Geometric and Dynamical aspects of fluid motion

Outline: Lecture 1 : The Euler equations

Lecture 2 : Long time dynamics of two-dimensional inviscid fluids

Lecture 3 : Transition to turbulence and a problem of Kolmogorov

Lecture 4 : Phenomenology and Mathematics of three-dimensional turbulence

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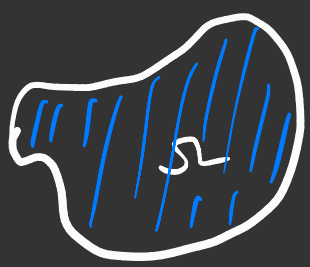
Dynamics of a two dimensional ideal fluid.

Ideal fluid in 2D; $u = (u_1, u_2)$



$$\partial_t u + u \cdot \nabla u = -\nabla p \quad \text{in } \Omega$$
$$\nabla \cdot u = 0 \quad \text{in } \Omega$$
$$u \cdot \hat{n} = 0 \quad \text{on } \partial\Omega$$

Every solenoidal 2D velocity can be represented by

$\{\psi = 1\}$

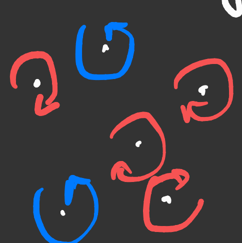
$$u = \nabla^\perp \psi \qquad \nabla^\perp = \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix}$$

streamfunction

Thus, u is pointwise parallel to isolines of ψ.

$$u \cdot \hat{n} = 0 \text{ on } \partial\Omega \iff \psi = \text{const on } \partial\Omega$$

Vorticity has one component $\vec{\omega} = \omega \hat{z}$ with



$$\omega = \nabla^\perp \cdot u = \partial_1 u_2 - \partial_2 u_1$$

in terms of streamfunction $\omega = \Delta \psi$

$$\boxed{\begin{aligned} \partial_t \omega + u \cdot \nabla \omega &= 0 \\ u &= \nabla^\perp \Delta_D^{-1} \omega \end{aligned}} \iff \boxed{\begin{aligned} \omega(t) &= \omega_0 (X_t^{-1}) \\ \dot{X}_t &= u(X_t) \end{aligned}}$$

Invariants of the motion:

Energy: $\frac{d}{dt}\int_\Omega \frac{1}{2}|u|^2\,dx = 0$ ← uses $u\cdot\hat{n} = 0$ on $\partial\Omega$, $\nabla\cdot u = 0$ in Ω

Casimirs: $\frac{d}{dt}\int_\Omega f(\omega)\,dx = 0 \quad \forall$ continuous $f:\mathbb{R}\to\mathbb{R}$

in particular $\|\omega(t)\|_{L^p} = \|\omega_0\|_{L^p} \quad \forall\, p\in[1,\infty]$.

Momentum: $\frac{d}{dt}\int_\Omega u\,dx = 0$ ← on $\Omega = \mathbb{T}^2, \mathbb{T}\times[0,1], D$

These facts allow one to define a dynamics

THEOREM: Let $\Omega\subseteq\mathbb{R}^2$ and $\omega_0\in C^\alpha(\Omega)$. Then there exists a unique solution $\omega\in C^\alpha((-\infty,\infty)\times\Omega)$ with

$$\|\omega(t)\|_{C^\alpha} \leq \left(\frac{\|\omega_0\|_{C^\alpha}}{\|\omega_0\|_{L^\infty}}\right)^{\exp(c\|\omega_0\|_{L^\infty}t/\alpha)}$$

