A Entropic Characterization of Navier-Stokes

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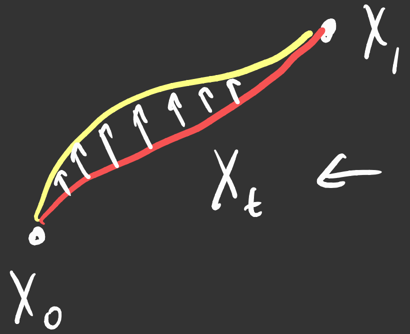
Variational Principle for Euler / Burgers

Let M be a compact manifold without boundary (e.g $\mathbb{T}^d$)

Fix X_0 and $X_1 \in SDiff(M)$ ($Diff(M)$)



$\leftarrow$ $\{X_t\}_{t\in[0,1]}$ is a path between X_0, X_1,

Let $v(x,t)$ be a vectorfield defined in a neighborhood of $\{X_t\}$, such that

- $v(x,0) = v(x,1) = 0$
- $\operatorname{div} v = 0$ (in case of SDiff)

Let X_t^ε be defined by

$$\frac{d}{d\varepsilon} X_t^\varepsilon = v(X_t^\varepsilon, t)$$

$$X_t^\varepsilon\Big|_{\varepsilon=0} = X_t$$

Then, $\left.\dfrac{d}{d\varepsilon} X_t^\varepsilon\right|_{\varepsilon=0} = v(X_t, t)$ is a variation field.

We model our fluid as a continuum of particles with unit density, whose only salient feature is kinetic energy. By Hamilton's principle, the fluid equations arise as a description of the path that extremizes the kinetic energy among all admissible paths in the configuration space:

$$S[\{X_t^\varepsilon\}] = \frac{1}{2}\int_0^1\int_M |\dot{X}_t^\varepsilon(a)|^2 \, dV_a \, dt$$

$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} S[\{X_t^\varepsilon\}] = \int_0^1\int_M \dot{X}_t(a) \cdot \frac{d}{d\varepsilon}\dot{X}_t^\varepsilon(a)\Big|_{\varepsilon=0} dV_a \, dt$$

$$= -\int_0^1\int_M \ddot{X}_t(a) \cdot \frac{d}{d\varepsilon}X_t^\varepsilon(a)\Big|_{\varepsilon=0} dV_a \, dt$$

$$= -\int_0^1\int_M \ddot{X}_t(a) \cdot v(X_t(a), t) \, dV_a \, dt$$

$\det \nabla X_t^{-1}(x)$ 1 is sdiff

$$= -\int_0^1\int_M (\ddot{X}_t \circ X_t^{-1})(x) \cdot v(x,t) \, \cancel{J_t(x)} \, dV_x \, dt$$

Since v is arbitrary and div-free by Helmholtz-Hodge $\ddot{X}_t \circ X_t^{-1} = -\operatorname{grad} p$

If v is arbitrary, we get Burgers $\ddot{X}_t = 0$.

Note, introducing u via $\dot{X}_t = u(X_t, t)$, Euler reads

$$\partial_t u + u \cdot \nabla u = -\nabla p$$
$$\nabla \cdot u = 0$$

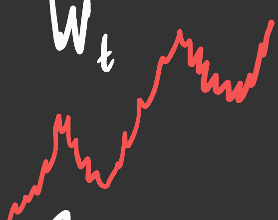
We now aim to give a similar variational picture for viscouse equations, such as Navier-Stokes

$$\partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u$$
$$\nabla \cdot u = 0$$

The additional term model molecular friction. As such, it should have a probabilistic origin.

Let's introduce Brownian Motion: (Wiener process)

W_t



Paths are continuous almost surely (just shy of $C_t^{1/2}$ regular), since

- $W_0 = 0$ a.s.
- W has independent increments, e.g.

