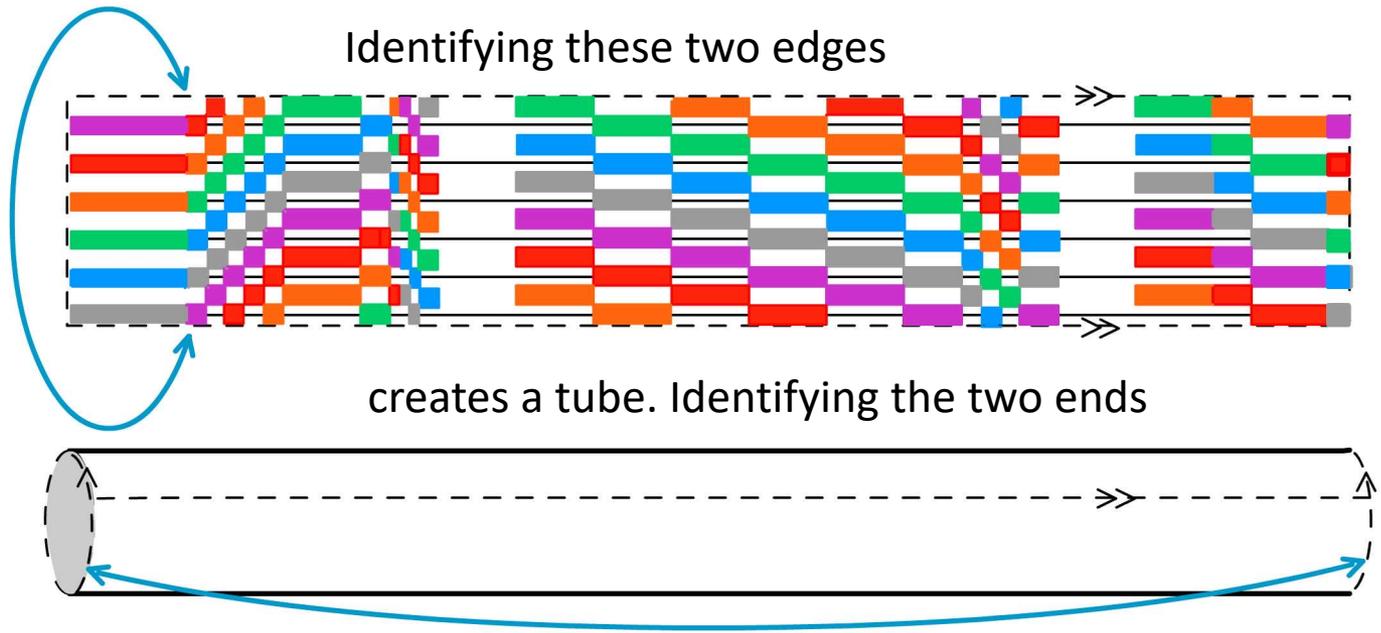


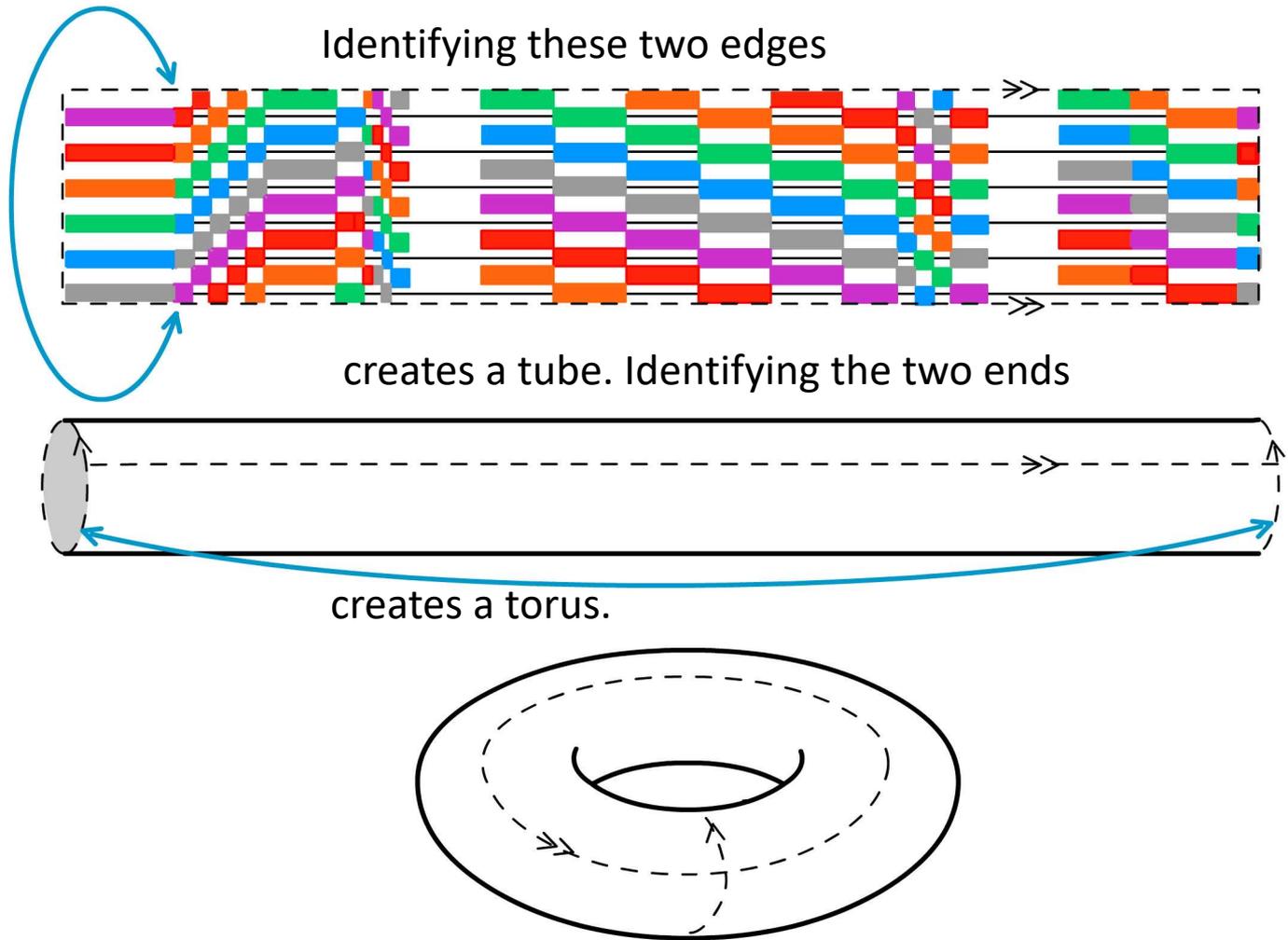


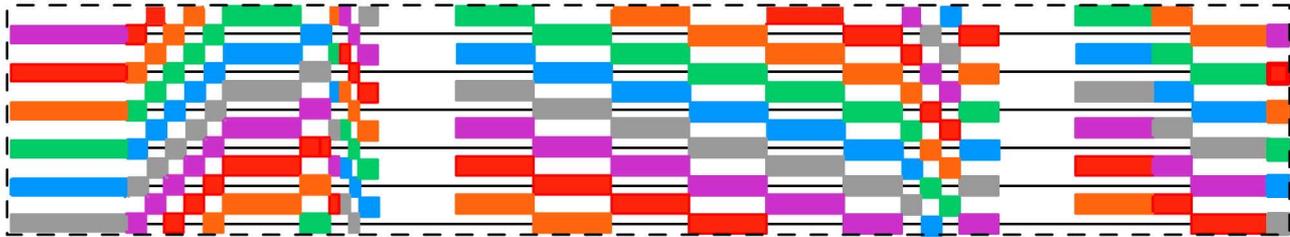
