Lecture 8: Inviscid Burgers equation

Equation arises as geodesic on Diff(M):

$$A(X) = \frac{1}{2}\int_{t_1}^{t_2}\int_M |\dot{X}_t(a)|^2 \, da \, dt$$

$$\delta A(X) = 0 \iff \ddot{X}_t = 0$$

The equation for $M \subseteq \mathbb{R}^n$ (without boundary) is:

Initial value problem
$$\begin{cases} \ddot{X}_t(a) = 0 \\ \dot{X}_0(a) = u_0(a) \\ X_0(a) = a \in M \end{cases}$$

is called the Burgers equation. If M is flat, then trajectories X_t themselves sweep out geodesics on M. Provided X is a diffeomorphism;

Eulerian form:

$$\dot{X}_t(a) = u(X_t(a), t)$$

$$\partial_t u + u \cdot \nabla u = 0$$

$$u\big|_{t=0} = u_0$$

$u : M \times \mathbb{R} \to \mathbb{R}^d$

Two point problem for Burgers (remark)

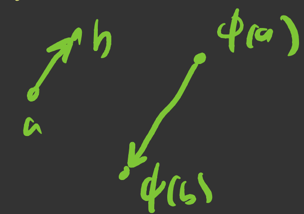
Fix $\varphi \in \mathrm{Diff}(M)$. Burgers geodesic between φ and id:

$$\text{Two-point problem} \begin{cases} \ddot{X}_t(a) = 0 \\ X_0(a) = a \\ X_1(a) = \phi(a) \end{cases} \iff X_t(a) = (1-t)a + t\,\Phi(a)$$

Suppose $\exists\, t_* > 0$ s.t. $X_t(a) = X_t(b)$ for $a \neq b$. This happens iff

$$a - b = \frac{-t_*}{1-t_*}\left(\Phi(a) - \Phi(b)\right).$$

requires anti alignment



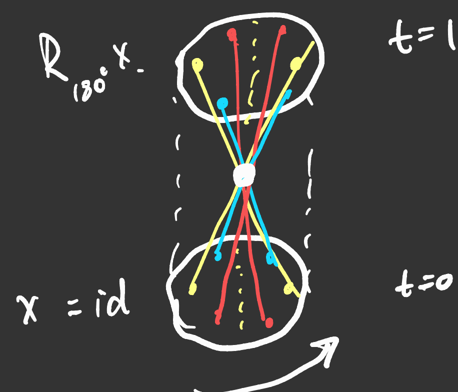
note $f(t) := \frac{-t}{1-t} \in (-\infty, 0)$

Remark, in 1d, say $a > b$, then $\phi(a) > \phi(b)$

Thus $\frac{a-b}{\phi(a) - \varphi(b)} > 0$ and so there cannot be collisions.

In 1d, the two point problem is always solvable

Remark in 2D: solid body has collisions once a half turn is made.



Singularity formation v.s. Global wellposedness

$$\ddot{X}_t(a) = 0 \implies X_t(a) = a + t u_0(a)$$

Introducing $\dot{X}_t(a) = u(X_t(a), t)$, we see

$$\frac{d}{dt} u(X_t(a), t) = 0 \implies u(X_t(a), t) = u_0(a)$$

Combining, we have

$$u(a + t u_0(a), t) = u_0(a)$$

Differentiating in the label "a",

$$\nabla_a \left[u(a + t u_0(a), t) \right] = \nabla_a u_0(a)$$

Thus

$$\left(I + t \nabla_a u_0(a) \right) \cdot [\nabla_x u](a + t u_0(a), t) = \nabla_a u_0(a)$$

which implies

$$\boxed{\nabla_x u(a + t u_0(a), t) = \left(I + t \nabla_a u_0(a) \right)^{-1} \nabla_a u_0(a)}$$

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<u>Remarks:</u> Let $A_0 = \nabla_a u_0(a)$, $A(t) = \nabla_x u(a + t u_0(a), t)$.

