Approach 1

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particle labelling   incompressibility

$$S[u,\lambda,\rho,p] = \int_0^T \int_{\mathbb{T}^d} \left(\frac{1}{2}\rho|u|^2 + \mu\cdot(\partial_t a + u\cdot\nabla a) + p(1-\rho) + \lambda(\partial_t \rho + \operatorname{div}(u\rho)) \right) dx\, dt$$

mass density

$$u, \mu : \mathbb{T}^d \times \mathbb{R}^+ \to \mathbb{R}^d, \quad \lambda, p, \rho : \mathbb{T}^d \times \mathbb{R}^+ \to \mathbb{R}$$

$$\delta S[u,\lambda,\rho,p] = \int_0^T \int_{\mathbb{T}^d} (\rho u + \nabla\mu\cdot a - \rho\nabla\lambda)\cdot \delta u$$

$$+ \int_0^T \int_{\mathbb{T}^d} \left(-\frac{|u|^2}{2} - \partial_t\lambda - u\cdot\nabla\lambda + p\right)\delta\rho$$

$$+ \int_0^T \int_{\mathbb{T}^d} (1-\rho)\,\delta p$$

$$- \int_0^T \int_{\mathbb{T}^d} (\partial_t \mu + \operatorname{div}(u\mu))\cdot\delta a$$

$$+ \int_0^T \int_{\mathbb{T}^d} (\partial_t a + u\cdot\nabla a)\cdot\delta\mu$$

$$+ \int\int (\partial_t \rho + \operatorname{div}(u\rho))\cdot\delta\lambda$$

1) $\rho u + \nabla\mu \cdot a - \rho\nabla\lambda = 0$

2) $-\frac{|u|^2}{2} - \partial_t\lambda - u\cdot\nabla\lambda + P = 0$

3) $1 - \rho = 0$

4) $\partial_t \mu + \mathrm{div}(u\mu) = 0$

5) $\partial_t a + u\cdot\nabla a = 0$

6) $\partial_t \rho + \mathrm{div}(u\rho) = 0$

Consequences:

3 $\Longrightarrow$ $\rho = 1$

3 + 6 $\Longrightarrow$ $\mathrm{div}\, u = 0$

1) $u + \nabla\mu \cdot a - \nabla\lambda = 0$

2) $-\frac{|u|^2}{2} - \partial_t\lambda - u\cdot\nabla\lambda + P = 0$

4) $\partial_t\mu + u\cdot\nabla\mu = 0$

5) $\partial_t a + u\cdot\nabla a = 0$

NOTE: I will ignore total gradients

$$\partial_t (1) = \partial_t u + \partial_t(\nabla\mu\cdot a) - \nabla\partial_t\lambda$$

$$= \partial_t u + \overset{(a)}{\nabla\partial_t\mu\cdot a} + \overset{(b)}{\nabla\mu\cdot\partial_t a}$$

$$- \nabla\left(P - \frac{|u|^2}{2} - u\cdot\nabla\lambda\right)$$

$$(a) + (b) = -\nabla\mu\cdot(u\cdot\nabla a) - \nabla(u\cdot\nabla\mu)\cdot a$$

$$= -\nabla\mu\cdot(\nabla\lambda\cdot\nabla a) + \nabla\mu(\nabla\mu\cdot a\cdot\nabla a)$$

$$- \nabla(\nabla\lambda\cdot\nabla\mu)\cdot a + \nabla(\nabla\mu\cdot a\cdot\nabla\mu)\cdot a$$