Gunter, 1926
Lichtenstein, 1925

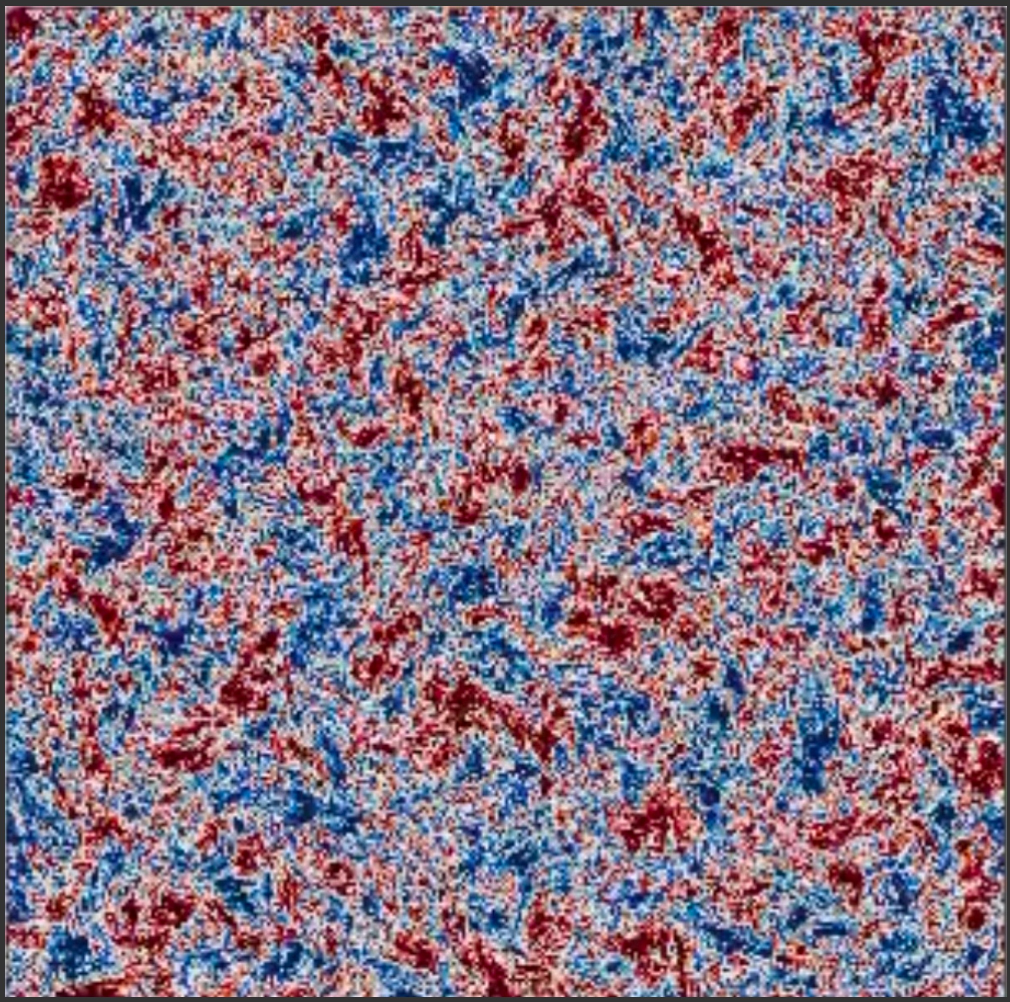
In fact (Yudovich, 1963), if $\omega_0\in L^\infty(\Omega)$ then the exists a unique weak solution $\omega\in L^\infty((-\infty,\infty)\times\Omega)$.

Moreover the solution depends continuously on the data; i.e.

$$\omega_0^n \xrightarrow{*} \omega_0 \quad \text{then} \quad \omega(t;\omega_0^n) \xrightarrow{*} \omega(t;\omega_0).$$

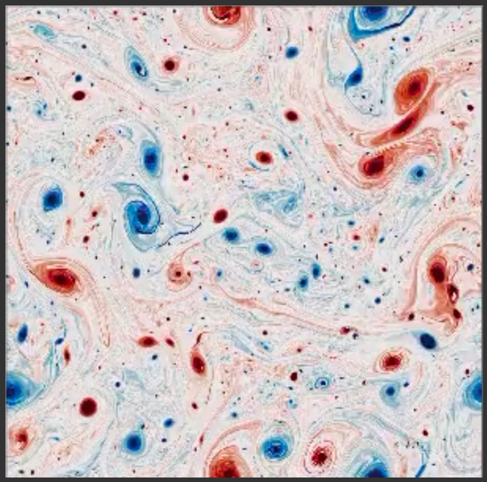
Yudovich space is very important for longtime behavior!

Start with a random, smooth, vorticity distribution on $\mathbb{T}^2$.