1. If A_0 has no real eigenvalues (excepting possibly 0) then the solution to Burgers exists for all time $t \in \mathbb{R}$.

   Impossible if $d=1$!

2. $A(t)$ is determined for all time $t > 0$ (and $t<0$) if A_0 has no negative (corresp. no positive) eigenvalues.

   Compressing in one or other time direction.

3. Incompressible motions!

   Such are called asymptotic geodesics, since they are geodesic both on the constrained space $D_\mu(M)$ as well as $D(M)$

   ↑ submanifold of $D(M)$.

Asymptotic geodesics

Recall $X_t(a) = a + t u_0(a)$

$$\nabla X_t(a) = I + t A_0 \qquad (A_0 = \nabla_a u_0(a))$$

If $X_t \in D_\mu(M)$, then $\det \nabla X_t(a) = 1$ for all $t \in \mathbb{R}$

In this case, we must have

$$\det(I + t A_0) = 1 \qquad \text{for all } t \in \mathbb{R}$$

Lemma: let $N \in M^{n,n}(\mathbb{R})$. Then

$$\det(I + tN) = 1 \quad \forall t \in \mathbb{R} \iff N \text{ Nilpotent}$$

Proof: Assuming $\det(I + tN) = 1$ we have:

$$0 = \frac{d}{dt} \det(I + tN) = \operatorname{tr}\left((I + tN)^{-1} N \right) \quad \forall t.$$

Evaluating at $t = 0$, we find $\operatorname{tr} N = 0$. Differentiating again:

$$0 = \frac{d}{dt} \operatorname{tr}\left((I + tN)^{-1} N \right)\Big|_{t=0} = \operatorname{tr}\left((I - tN)^{-1} N^2 \right)\Big|_{t=0} = \operatorname{tr}(N^2)$$

Continuing, $\operatorname{tr}(N^k) = 0$ for any $k \in \mathbb{N}$.

(we need only $k = 1, \dots, n$)

Now, suppose for contradiction, that N is not nilpotent. * If all eig $=0$, then nilpotent

Then N has some non-zero eigenvalues $\lambda_1, \dots, \lambda_r$.

Let m_i be the multiplicity of $\lambda_i \in \mathbb{C}$. Then

$$\begin{cases} m_1\lambda_1 + \dots + m_r\lambda_r = 0 \\ \vdots \\ m_1\lambda_1^r + \dots + m_r\lambda_r^r = 0 \end{cases}$$

* Schur decomposition:
$A \in \mathbb{C}^{n \times n}$, then $A = UTU^{-1}$,
T upper triangular, diagonal are eigenvalues.
Suppose that $\lambda = 0$. Then
$T^n = 0$. Thus $A^n = UT^nU^{-1} = 0$

Thus

$$(*) \quad \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \dots & \lambda_r \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \dots & \lambda_r^2 \\ \vdots & & & & \\ \lambda_1^r & \lambda_2^r & \lambda_3^r & \dots & \lambda_r^r \end{pmatrix} \begin{pmatrix} m_1 \\ \vdots \\ m_r \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

But this is a Vandermonde matrix, so $\prod_{0 \le i < j \le r} (\lambda_i - \lambda_j)$

$$\det \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \dots & \lambda_r \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \dots & \lambda_r^2 \\ \vdots & & & & \\ \lambda_1^r & \lambda_2^r & \lambda_3^r & \dots & \lambda_r^r \end{pmatrix} = \lambda_1 \cdots \lambda_r \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_r \\ \vdots & & & \\ \lambda_1^{r-1} & \dots & \dots & \lambda_r^{r-1} \end{pmatrix}$$

$\neq 0$ since $\lambda_i \neq \lambda_j$ for $i \neq j$

Thus $(*)$ has unique solution: $m_i = 0$. Contradiction.

For the other direction, suppose that N is nilpotent. Then $N^k = 0$ for some $k \in \mathbb{N}$.