  $W_{t+u} - W_t$ for any $u \geq 0$ is independent of past values W_s, s.t.
- Increments are Gaussian, e.g.

  $$W_{t+u} - W_t \sim N(0, u) = \sqrt{u}\, N(0,1)$$

  mean 0, variance u. $f_{W_t}(x) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}}$

$$\mathbb{E}\left[(W_{t+dt} - W_t)^2 \right] = dt, \quad \text{“ } dW_t = \sqrt{dt} \text{ ”}$$

Stochastic integration (Ito integral)

adapted to $\mathcal{F}_t$, i.e. causal, cannot see into future

$$\int_0^t g_{t'}\, dW_{t'} = \lim_{N\to\infty} \sum_{i=1}^{N} g_{t_{i-1}} \left(W_{t_i} - W_{t_{i-1}} \right)$$

converges in probability (weakly)

$$\lim_{N\to\infty} P(|X_N - X| > \varepsilon) = 0$$

Note that if

Conditional expectation given full knowledge of history $[0,s]$

$$h(t) := \int_0^t g_{t'}\, dW_{t'}, \quad \text{then} \quad \mathbb{E}\left[h(t) \mid \mathcal{F}_s \right] \quad \text{is}$$

$$\mathbb{E}\left[h(t) \mid \mathcal{F}_s \right] = \mathbb{E}\left[h(t) - h(s) \mid \mathcal{F}_s \right] + \mathbb{E}\left[h(s) \mid \mathcal{F}_s \right]$$

$$= 0 + h(s)$$

since increments are iid and mean zero

Thus, Ito integrals are Martingales, e.g. a fair game.

In particular, Ito integrals are mean zero:

$$\mathbb{E}\left[\int_0^t g_{t'}\, dW_{t'} \right] = 0, \quad \text{since } h(0) = 0$$

Itô's formula. Fix $f \in C^1_t C^2_x$.

Let $dX_t = u(X_t, t)\,dt + \sqrt{2\nu}\,dW_t$, $\quad X_0(a) = a$

Consider $f(X_t(a), t)$. Let us keep terms up to order dt:

$$d f(X_t, t) = \partial_t f\big|_{X_t} dt + dX_t \cdot \nabla f\big|_{X_t} + \frac{1}{2} \mathrm{Hess} f\big|_{X_t}(dX_t, dX_t) + O(dt^2)$$

$$\approx \left(\partial_t f + u \cdot \nabla f + \nu \Delta f\right)\big|_{X_t} dt + \sqrt{2\nu}\, dW_t \cdot \nabla f\big|_{X_t}$$

This is Itô's formula:

$$\boxed{df(X_t, t) = \left(\partial_t f + u \cdot \nabla f + \nu \Delta f\right)\big|_{X_t} dt + \sqrt{2\nu}\, dW_t \cdot \nabla f\big|_{X_t}}$$

If we instead use time reversed Brownian motion

$W_T = 0$, $\quad W_{T,t} \quad 0 \leq t \leq T \qquad X_{T,t}$

$$\boxed{df(X_{T,t}, t) = \left(\partial_t f + u \cdot \nabla f - \nu \Delta f\right)\big|_{X_{T,t}} dt + \sqrt{2\nu}\, dW_t \cdot \nabla f\big|_{X_{T,t}}}$$

REMARK: If $\partial_t T = \nu \Delta T$, then $u = 0$ and $X_{T,T}(x) = x$ gives

$$d\, T(X_{T,t}, t) = \sqrt{2\nu}\, dW_{T,t} \cdot \nabla T\big|_{X_{T,t}} \implies T(x,t) = \mathbb{E}\left[T_0(X_{T,0}(x))\right]$$

Feynman-Kac

Namely:

$$T(x,t) = \int_{\mathbb{R}^d} T_0(a)\, p(a,0; x,t)\, da = \int_{\mathbb{R}^d} T_0(a) \frac{1}{(4\pi\nu t)^{d/2}} \exp\left(\frac{-|x-a|^2}{4\nu t}\right) da$$