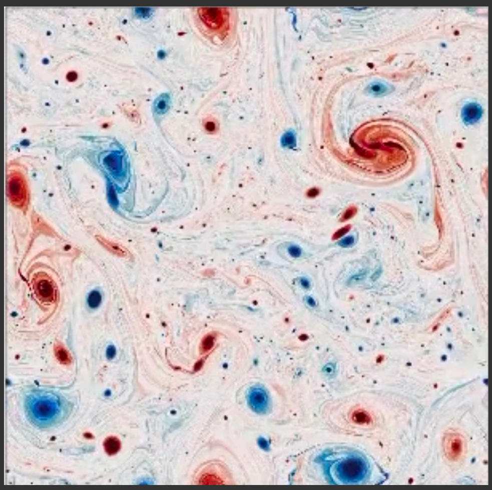
What happens Next?

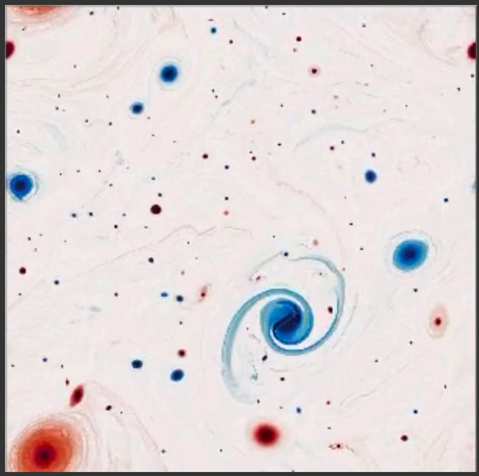
Features of long time



Aggregation of like signed vortices

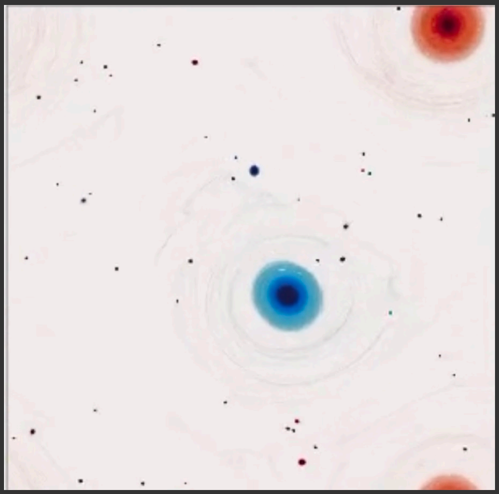
(inverse energy cascade)





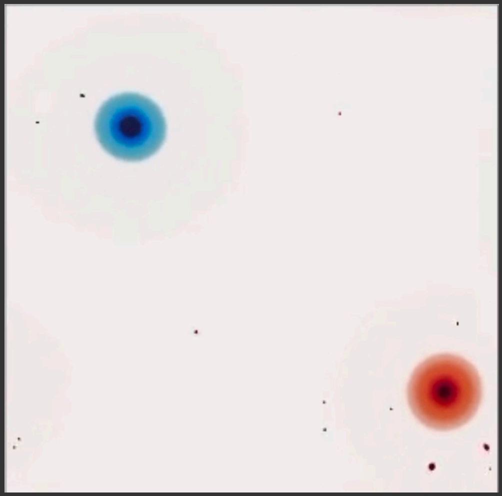
Merger

(strongly nonlinear process)



Mixing of weak vorticity

(inviscid damping)



Weak convergence to some prefered state

often looking like a (locally) steady state

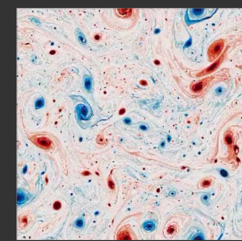
Why should like signed vorticies aggregate? ⑤

$\Omega : \mathbb{R}^+ \to \mathbb{R}$

compact support,

$\int_0^\infty \Omega(r)\, r\, dr = \frac{\Gamma}{2\pi}$

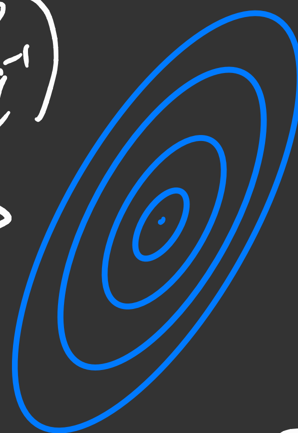
$\nabla \Phi$ if $u = \sigma(x, -y)$.