We claim

$$\det(I + N) = 1$$

Suppose, for contradiction, that $\det(I + N) \neq 1$.
Then $I + N$ must have a non-unity eigenvalue

$$\lambda u = (I + N)u = u + Nu \qquad u \neq 0.$$

Thus $$Nu = (\lambda - 1)u$$
Note that $(\lambda - 1)u \neq 0$. Thus, by induction:

$$N^k u = (\lambda - 1)^k u \qquad \text{for all } k \in \mathbb{N}.$$

This contradicts N being nilpotent. Thus all $\lambda = 1$.

Note, since $I + N$ has all eigenvalues one bigger than N, it follows that all eigenvalues of N are zero.

Return to <u>Asymptotic geodesics</u>

Recall
$$X_t(a) = a + t u_0(a)$$
$$\nabla X_t(a) = I + t A_0 \qquad (A_0 = \nabla_a u_0(a))$$

we must have

$$\det(I + t A_0) = 1 \qquad \text{for all } t \in \mathbb{R}$$

This holds <u>iff</u> A_0 is Nilpotent. In which case

$$(I + t A_0)^{-1} = \sum_{m=0}^{k} (-A_0)^m \qquad (A_0^k = 0)$$
$$= I - A_0 + A_0^2 - A_0^3 + \dots + (-A_0)^k$$

Using,

$$\boxed{\nabla_x u(a + t u_0(a), t) = (I + t \nabla_a u_0(a))^{-1} \nabla_a u_0(a)}$$

We see

$$(\nabla_x u)(X_t(a), t) = \sum_{m=0}^{d-2} (-t)^m (\nabla u_0(a))^{m+1}$$

In dim d, growth like
$\|\nabla u\|_{L^\infty} \sim t^{d-2}$.

Examples of Nilpotent data

$d=2$

$$u_0(a_1,a_2) = \begin{pmatrix} v(a_2) \\ 0 \end{pmatrix}$$

$$\nabla_a u_0(a) = \begin{pmatrix} 0 & v'(a_2) \\ 0 & 0 \end{pmatrix}$$

$d=3$

$$u_0(a_1,a_2,a_3) = \begin{pmatrix} u_1(a_2,a_3) \\ u_2(a_3) \\ 0 \end{pmatrix}$$

$$\nabla u_0(a) = \begin{pmatrix} 0 & \partial_2 u_1 & \partial_3 u_1 \\ 0 & 0 & \partial_3 u_2 \\ 0 & 0 & 0 \end{pmatrix}$$

In this case

$$\begin{aligned} \dot{X}_1(a) &= u_1(X_2, X_3) \\ \dot{X}_2(a) &= u_2(X_3) \\ \dot{X}_3(a) &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} X_1(a) &= a_1 + u_1(a_2 + t u_2(a_3), a_3) \\ X_2(a) &= a_2 + t u_2(a_3) \\ X_3(a) &= a_3 \end{aligned}$$

$$\ddot{X}_1(a) = \dot{X}_2 \partial_2 u_1 + \dot{X}_3 \partial_3 u_1 = u_2(X_3) \cdot \nabla_2 u_1(X_2, X_3)$$

$\partial_t u_1''(x_2, x_3)$

Questions:

① In dimension $d=2$, classify all pressureless solutions.

Are they all steady? If steady, are they all shear flows? At what regularity?

② One can construct solutions that blow-up in finite time, in infinite space dim.

Take data

$$u_0(a) = \begin{pmatrix} u_1(a_2) \\ u_2(a_3) \\ u_3(a_4) \\ \vdots \\ u_{d-1}(a_d) \\ 0 \end{pmatrix}$$

Note: $\|\nabla u_0\|_{L^\infty} := \sup_x \max_{1\le i\le d} \sum_{j=1}^{d} |(\nabla u_0)_{ij}| < \infty$ ind of d.