Identifying these two edges



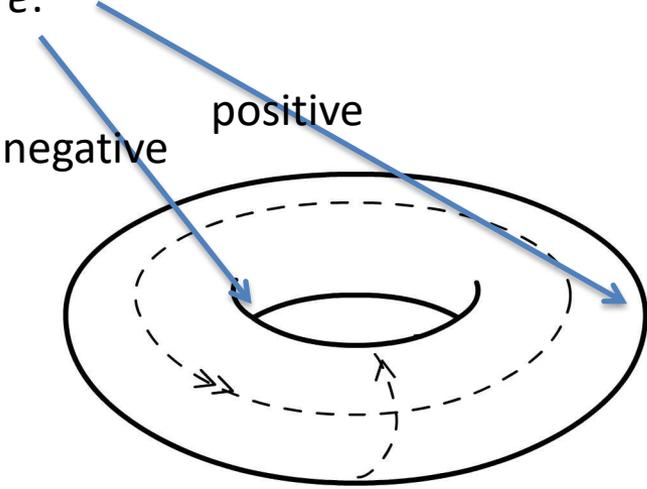


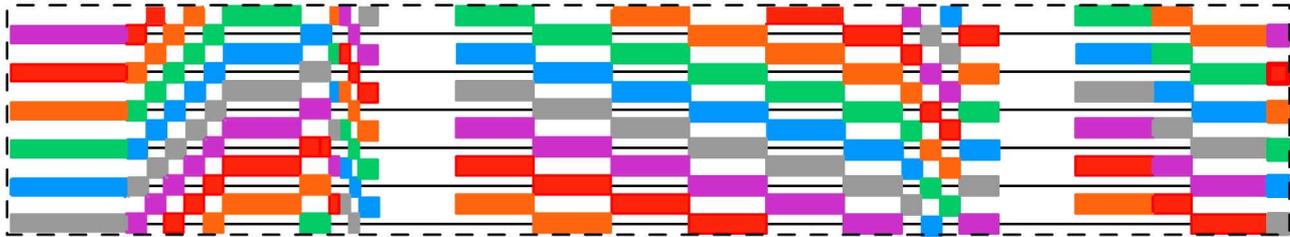




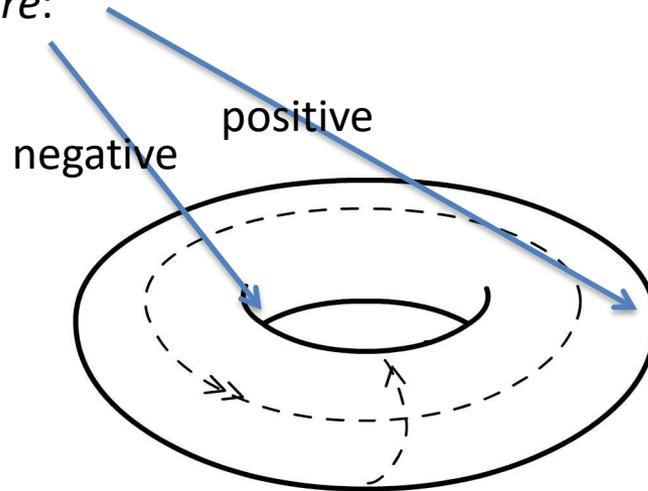


Note that our geometric picture of a smooth torus in 3-space has *curvature*:

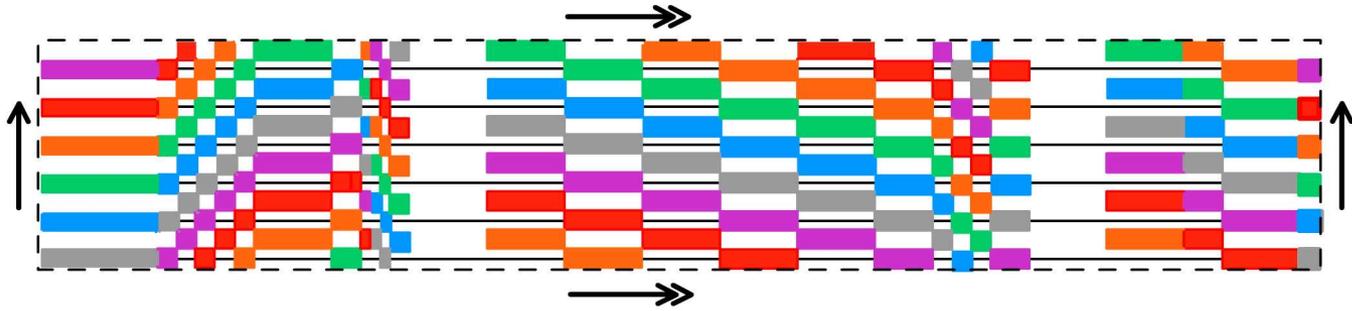




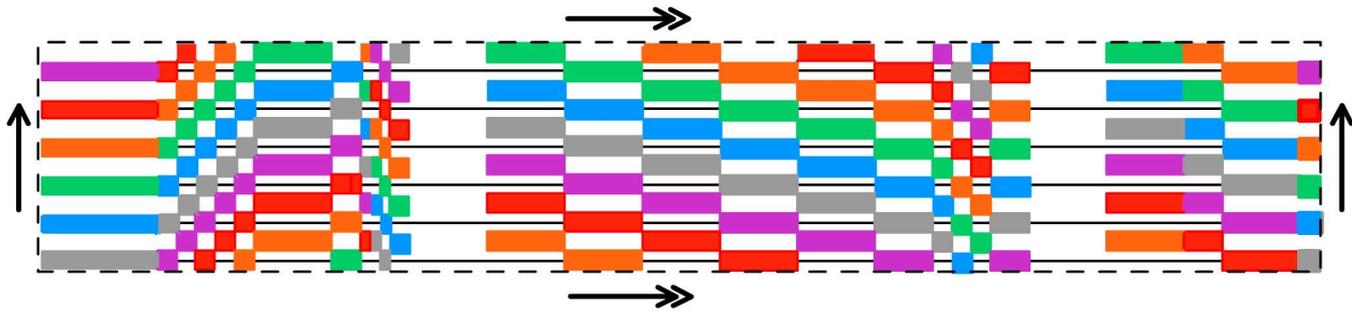
Note that our geometric picture of a smooth torus in 3-space has *curvature*:



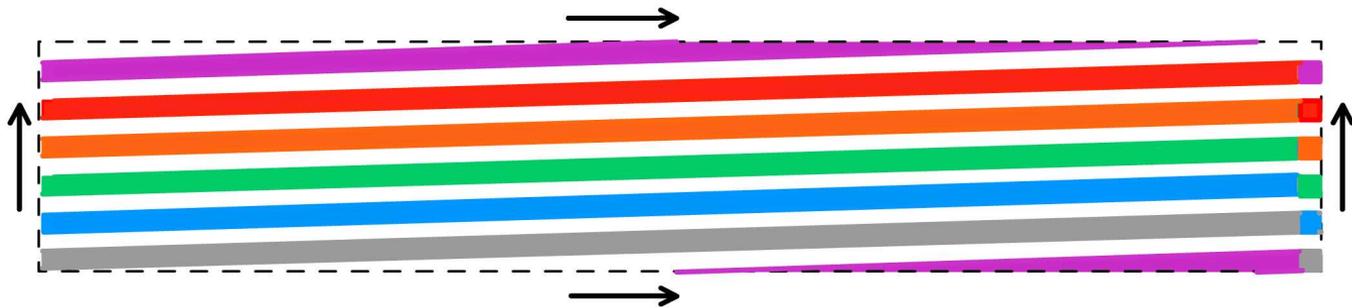
This does not make sense for a musical score.



The score of Canon 5 is a *flat* torus (no curvature), obtained by identifying opposite edges of a rectangle. It is out there, but it cannot be smoothly exhibited in 3-dimensional space.