$A = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$

$\det A = 1$

$q = e^{\sigma t}$



$\omega(x) = \Omega(|x|)$

$\omega(x) = \Omega(|Ax|)$

$$\psi(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln|x - x'|\, \omega(x')\, dx'$$

$$E = -\frac{1}{2} \int_{\mathbb{R}^2} \psi(x)\, \omega(x)\, dx$$

renormalized energy, since $u \notin L^2$ for cmpt supp vorticity

$$= -\frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x - x'|\, \omega(x')\, \omega(x)\, dx'\, dx$$

$$\Delta E = E_{\text{ellipse}} - E_{\text{disk}}$$

$$= \frac{1}{4\pi} \iint_{\mathbb{R}^2 \mathbb{R}^2} \ln|x - x'| \Big(\Omega(|x|)\Omega(|x'|) - \Omega(|Ax|)\Omega(|Ax'|) \Big) dx\, dx'$$

$$= -\frac{1}{4\pi} \iint_{\mathbb{R}^2 \mathbb{R}^2} \ln\left(\frac{|A^{-1}(x - x')|}{|x - x'|} \right) \Omega(|x|)\Omega(|x'|)\, dx\, dx'$$

$$= -\frac{\Gamma^2}{4\pi} \ln\left(\frac{q^{-1} + q}{2} \right)$$

Stretching makes energy go down!

Now, say you have two radial vortices, far apart

The velocity of each vortex looks

$$u_i(x) = \Gamma_i \frac{(x-p_i)^\perp}{|x-p_i|^2} \qquad i=1,2$$

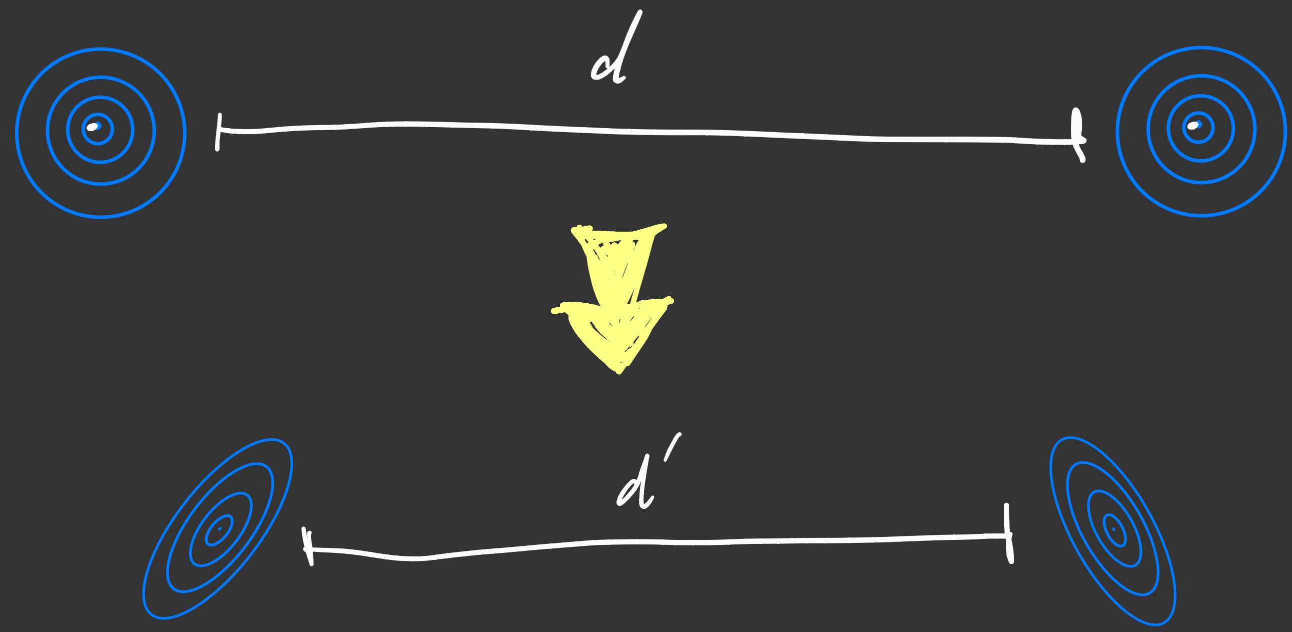
This velocity, evaluated at the other vortex, is

$$u_1(p_2+\varepsilon x) = \Gamma_1 \frac{(p_2-p_1+\varepsilon x)^\perp}{\|p_2-p_1+\varepsilon x\|^2}$$