On other hand

$$\|\nabla u(0,t)\|_{L^\infty} = \sum_{i=0}^{d} t^i = \frac{1-t^{d-1}}{1-t}$$

as $d\to\infty$, this blows up at time 1!

Singularity formation

Let's focus on $d=2$ and $M=\mathbb{R}$. We have

$$u_x(X_t(a),t) = \frac{u_0'(a)}{1+u_0'(a)}.$$

Thus the first singularity emerges from the label a_* at which $u_0'(a_*)$ is most negative.

The time of blowup is explicit

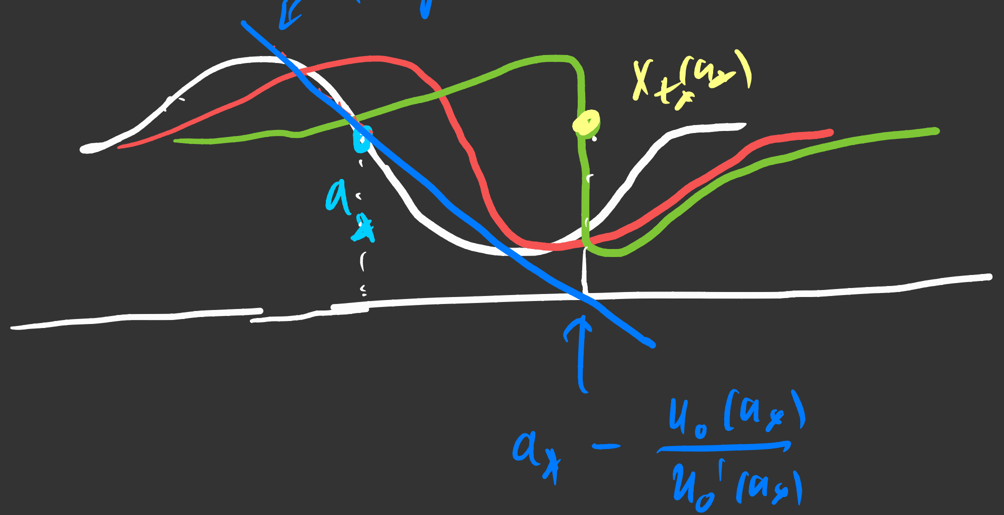
$$t_* = \frac{1}{-u_0'(a_*)}$$

the location is also

$$X_{t_*}(a_*) = a_* + t_* u_0(a_*).$$
$$= a_* - \frac{u_0(a_*)}{u_0'(a_*)}$$

Remark: this formula looks like Newton's method!

tangent line : slope $= u_0'(a_*)$



Note that for $|a-a_*| \ll 1$, we have

$$u_0'(a) = u_0'(a_*)$$ (<0, by assumption)

$$+ \cancel{u_0''(a_*)(a-a_*)} \to 0$$ since a_* is minimum of $u_0'(a_*)$

$$+ C(a-a_*)^2$$ ($C > 0$, since minimum)

$$+ \mathcal{O}((a-a_*)^3)$$ (assuming $u_0 \in C^4$)

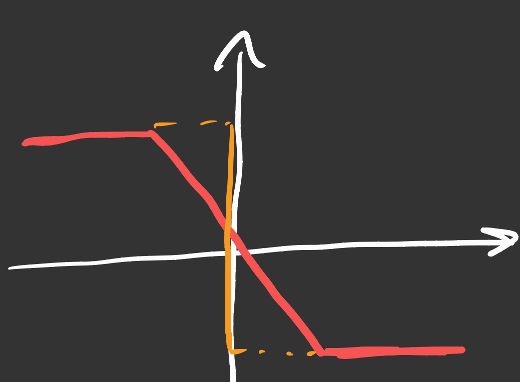
Thus, at time t_*,

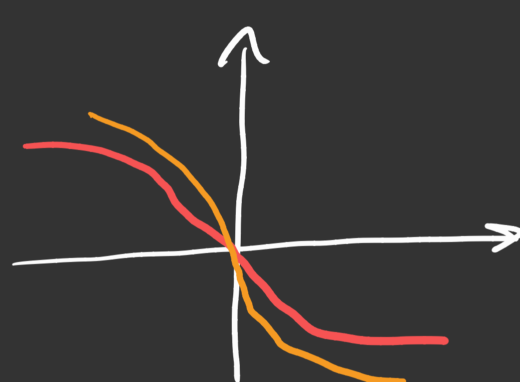
$u_x(X_{t_*}(a), t_*)$ is finite $\forall\, a \neq a_*$.