Brownian motion induces a measure on the space of continuous paths. One can approximate this by

$$P(W_{t_1}, \dots, W_{t_N}) = (\text{const}) \exp\left(-\frac{1}{2\Delta t} \sum_{i=1}^{N} |W_{t_n} - W_{t_{n-1}}|^2\right)$$

$$= (\text{const}) \prod_{n=1}^{N} \exp\left(-\frac{1}{2\Delta t} |W_{t_n} - W_{t_{n-1}}|^2\right)$$

This is the joint density function for Brownian motion at a discrete set of times.

$$P(W_{t_1}, \dots, W_{t_N})\, dW_{t_1} \cdots dW_{t_N} \longrightarrow \text{“}\mathcal{D}W(W)\text{”}$$

Wiener Measure

How does this measure transform under translation?

$$\tilde{W}_{t_i} = W_{t_i} + h_{t_i}, \qquad \Delta W_{t_i} = W_{t_i} - W_{t_{i-1}}$$

$$\exp\left(-\frac{1}{2\Delta t} |\Delta W_{t_i}|^2\right) = \exp\left(-\frac{1}{2\Delta t} |\Delta\tilde{W}_{t_i} - \Delta h_{t_i}|^2\right)$$

Gaussian

$$= \exp\left(-\frac{1}{2\Delta t}\left(|\Delta h_{t_i}|^2 - 2\,\Delta h_{t_i} \Delta W_{t_i}\right)\right) \exp\left(-\frac{1}{2\Delta t} |\Delta\tilde{W}_{t_i}|^2\right)$$

Thus

$$\prod_{i=1}^{N} \frac{1}{(2\pi\Delta t_i)^{d/2}} \exp\left(-\frac{|\Delta W_{t_i}|^2}{2\Delta t_i}\right) \xrightarrow{N\to\infty} \exp\left(\int_0^t h_s'\, dW_s - \frac{1}{2}\int_0^t |h_s'|^2 ds\right) \tilde{\mu}$$

$$\text{“}\tilde{\mu} = \mathcal{D}W(\tilde{W})\text{”}$$

For these computations to make sense, we require that $h \in H^1_t$. If

$$d\tilde{w}_t = u(\tilde{w}_t, t)\,dt + dW_t$$

then $h(t) = \int_0^t u(\tilde{w}_s, s)\,ds$, so $h'(t) = u(\tilde{w}_t, t)$

and $h \in H^1 \Rightarrow h' \in L^2$. This means we should have $u \in L^2$.

Thus we arrive formally at Girsanov Theorem.

If $dX_t = u(X_t, t)\,dt + \sqrt{2\nu}\,dW_t$, then

$$W_t = \frac{1}{2\nu}\left[X_t(a) - a - \int_0^t u(X_s(a), s)\,ds\right]$$

so

$$DW(W) = \exp\left(\frac{1}{2\nu}\left(\int_0^t u(X_s, s)\,dX_s - \frac{1}{2}\int_0^t |u(X_s, s)|^2 ds\right)\right) DW_\nu(X)$$

Wiener measure under which W_t is Brownian motion

Wiener measure under which X is a Brownian motion

A more standard way to write is

measure under which W_t is Brownian motion

$$\frac{d\mathbb{P}_t}{d\mathbb{Q}_t}(\omega) = \exp\left(\frac{1}{2\nu}\left(\int_0^t u(X_s, s)\, dX_s - \frac{1}{2}\int_0^t |u(X_s,s)|^2\, ds\right)\right)$$

measure under which X_t is Brownian motion.

We now introduce the concept of Relative entropy.

Let p and q be two distributions functions on $\mathbb{R}$.

$$K(p|q) := \int_{\mathbb{R}} p(x) \log\left(\frac{p(x)}{q(x)}\right) dx$$

Measures how different p and q are. A relative disorder.