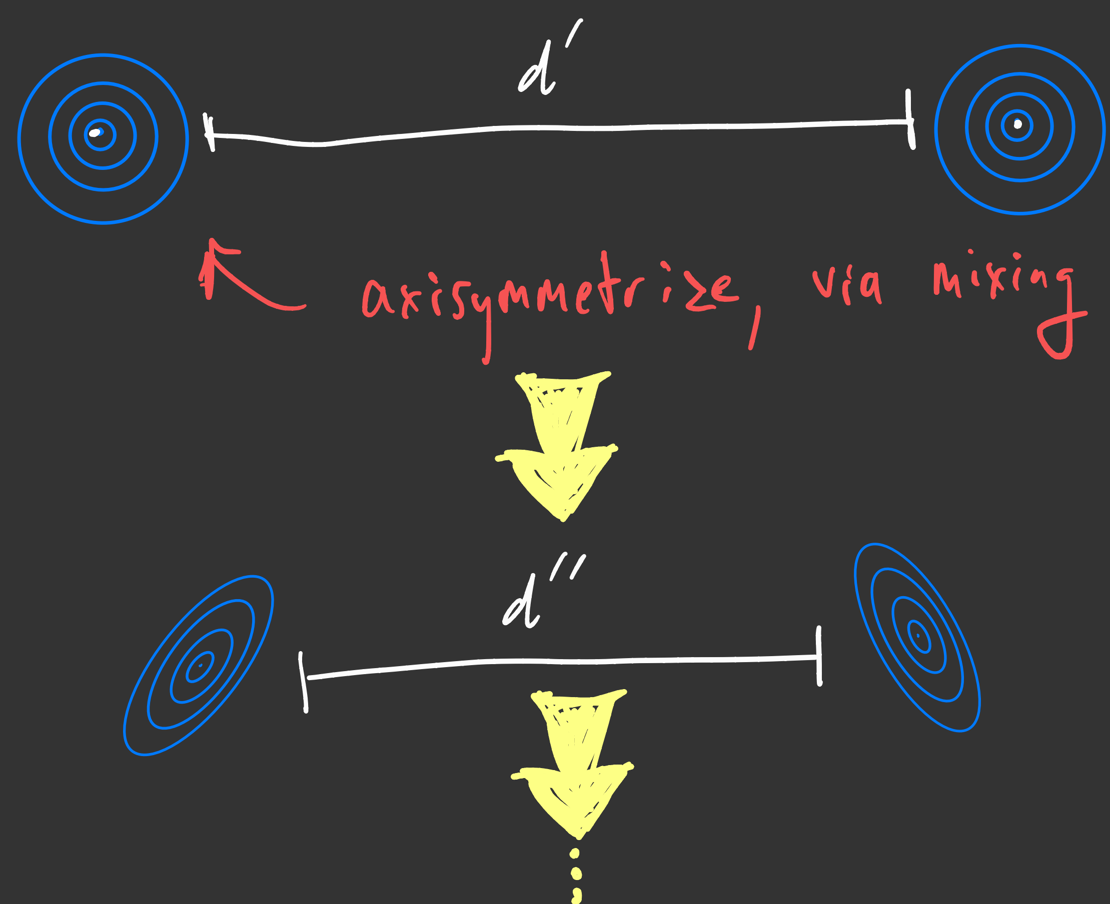
$$= \frac{\Gamma_1}{d^2}(p_2-p_1)^\perp + \varepsilon \Gamma_1 A_{p_1,p_2} x + \mathcal{O}(\varepsilon^2)$$

$$\frac{x^\perp}{d^2} - 2\frac{(p_2-p_1)^\perp}{d^2}(p_2-p_1)\cdot x = \frac{1}{d^2}\left(J - 2(p_2-p_1)^\perp \otimes (p_2-p_1)\right)x$$

$$= Ax$$



The distance d' must be smaller than d, since the individual energies of each vortex goes down. Thus, to keep net energy constant, the two vortices must get closer.