First singularity occurs at a single point in spacetime

<u>Define:</u> A data is a global minimum and generic if

- $u_0 \in C^4$
- $u_0'(a_*) < 0$
- $u_0''(a_*) = 0$
- $u_0'''(a_*) > 0$





What regularity should we expect?

Consider data

$$u_0(a) = -a + a^n \quad , \qquad n \geq 3, \quad n \text{ odd}$$

Note that $a_* = 0$ is global minimum

$$u_0'(0) = -1 < 0$$

$$u_0''(0) = 0$$

$$u_0'''(0) = n \cdot (n-1)(n-2) \, a^{n-3}$$

$$= \begin{cases} 6 & \text{if } n = 3 \quad \text{generic} \\ 0 & \text{if } n > 3 \quad \text{non-generic} \end{cases}$$

Time of blowup is

$$t_* = -\frac{1}{u_0'(a_*)} = 1$$

The flowmap is then is

$$X_{t_*}(a) = a + u_0(a) = a^n$$

Recalling that

$$u(X_t(a), t) = u_0(a)$$

at time $t_* = 1$, we have

$$u(a^n, 1) = u_0(a)$$

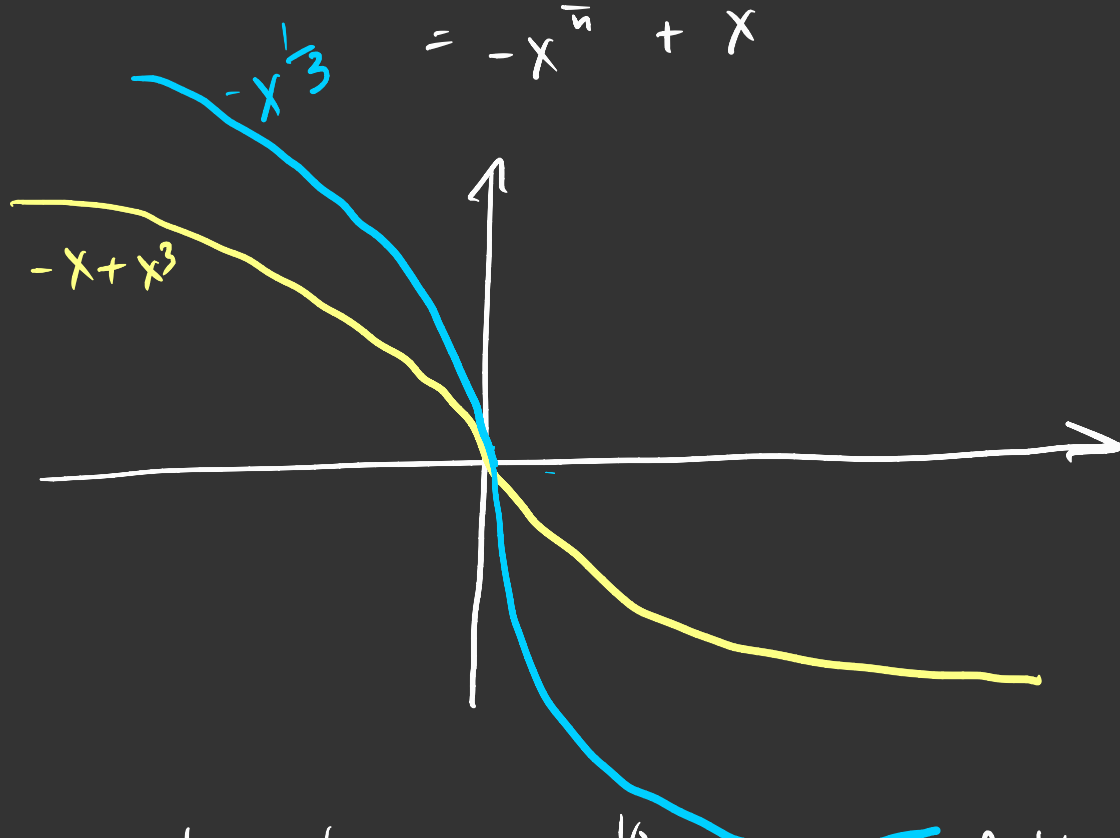
Inverting the homeomorphism a^n gives

$$X_{t_*}^{-1}(x) = x^{\frac{1}{n}}.$$

Thus

$$u(x,1) = u_0(x^{\frac{1}{n}})$$

$$= -x^{\frac{1}{n}} + x$$



Thus, we expect that $u(\cdot, t_*) \in C^{1/3}$ at time of blowup!

Self-similarity

Note that if

$$X_t(a) = a + t\,u_0(a)$$

then $y = A_t(x)$, the inverse, solves