We wish to compare this to

$$u\cdot\nabla u = (\nabla\lambda - \nabla\mu\cdot a)\cdot\nabla(\nabla\lambda - \nabla\mu\cdot a)$$

$$= \nabla\frac{|\nabla\lambda|^2}{2} - \nabla\otimes\nabla\lambda\cdot\nabla\mu\cdot a$$

$$- \nabla\lambda\cdot\nabla(\nabla\mu\cdot a) + \nabla\mu\cdot a\cdot\nabla(\nabla\mu\cdot a)$$

$$u\cdot\nabla u = \nabla\frac{|\nabla\lambda|^2}{2} - \nabla\otimes\nabla\lambda\cdot\nabla\mu\cdot a$$
$$- \nabla\lambda\cdot\nabla(\nabla\mu\cdot a) + \nabla\mu\cdot a\cdot\nabla(\nabla\mu\cdot a)$$
$$= \nabla\frac{|\nabla\lambda|^2}{2} - \nabla\otimes\nabla\lambda\overset{⑤}{\cdot}\nabla\mu\cdot a$$
$$- \nabla\lambda\cdot\overset{③}{\nabla\otimes\nabla\mu}\cdot a - \nabla\mu\cdot(\nabla\lambda\cdot\nabla a)$$
$$+ \nabla\mu\cdot a\cdot\overset{④}{\nabla\otimes\nabla\mu}\cdot a + \nabla\mu(\nabla\mu\cdot a\cdot\nabla a)$$

$$③ = \partial_i\lambda\,\partial_i\partial_k\mu_\ell a_\ell$$
$$= \partial_k(\partial_i\lambda\,\partial_i\mu_\ell a_\ell) - \partial_i\partial_k\lambda\,\partial_i\mu\,a_\ell$$
$$- \partial_i\lambda\,\partial_i\mu_\ell\,\partial_k a_\ell$$

$$④ = \partial_i\mu_j a_j\,\partial_i\partial_k\mu_\ell a_\ell$$
$$= \partial_i\mu_j a_j\,\partial_k(\partial_i\mu_\ell a_\ell) \quad " \nabla\frac{|\nabla\mu\cdot a|^2}{2}$$
$$- \partial_i\mu_j a_j\,\partial_i\mu_\ell\,\partial_k a_\ell$$

$$⑤ + ③ + ④ = \nabla q - \partial_i\lambda\,\partial_i\mu_\ell\,\partial_k a_\ell$$
$$- \partial_i\mu_j a_j\,\partial_i\mu_\ell\,\partial_k a_\ell$$

5t

3 + 4 $= \nabla q - \partial_i \lambda \, \partial_i \mu_\ell \, \partial_k a_\ell$

$- \partial_i \mu_j \, a_j \, \partial_i \mu_\ell \, \partial_k a_\ell$

should be compared to (from pg 3)

$- \nabla(\nabla\lambda \cdot \nabla\mu) \cdot a + \nabla(\nabla\mu \cdot a \cdot \nabla\mu) \cdot a$

$= \partial_k(\partial_i \lambda \, \partial_i \mu_\ell) a_\ell + \partial_k(\partial_i \mu_j \, a_j \, \partial_i \mu_\ell) a_\ell$

$= \nabla \tilde{q} - \partial_i \lambda \, \partial_i \mu_\ell \, \partial_k a_\ell - \partial_i \mu_j \, a_j \, \partial_i \mu_\ell \, \partial_k a_\ell$

Thus

some scalar

$u \cdot \nabla u = \nabla q - \nabla(\nabla\lambda \cdot \nabla\mu) \cdot a + \nabla(\nabla\mu \cdot a \cdot \nabla\mu) \cdot a$

$- \nabla\mu \cdot (\nabla\lambda \cdot \nabla a) + \nabla\mu(\nabla\mu \cdot a \cdot \nabla a)$

Compare this with a + b on $\boxed{\text{pg } 3.}$ - have

a + b $= u \cdot \nabla u + \nabla \tilde{q}$

some scalar.

Combining, we obtain Euler equations:

$$\partial_t u + u \cdot \nabla u = \nabla p$$
$$\nabla \cdot u = 0$$

Note that, this u has the structure

$$(*) \quad u = \nabla \lambda - \nabla \mu \cdot a$$

where
$$\partial_t \mu + u \cdot \nabla \mu = 0$$
$$\partial_t a + u \cdot \nabla a = 0$$

REMARK: If μ and a are scalars, then $(*)$ is the Clebsch representation.