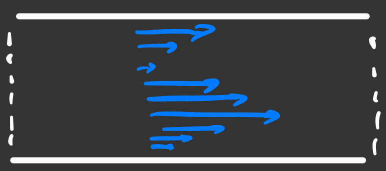


Stationary States, Structure and (in)stability

defined by $u \cdot \nabla \omega = 0$ i.e. $\nabla \omega$ and $\nabla \psi$ are colinear

$u = \nabla^\perp \psi$

Here are some special families



$u(x,y) = v(y)\, e_x$ Shear flows

$u(r,\theta) = v(r)\, e_\theta$ Circular flows

Another large subclass of stationary states are

$$\omega = F(\psi) \qquad F \in Lip$$

Then, the streamfunction is determined by elliptic problem

$$\Delta \psi = F(\psi) \quad \text{in } \Omega$$
$$\psi = 0 \quad \text{on } \partial\Omega$$

This equation can have, none, one or many solutions depending on F. (consider, e.g, $F(x) = \lambda x$)?

$$\boxed{\text{QUESTION: Which are stable, and what shape do they take?}}$$

Variational description of steady states

V.I. Arnold gave two variational characterizations of steady states. They are critical points of either

co-adjoint orbit

a) $E'_{\omega_0}[X] = 0 \iff X_* \omega_0$ is a steady state vorticity

where

$$E_{\omega_0}[X] = \frac{1}{2}\int_M |K[X_* \omega_0]|^2 \, dx$$

for $X \in D_\mu(M)$. Here $K[\omega] = -\Delta^{-1} \operatorname{curl} \omega$.

Note, from our previous computations, we expect energy maximizers to have concentrated vorticity, and minimizers to have dispersed.

adjoint orbit

b) $E'_{u_0}[X] = 0 \iff X_* u_0$ is a steady state velocity

$$E_{u_0}[X] = \frac{1}{2}\int_M |X_* u_0|^2 \, dx.$$

First variations : $\delta E_{\omega}[X]=0$

(a) $$0 = E'_{\omega_0}[X]\xi = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E_{\omega_0}[X^{\varepsilon}]$$

$\omega = X_* \omega_0$

$u = K[\omega]$

$$= \int_M u \cdot K\left[\{\eta,\omega\}\right] dx$$

$\frac{d}{d\varepsilon} X^{\varepsilon}\Big|_{\varepsilon=0} = \xi$

$= \nabla^{\perp}\eta$

$$= \int_M \psi \,\{\eta,\omega\}\, dx$$

$$= -\int_M \{\psi,\omega\}\,\eta\, dx = 0$$

Holds for all $\eta \Rightarrow \{\psi,\omega\}=0$ (steady state).

(b) $$E'_{u_0}[X]\xi = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{1}{2}\int_M |X^{\varepsilon}_* u|^2 dx = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{1}{2}\int_M |\nabla(\psi\circ X^{\varepsilon})|^2$$

$$= \int_M \nabla\psi \cdot \nabla\{\eta,\psi\} dx = \int_M \{\omega,\psi\}\,\eta\, dx.$$

Second Variations

a)
$$E''_{w_0}[x](\xi,\xi) = \int_M u\cdot K[\{\eta,\{h,\omega\}\}]\,dx \qquad \begin{matrix}\xi=\nabla^\perp\eta\\ \xi=\nabla^\perp h\end{matrix}$$

$$+\int_M K[\{\eta,\omega\}]\,K[\{h,\omega\}]\,dx$$

Same as Hessian at a critical point

$$=\int_M \left[\Delta^{-1/2}\{\eta,\omega\}\,\Delta^{-1/2}\{h,\omega\} + \{\eta,\psi\}\{h,\omega\}\right]dx$$

If $\psi = G(\omega)$, then

$$E''_{w_0}[x](\xi,\xi) = \int_M \left[\left|\Delta^{-1/2}\{\eta,\omega\}\right|^2 + G'(\omega)\left|\{\eta,\omega\}\right|^2\right]dx$$

If $G=F^{-1}$, $G' = \frac{1}{F'(F^{-1})}$, then $G'>0 \Leftrightarrow F'>0$.

- If $G'>0$ then $E''_{w_0}>0$, so energy minimizes
- If $G'<-\frac{1}{\lambda_1}$, $E''_{w_0}<0$ so energy maximizes.

