MAT 310 Fall 2013 Review for Midterm 2

2.3. Composition of linear transformations, matrix multiplication. Understand that when $T: V \to W$ and $U: W \to Z$ are composed as functions (i.e. $UT(v) = U(T(v))$), if T and U are linear so is their composition UT. Know the definition of matrix multiplication (page 87), and be able to show that if $\alpha = (v_1, \dots, v_n)$ and $\beta = (w_1, \dots, w_m)$ are ordered bases for V and W, and if $\gamma = (x_1, \dots, x_p)$ is an ordered basis for Z then $[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$ (Theorem 2.11).

Understand also that the i-th column of $[T]_\alpha^\beta$ is made up of the coefficients $a_{ji}, j = 1 \dots m$ occurring when $T(v_i)$ is written as a linear combination $a_{1i}w_1 + \cdots + a_{mi}w_m$, and that if $v = c_1 v_1 + \cdots c_n v_n$ is an arbitrary vector in V, then the coefficients of $T(v) = d_1 w_1 + \cdots d_m w_m \in W$ are given by the matrix product $[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha$, where $[v]_\alpha$ is the *column vector* ($n \times 1$ matrix) $(c_1, \dots, c_n)$ and $[T(v)]_\beta$ is the column vector ($m \times 1$ matrix) $(d_1, \dots, d_m)$. *Exercises 12, 14ab.*

2.5 Understand how change of bases affects the matrix $[T]_\beta^\gamma$ of a linear transformation $T: V \to W$. Namely if β' is a new basis for V and γ' a new basis for W write T as the composition $T = I_W \circ T \circ I_V$:

$$V \xrightarrow{I_V} V \xrightarrow{T} W \xrightarrow{I_W} W$$

going from V (basis β') to V (basis β) to W (basis γ) to W (basis γ'), so

$$[T]_{\beta'}^{\gamma'} = [I_W]_\gamma^{\gamma'} [T]_\beta^\gamma [I_V]_{\beta'}^\beta.$$

Also, be able to calculate $[I_V]_{\beta'}^\beta$: suppose $\beta = (v_1, \dots, v_n)$ and $\beta' = (v_1', \dots, v_n')$; then the first column of $[I_V]_{\beta'}^\beta$ is the column of coefficients obtained when v_1' is written as a linear combination of $v_1, \dots, v_n$, i.e. it's the vector v_1' written in the basis $(v_1, \dots, v_n)$. Similarly second column of $[I_V]_{\beta'}^\beta$ is the vector v_2' written in the basis $(v_1, \dots, v_n)$, etc. This is especially simple when β is the standard basis.

On the other hand if you know $[I_W]_{\gamma'}^\gamma$, the matrix $[I_W]_\gamma^{\gamma'}$ can be retrieved by inverting $[I_W]_{\gamma'}^\gamma$, since $[I_W]_\gamma^{\gamma'} [I_W]_{\gamma'}^\gamma = [I_W]_\gamma^\gamma = I$, the identity matrix. *Examples 1, 2; Corollary, p.115; Problem 6d.*

3.1 Understand the three kinds of elementary row operations, and how each of them can be carried out on a matrix A by *left*-multiplying A with the appropriate *elementary matrix* (which is the matrix obtained by applying that row operation to the identity matrix!). *Theorem 3.1, Example 2.* Be able to invert an elementary matrix on inspection (be able to prove *Theorem 3.2*; pay attention to type 3.) Understand this paragraph when "row" $\to$ "column" and "left" $\to$ "right." *Problems 2, 4, 7.*

3.2 Remember that the *rank* of a linear transformation is the dimension of its range. Understand that the rank of an $m \times n$ matrix A is the rank of $L_A: \mathbf{F}^n \to \mathbf{F}^m$, and *Theorem 3.3.* Understand why the rank of A is the maximum number of linearly independent columns in A

(*Theorem 3.5, Examples 1, 2*). Understand the content of *Theorem 3.6*: After an appropriate change of basis, an arbitrary linear $T\colon \mathbf{F}^n \to \mathbf{F}^m$ becomes $(a_1, a_2, \ldots, a_r, a_{r+1}, \ldots, a_n) \to (a_1, a_2, \ldots, a_r, 0, \ldots, 0)$ where r is the rank of T. Understand how this theorem and its proof imply *Corollary 2: rank*(A^t) = *rank*(A) and *Corollary 3: Every invertible matrix is a product of elementary matrices* which are both very useful.

Know how to compute the inverse of an invertible matrix A by forming the augmented matrix (A, I) and row-reducing it to get (I, A^{-1}). *Examples 5, 6*: if A cannot be row-reduced to I, then A is not invertible.

4.1 Understand that the determinant of a 2×2 matrix $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$, that A is invertible iff $\det A \neq 0$, and the connection between determinants and area: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} =$ the area of the parallelogram spanned by the vectors (a, b) and (c, d) with a plus sign if $((a, b), (c, d))$ is a right-handed system and minus otherwise. *Problems 2, 3, 4.*

4.2 Understand how to carry out the inductive calculation of the determinant of a large square matrix, and be able to do it for 3×3 and 4×4 matrices. *Examples 1, 2, 3.* Understand the proof of *Theorem 4.3: the function* $A \to \det A$ *is linear in each row separately.* Understand the statement of the *Lemma* on pp. 213-214, and how it and Theorem 4.3 imply *Theorem 4.4: determinant can be calculated by cofactor expansion along any row.* Understand how this implies that if two rows of A are identical, then $\det A = 0$; and furthermore that if A' is derived from A by interchanging 2 rows, then $\det A' = -\det A$. Be able to use these concepts to prove *Theorem 4.6:* $\det$ *is invariant under type-3 elementary row operations* and its *Corollary p. 217: if* A *is* $n \times n$ *and rank*$(A) < n$ *then* $\det A = 0$. Be able to calculate the determinant of a large matrix by using row operations to simplify the problem. *Examples 5,6. Problems 9,21.*