Clebsch does not represent all divergence-free fields since they have zero helicity:

$$\omega = \operatorname{curl} u = -\nabla \mu \times \nabla a \Rightarrow \boxed{\omega \cdot u = 0.}$$

But if μ and a are vectors, as above, then $(*)$ represents a general incompressible field?

Claim 1: Given a pair of M-uples, $M \geq 1$ integer,

$$\mu = (\mu_1, \mu_2, \dots, \mu_M)$$

$$a = (a_1, a_2, \dots, a_M)$$

such that the following hold:

$$\partial_t \mu + u \cdot \nabla \mu = 0,$$

$$\partial_t a + u \cdot \nabla a = 0.$$

If u is given by the formula

$$u := \nabla \mu \cdot a + \nabla \lambda$$

with some scalar function λ, then u solves Euler

$$\partial_t u + u \cdot \nabla u = -\nabla p$$

where

$$p := (\partial_t + u \cdot \nabla) \lambda + \frac{1}{2} |u|^2$$

Note, demanding $\nabla \cdot u = 0$ fixes p as $-\Delta p = \operatorname{div}(u \cdot \nabla u)$.
This gives the equation for λ by combining with above.

Proof of Claim 1:

We use the notation

$$D_t := \partial_t + u \cdot \nabla$$

We require the commutator identity: let f be scalar,

$$D_t \nabla f = \nabla D_t f - \nabla u \cdot \nabla f.$$

$\partial_i u_j \partial_j f$

We now differentiate the expression

$$u := \nabla \mu \cdot a - \nabla \lambda .$$

Computation:

$$D_t u = D_t(\nabla \mu \cdot a) + D_t(\nabla \lambda)$$

$$= D_t \nabla \mu \cdot a + \nabla \mu \cdot D_t a + D_t \nabla \lambda$$

using $D_t \mu = D_t a = 0$

$$= -(\nabla u \cdot \nabla \mu) \cdot a + \nabla u \cdot \nabla \lambda - \nabla D_t \lambda$$

$$= -\nabla u \cdot u - \nabla D_t \lambda$$

$$= -\nabla \left(D_t \lambda + \tfrac{1}{2} |u|^2 \right).$$

This completes the proof.

Claim 2: Let u be a smooth solution of Euler

$$\partial_t u + u \cdot \nabla u = -\nabla p$$
$$\nabla \cdot u = 0.$$

Let trajectories be defined

$$\frac{d}{dt} X_t(x) = u(X_t(x), t)$$
$$X_0(x) = x$$

and the 'back-to-labels' map $A_t := X_t^{-1}$. Then

$$u := \nabla \mu \cdot a - \nabla \lambda.$$

where

$$\mu := A_t(x)$$
$$a := u_0(A_t(x))$$
$$\lambda := \tilde{\lambda}_t(A_t(x)),$$
$$\tilde{\lambda} := \int_0^t \left(p(X_s(x), t) - \frac{1}{2}|u(X_s(x), t)|^2 \right) ds$$

Proof of Claim 2:

Note that, by definition

$$A_t(X_t(x)) = x.$$

Differentiating in time and using chain rule, we find

$$\partial_t A(X_t(x)) + \dot{X}_t(x) \cdot \nabla A_t(X_t(x)) = 0$$

using the equation $\dot{X}_t = u(X_t)$, together with the fact that X_t is a diffeomorphism from the fluid domain to itself, we deduce

$$D_t \mu = D_t A_t = 0.$$

$$D_t a = D_t u_0(A_t(x)) = 0.$$

Finally, note that from the definition of

$$D_t \lambda := \partial_t \tilde{\lambda}(A_t(x)) + \cancelto{0}{D_t A_t \nabla \tilde{\lambda}}$$

$$= p - \tfrac{1}{2}|u|^2.$$

Claim 2 follows by applying Claim 1. $\blacksquare$