$$x = y + t\,u_0(y).$$

For the special data: $x = (1-t)\,y + t\,y^3$

Subsitute $z = \left(\frac{t}{1-t}\right)^{1/2} y.$ Then

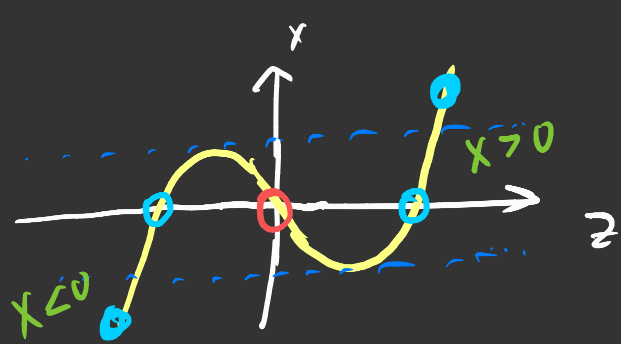
$$t y^3 = \frac{(1-t)^{3/2}}{\sqrt{t}} z^3 \qquad (1-t)\,y = \frac{(1-t)^{3/2}}{\sqrt{t}} z$$

$$z^3 + z = \frac{\sqrt{t}}{(1-t)^{3/2}}\,x \equiv X$$

Thus

$$z = F(X),$$ F explicit

$F'(0) = -1$



In fact $z^3 + z = X$ is solved by

$$z = F(x) = \left[\frac{x}{2} + \left(\frac{1}{27} + \frac{x^2}{4} \right)^{1/2} \right]^{1/3} - \left[-\frac{x}{2} + \left(\frac{1}{27} + \frac{x^2}{4} \right)^{1/2} \right]^{1/3}$$

Thus

$$\begin{aligned} u(x,t) &= u_0\left(A_t(x)\right) \\ &= -A_t(x) + A_t(x)^3 \\ &= -\frac{(1-t)^{1/2}}{\sqrt{t}} F\left(\frac{\sqrt{t}\, x}{(1-t)^{3/2}} \right) + \frac{(1-t)^{3/2}}{t^{3/2}} F^3\left(\frac{\sqrt{t}\, x}{(1-t)^{3/2}} \right) \end{aligned}$$

and

$$u_x(x,t) = \underbrace{\frac{1}{1-t} F'\left(\frac{\sqrt{t}\, x}{(1-t)^{3/2}} \right)}_{\text{blowing up at } t=1 \text{ in a self-similar way}} + \underbrace{3(F'F)\left(\frac{\sqrt{t}\, x}{(1-t)^{3/2}} \right)}_{\text{bounded as } t \to 1.}$$