On this ideal torus the melodic line of the canon winds around six times in one direction, and once in the other.



A beautiful video has been posted on YouTube



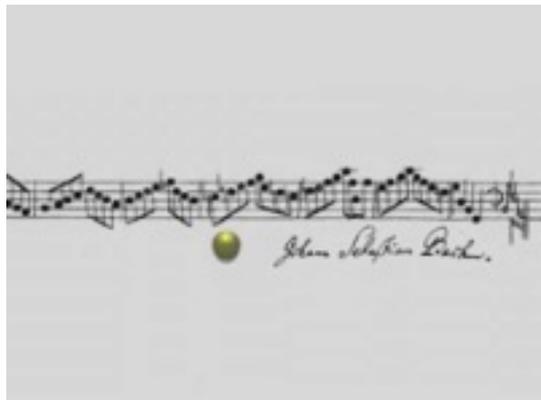
Explaining how the “Crab Canon” can be written on a Möbius strip.

The Crab Canon is also derived from the Royal Theme.
The two voices play the score starting from opposite ends.

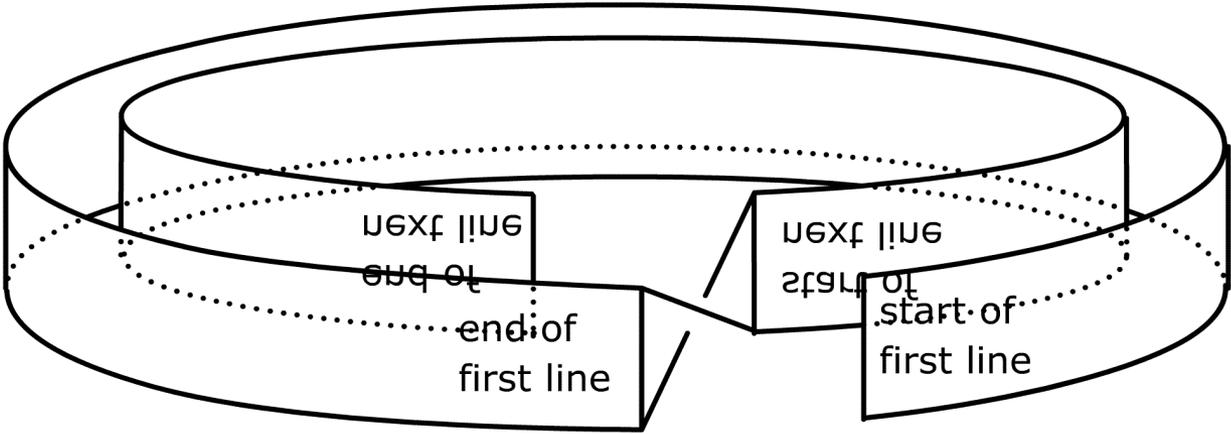
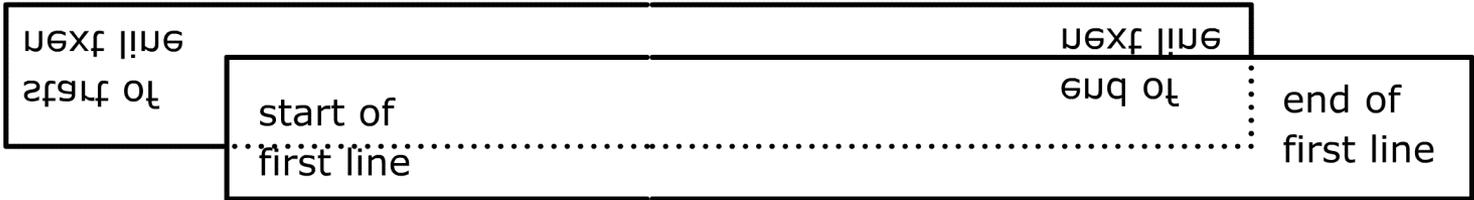
Alto clef, time and key signatures

