Hölder regularity of the Lagrangian velocity in turbulence

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Theorem 1. *Let $u \in L^\infty_t C^\alpha_x$ be a weak solution of the Euler equations. For each $a \in M$, consider any solution of*

$$\frac{\mathrm{d}}{\mathrm{d}t} X_t(a) = u(X_t(a), t), \qquad X_0(a) = a. \tag{1}$$

Define $v(t,a) := \frac{\mathrm{d}}{\mathrm{d}t} X_t(a)$. Then $v \in L^\infty_a C_t^{\frac{\alpha}{1-\alpha}}$.

Remark 1. If $\alpha = \frac{1}{3}$, then $\frac{\alpha}{1-\alpha} = \frac{1}{2}$, in agreement with the prediction of Landau and Lifshitz.

Remark 2. In fact, Phil Isett proved that if $u \in L^\infty_t C^\alpha_x$ be a weak solution of the Euler equations, then every particle trajectory of u is of class $C_t^{\frac{1}{1-\alpha}}$, see [1]. See also [2]. This is a stronger statement than what we prove here.

PROOF. Consider flow of a mollified field

$$\frac{\mathrm{d}}{\mathrm{d}t} X^\ell_t(a) = \bar{u}_\ell(X^\ell_t(a), t), \qquad X^\ell_0(a) = a. \tag{2}$$

Let $v^\ell(t,a) := \frac{\mathrm{d}}{\mathrm{d}t} X^\ell_t(a)$ and $\delta_\tau f(t,a) := f(\tau + t, a) - f(t,a)$. The natural time-scale of the Lagrangian velocity mollified at length-scale ℓ is the local eddy turnover time $\ell/\delta_\ell u$, defined precisely by

$$\tau_\ell := \frac{\ell^{1-\alpha}}{\|u\|_{L^\infty_t C^\alpha_x}} \tag{3}$$

where $\delta_\ell u := u(x+\ell, t) - u(x)$. Indeed, we have

Lemma 1. *Let X_t be any solution of (1). Then*

$$\|X_t(a) - X^\ell_t(a)\| \lesssim \ell \exp(t/\tau_\ell). \tag{4}$$

PROOF. Let us introduce $\delta X^\ell_t(a) := X_t(a) - X^\ell_t(a)$. Then $\delta X^\ell_0(a) = 0$ while

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \delta X^\ell_t(a) &= u(X_t(a), t) - \bar{u}_\ell(X^\ell_t(a), t) \\ &= u(X_t(a), t) - \bar{u}_\ell(X_t(a), t) + \bar{u}_\ell(X_t(a), t) - \bar{u}_\ell(X^\ell_t(a), t).\end{aligned}$$

Whence, using that $\|\nabla \bar{u}_\ell(\cdot, t)\| \lesssim \sup_{\ell' \leq \ell} \|\delta_{\ell'} u\|_{L^\infty_{t,x}}/\ell \leq {\tau_\ell}^{-1}$ and $|u - \bar{u}_\ell| \leq \sup_{\ell' \leq \ell} \|\delta_{\ell'} u\|_{L^\infty_{t,x}}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\delta X^\ell_t(a)\| \leq \sup_{\ell' \leq \ell} \|\delta_{\ell'} u\|_{L^\infty_{t,x}} + \|\nabla \bar{u}_\ell(\cdot, t)\|_{L^\infty} \|\delta X^\ell_t(a)\| \lesssim \frac{1}{\tau_\ell} \left(\ell + \|\delta X^\ell_t(a)\| \right).$$

The claimed result follows by Gronwall's inequality. □

Lemma 2. *Let X_t be any solution of (1) and v be the associated Lagrangian velocity. Then*

$$\|v^\ell(t, \cdot) - v(t, \cdot)\|_{L^\infty} \lesssim \frac{\ell}{\tau_\ell} \qquad \textit{for times} \qquad 0 \leq t \leq \tau_\ell. \tag{5}$$

PROOF. Fix $0 \leq t \leq \tau_\ell$ so that $\|\delta X_t^\ell\|_{L^\infty} \leq \ell$. Thus, on these timescales, we have

$$\begin{aligned}\|v^\ell(t,\cdot) - v(t,\cdot)\|_{L^\infty} &= \|\bar{u}_\ell(X_t^\ell(\cdot),t) - u(X_t(\cdot),t)\|_{L^\infty}\\ &= \|\bar{u}_\ell(X_t^\ell(\cdot),t) - u(X_t^\ell(\cdot) + \delta X_t^\ell(\cdot),t)\|_{L^\infty}\\ &= \|\bar{u}_\ell(\cdot,t) - u(\cdot + \delta X_t^\ell(A_t^\ell(\cdot)),t)\|_{L^\infty} \lesssim \sup_{\ell' \leq \ell} \|\delta_{\ell'} u\|_{L^\infty_{t,x}},\end{aligned}$$

where $A_t^\ell := (X_t^\ell)^{-1}$. □

Lemma 3. *Let X_t be any solution of (1) and v be the associated Lagrangian velocity. Then*

$$\|v^\ell(t+\tau,\cdot) - v^\ell(t,\cdot)\|_{L^\infty} \lesssim \frac{\ell}{\tau_\ell} \qquad \textit{for times} \qquad 0 \leq \tau \leq \tau_\ell. \tag{6}$$