$(-\lambda_1 < F' < 0)$

Rigidity and Symmetry

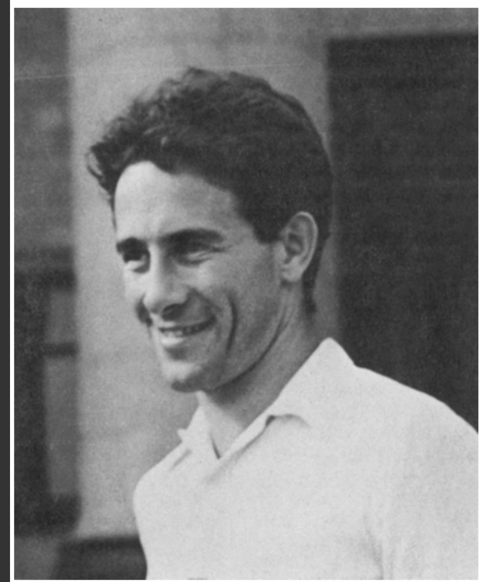
Arnol'd Stable steady States

$$\Delta \Psi_0 = F_0(\Psi_0)$$

where F_0 satisfies

$$-\lambda_1 < F_0'(\Psi) < 0 \quad \text{or} \quad 0 < F_0'(\Psi) < \infty$$

Then, $\omega_0 = \Delta \Psi_0$ is orbitally stable in L^2 under the Euler dynamics.



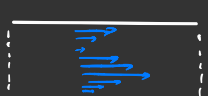
Vladimir I. Arnold, 1961
Photograph by Jürgen Moser

Such flows correspond to local maxima or minima of Energy on isovortical sheets, i.e.

$$\mathcal{O}_{\omega_0} := \{\omega : \exists\ \phi \in \mathrm{Diff}_\mu(\Omega) \text{ s.t. } \omega = \omega_0 \circ \phi\}$$

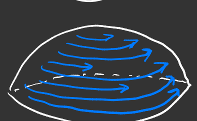
THEOREM: Let (M, g) be a compact two-dimensional Riemannian manifold with smooth bdry ∂M. Let ξ be a killing field for g tangent to the boundary.

If $u \in C^2(M)$ is Arnold stable, then $\mathcal{L}_\xi u = 0$.



- if M is periodic channel, u is shear

- if M is disk or annulus, u is radial



- if M is spherical cap, u is zonal

$\mathbb{T}^2$ 

- if M has no boundary, $\nexists\ u$

Other Results: Hamel-Nadirasvili, Gomez-Serrano, Park, Shi, Yao

Energetic assumptions imply symmetry.

First, lets understand how energy changes under deformation of the velocity field.

Return to our example  $u = \sigma(x, -y)$

$$\psi_A(x) \to \psi(|Ax|) \quad \det A = 1$$

$$E(\psi_A) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla \psi(|Ax|)|^2 \, dx$$

$$= \frac{1}{2}\int_{\mathbb{R}^2} |\psi'(|Ax|)|^2 \left|\frac{A^2 x}{|Ax|}\right|^2 dx$$

$A = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$

$$= \frac{1}{2}\int_{\mathbb{R}^2} |\psi'(|z|)|^2 \, \frac{|Az|^2}{|z|^2} \, dz$$

$$\frac{q^2 z_1^2 + q^{-2} z_2^2}{z_1^2 + z_2^2}$$

$$= -\left(\frac{q^{-2} + q^2}{2}\right) \int_{\mathbb{R}} |\psi'(\rho)|^2 \, d\rho$$

Thus, under volume preserving deformations of the velocity, stretching increases energy. Minimal energy states should be concentrated.

In fact...