Central to our study is

$$K(\mathbb{P}_t \| \mathbb{Q}_t) = \mathbb{E}_{\mathbb{Q}_t}\left[\ln\left(\frac{d\mathbb{P}_t}{d\mathbb{Q}_t}\right)\right] = \int \ln\left(\frac{d\mathbb{P}_t}{d\mathbb{Q}_t}\right) d\mathbb{Q}_t$$

Note if $\rho = \frac{d\mathbb{P}_t}{d\mathbb{Q}_t}$ is the "density", this is

$$K(\mathbb{P}_t \| \mathbb{Q}_t) = -\int \rho \log \rho \; DW(w)$$

the Shannon entropy formula.

Note that, using the Girsanov theorem, we have

$$K(\mathbb{P}_t \| \mathbb{Q}_t) = \mathbb{E}_{\mathbb{Q}}\left[\ln\left(\frac{d\mathbb{P}_t}{d\mathbb{Q}_t}\right)\right]$$

$$= \cancel{\frac{1}{2\nu}\mathbb{E}_{\mathbb{Q}}\left[\int_0^t u(X_s, s)\, dX_s\right]} \qquad 0$$

as this is an Ito integral under $\mathbb{Q}$.

$$- \frac{1}{2\nu}\mathbb{E}_{\mathbb{Q}}\left[\int_0^t |u(X_s, s)|^2 ds\right]$$

Thus we find

$$-2\nu K(\mathbb{P}_t \| \mathbb{Q}_t) = \frac{1}{2}\int_0^t \mathbb{E}_{\mathbb{Q}}\left[|u(X_s(a), s)|^2\right] ds$$

Averaging over all possible starting points,

$$\boxed{-2\nu \int_M K(\mathbb{P}_t \| \mathbb{Q}_t)\, da = \frac{1}{2}\int_0^t \int_M |u(x, s)|^2 dV_x\, ds}$$

Relative entropy and energy are equivalent!

max entropy $\Longleftrightarrow$ min energy

Action Principle for Navier Stokes

Note if

$$dX_t = u(X_t, t)\,dt + \sqrt{2\nu}\,dW_t$$
$$X_0(a) = a$$

then

$$u(X_t, t) = \lim_{h \to 0} \frac{\mathbb{E}[X_{t+h} \mid \mathcal{F}_t] - X_t}{h} \quad \text{a.s.}$$

For a general family of paths, define

$$DX_t := \lim_{h \to 0} \frac{\mathbb{E}[X_{t+h} \mid \mathcal{F}_t] - X_t}{h}$$

Fix two random diffeomorphisms X_0 and X_t. Let $\{X_t\}$ be a stochastic process connecting them

Consider the action:

$$S[\{X_t\}] = \frac{1}{2} \int_0^1 \int_M |DX_t|^2 \, dV_a \, dt$$

Variations are again made by

$$\frac{d}{d\varepsilon} X_t^\varepsilon = v(X_t^\varepsilon, t) \qquad \begin{matrix} v(x,0) = v(x,t) = 0 \\ \nabla \cdot v = 0 \end{matrix}$$

Now

$$0 = \frac{d}{d\varepsilon} S(\{X_t^\varepsilon\})\Big|_{\varepsilon=0} = -\int_0^1 \int_M DX_t \cdot \frac{d}{d\varepsilon} DX_t^\varepsilon \Big|_{\varepsilon=0} dV dt$$

$$= -\int_0^1 \int_M DX_t \cdot Dv(X_t, t) \, dV \, dt$$

Itô's formula

$$= -\int_0^1 \int_M u \circ X_t \cdot (\partial_t + \nabla_u + \nu \Delta) v \, dV \, dt$$

$$= \int_0^1 \int_M (\partial_t u + u \cdot \nabla u - \nu \Delta u) \cdot v \, dV dt$$

For this to hold for all solenoidal v, we must have

$$\partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u$$
$$\nabla \cdot u = 0$$

Navier-Stokes equations!