PROOF. From the Euler equation $\delta_\tau v^\ell(t,a) := v^\ell(t+\tau,a) - v^\ell(t,a)$ satisfies

$$\delta_\tau v^\ell(t,a) = \int_0^\tau a_\ell(s, X_s(a))\mathrm{d}s, \tag{7}$$

where $a_\ell = \nabla \bar{p}_\ell + \nabla \cdot \tau_\ell(u,u)$. It follows from standard commutator estimates that

$$\|a_\ell(t,\cdot)\|_{L^\infty} \lesssim \frac{1}{\ell} \sup_{\ell' \leq \ell} \|\delta_{\ell'} u\|^2_{L^\infty_{t,x}} = \frac{\ell}{\tau_\ell^2}. \tag{8}$$

The conclusion follows using that $\tau \leq \tau_\ell$. □

Combining the previous two lemmas, we have, we have

$$\sup_{0 \leq \tau \leq \tau_\ell} \|v(t+\tau,\cdot) - v(t,\cdot)\|_{L^\infty_a} \lesssim \frac{\ell}{\tau_\ell}. \tag{9}$$

Now, note that since $\ell = \left(\|u\|_{L_t^\infty C_x^\alpha} \tau_\ell\right)^{\frac{1}{1-\alpha}}$. Thus we finally obtain:

$$\sup_{0 \leq \tau \leq \tau_\ell} \|v(t+\tau,\cdot) - v(t,\cdot)\|_{L^\infty_a} \lesssim \left(\|u\|_{L_t^\infty C_x^\alpha}\right)^{\frac{1}{1-\alpha}} \tau_\ell^{\frac{\alpha}{1-\alpha}}. \tag{10}$$

Since $\tau_\ell \sim \ell^{1-\alpha} \to 0$ as $\ell \to 0$, the statement holds. □

On the following page we include an excerpt from Landau and Lifshitz book on fluid dynamics, where this improved regularity of the Lagrangian velocity is discussed.

References

[1] Philip Isett. Regularity in time along the coarse scale flow for the incompressible Euler equations. Transactions of the American Mathematical Society 376.10 (2023): 6927-6987.

[2] G. Eyink. Turbulence theory. Course notes. The Johns Hopkins University, 2007-08.

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Let us determine the order of magnitude v_λ of the turbulent velocity variation over distances of the order of λ. It must be determined only by ε and, of course, the distance λ itself.† From these two quantities we can form only one having the dimensions of velocity, namely $(\varepsilon\lambda)^{\frac{1}{3}}$. Hence we can say that the relation

$$v_\lambda \propto (\varepsilon\lambda)^{\frac{1}{3}} \tag{33.6}$$

must hold. We thus find that the velocity variation over a small distance is proportional to the cube root of the distance (*Kolmogorov and Obukhov's law*). The quantity v_λ may also be regarded as the velocity of turbulent eddies whose size is of the order of λ: the variation of the mean velocity over small distances is small compared with the variation of the fluctuating velocity over those distances, and may be neglected.

The relation (33.6) may be obtained in another way by expressing a constant quantity, the dissipation ε, in terms of quantities characterizing the eddies of size λ; ε must be proportional to the squared gradient of the velocity v_λ and to the appropriate turbulent viscosity coefficient $\nu_{\text{turb},\lambda} \propto v_\lambda \lambda$:

$$\varepsilon \propto \nu_{\text{turb},\lambda}(v_\lambda/\lambda)^2 \propto v_\lambda{}^3/\lambda,$$

whence we obtain (33.6).

Let us now put the problem somewhat differently, and determine the order of magnitude v_τ of the velocity variation at a given point over a time interval τ which is short compared with the time $T \sim l/u$ characterizing the flow as a whole. To do this, we notice that, since there is a net mean flow, any given portion of the fluid is displaced, during the interval τ, over a distance of the order of τu, u being the mean velocity. Hence the portion of fluid which is at a given point at time τ will have been at a distance τu from that point at the initial instant. We can therefore obtain the required quantity v_τ by direct substitution of τu for λ in (33.6):

$$v_\tau \propto (\varepsilon\tau u)^{\frac{1}{3}}. \tag{33.7}$$

The quantity v_τ must be distinguished from v_τ', the variation in velocity of a portion of fluid as it moves about. This variation can evidently depend only on ε, which determines the local properties of the turbulence, and of course on τ itself. Forming the only combination of ε and τ that has the dimensions of velocity, we obtain

$$v_\tau' \propto (\varepsilon\tau)^{\frac{1}{2}}. \tag{33.8}$$

Unlike the velocity variation at a given point, it is proportional to the square root of τ, not to the cube root. It is easy to see that, for τ small compared with T, v_τ' is always less than v_τ.‡

Using the expression (33.1) for ε, we can rewrite (33.6) and (33.7) as

$$\left.\begin{aligned} v_\lambda &\propto \Delta u(\lambda/l)^{\frac{1}{3}}, \\ v_\tau &\propto \Delta u(\tau/T)^{\frac{1}{3}}. \end{aligned}\right\} \tag{33.9}$$

This form shows clearly the similarity property of local turbulence: the small-scale characteristics of different turbulent flows are the same apart from the scale of measurement of lengths and velocities (or, equivalently, lengths and times).††

† The dimensions of ε are erg/g sec = cm^2/sec^3, and do not include mass; the only quantity involving the mass dimension is the density ρ. The latter is therefore not involved in quantities whose dimensions do not include mass.

‡ The inequality $v_\tau' \ll v_\tau$ has in essence been assumed in the derivation of (33.7).

†† In this connection, the term *self-similarity* is often used in recent literature.

FIGURE 1. Pg 133 of Landau and Lifshitz