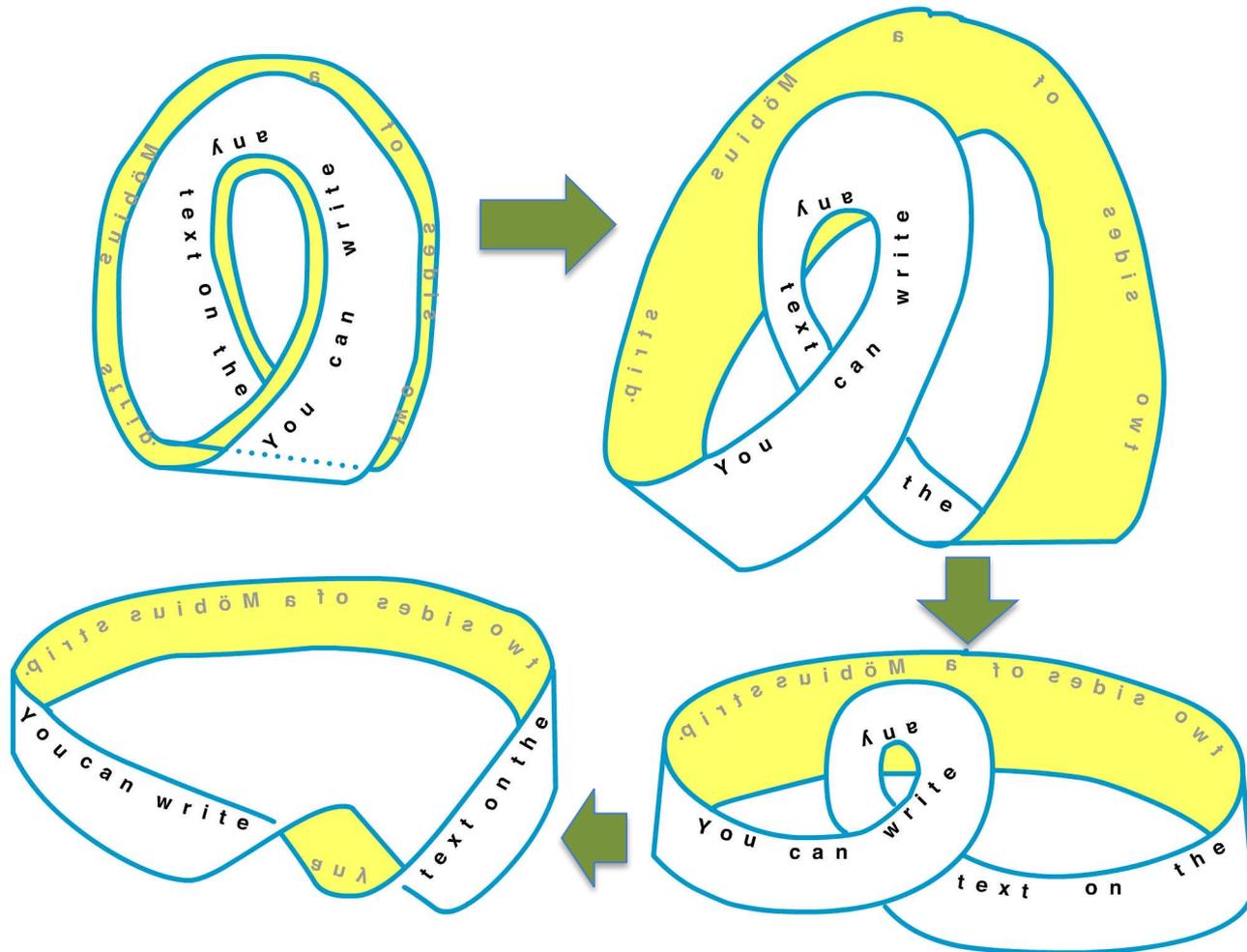


Backwards alto clef, time and key signatures



The problem is that this procedure works with *any* one-line text.





(You can write any text on the two sides of a Möbius strip.)