Theorem (Arnold) Let M = disk. Consider any smooth velocity u_0 with one non-degenerate zero. Then $\exists!$ global minimizer u_* on $\mathcal{O}_{u_0}$, and u_* is circular.

$$E_{u_0}[X] = \frac{1}{2}\int_M |\nabla(\psi_0 \circ X)|^2 \, dx$$

$$= \frac{1}{2}\int_{\operatorname{rang}\psi_0} dc \oint_{\{\psi_0\circ X = c\}} |\nabla(\psi_0\circ X)| \, dl$$

$$\geq \frac{1}{2}\int_{\operatorname{rang}\psi_0} dc \, \frac{\left(\operatorname{length}(\{\psi_0\circ X = c\})\right)^2}{\mu(c)}$$

isoperimetric inequality!

$$\geq \frac{1}{2}\int_{\operatorname{rang}\psi_*} \frac{\left(\operatorname{length}\{\psi_* = c\}\right)^2}{\mu(c)} .$$

$\mu(c)^{1/2}$, sqrt of travel time

where we used

$$\operatorname{length}(\{\psi_0 = c\}) = \oint_{\{\psi_0 = c\}} dl \leq \left(\oint_{\{\psi_0\circ X = c\}} |\nabla(\psi_0\circ X)| \, dl\right)^{1/2} \left(\oint_{\{\psi_0\circ X = c\}} \frac{dl}{|\nabla(\psi_0\circ X)|}\right)^{1/2} \leq \mu(c)^{1/2}\left(\oint_{\{\psi_0\circ X = c\}} |\nabla(\psi_0\circ X)| \, dl\right)^{1/2}$$

$$\frac{d}{dc} A(\{\psi_0 X \leq c\}) = \frac{d}{dc}\int_0^c dc' \oint_{\{\psi_0 X = c'\}} \frac{dl}{|\nabla\psi_0 X|} = \mu(c) .$$

Non-stagnant steady flows on channel are shear

(Nadirashvili - Hamel). Assume $u \in C^2(M)$, $u \neq 0$.

Fix $v \in S^1$ and let $\theta(x)$ be the angle (Farina, 2003)

$$\theta_v(x) = \arccos\left(\frac{u(x)\cdot v}{|u(x)|}\right)\operatorname{sgn}\left(u(x)\cdot v^\perp\right)$$

with convension $\arccos(1)=0$. Compute

$$\arccos'(z) = \frac{-1}{\sqrt{1-z^2}}$$

$$\operatorname{sgn}'(z) = 2\delta_0(z)$$

$$(u\cdot v^\perp)^2 = |u|^2|v|^2 - (u\cdot v)^2$$

$$\nabla|u| = \frac{\nabla u\cdot u}{|u|}$$

$$\nabla\theta_v(x) = \frac{-1}{\sqrt{1-\frac{(u\cdot v)^2}{|u|^2}}}\,\nabla\left(\frac{u(x)\cdot v}{|u(x)|}\right)\operatorname{sgn}(u\cdot v^\perp)$$

$$= \frac{|u|}{u\cdot v^\perp}\left[\frac{\nabla u\cdot v}{|u|} - \frac{u\cdot v}{|u|^2}\frac{\nabla u\cdot u}{|u|}\right]$$

$$= \frac{|u|}{u\cdot v^\perp}\left[\cancel{\frac{\nabla u\cdot u}{|u|}\frac{(u\cdot v)}{|u|^2}} + \frac{\nabla u\cdot u^\perp}{|u|}\frac{(u^\perp\cdot v)}{|u|^2} - \cancel{\frac{u\cdot v}{|u|^2}\frac{\nabla u\cdot u}{|u|}}\right]$$

$$= -\frac{\nabla u\cdot u^\perp}{|u|^2}$$

Thus:

$$\operatorname{div}(|u|^2\nabla\theta_v) = -\operatorname{div}(\nabla u\cdot u^\perp) = -\Delta u\cdot u^\perp$$

$$= \Delta\nabla^\perp\psi\cdot\nabla\psi = \{\omega,\psi\} \; !$$

$$\{f,g\} \overset{?}{=} \nabla^\perp f\cdot\nabla g$$

Thus we arrive at the beautiful formula:

$$\boxed{\operatorname{div}\left(|u|^2 \nabla \theta_v\right) = \{\omega, \psi\}}$$

As such, if u is a steady state, we find

$$\operatorname{div}\left(|u|^2 \nabla \theta_v\right) = 0$$

Suppose our domain is the channel $M = [0,1] \times \mathbb{T}$.
Let $v = e_1$. Then $\frac{u \cdot e_1}{|u|} = 1$ at $y = 0, 1$. As such
$\theta|_{y=0,1} = 0$ and we have

$$0 = \int_M \theta \operatorname{div}\left(|u|^2 \nabla \theta\right) dx = -\int_M |u|^2 |\nabla \theta|^2 dx$$

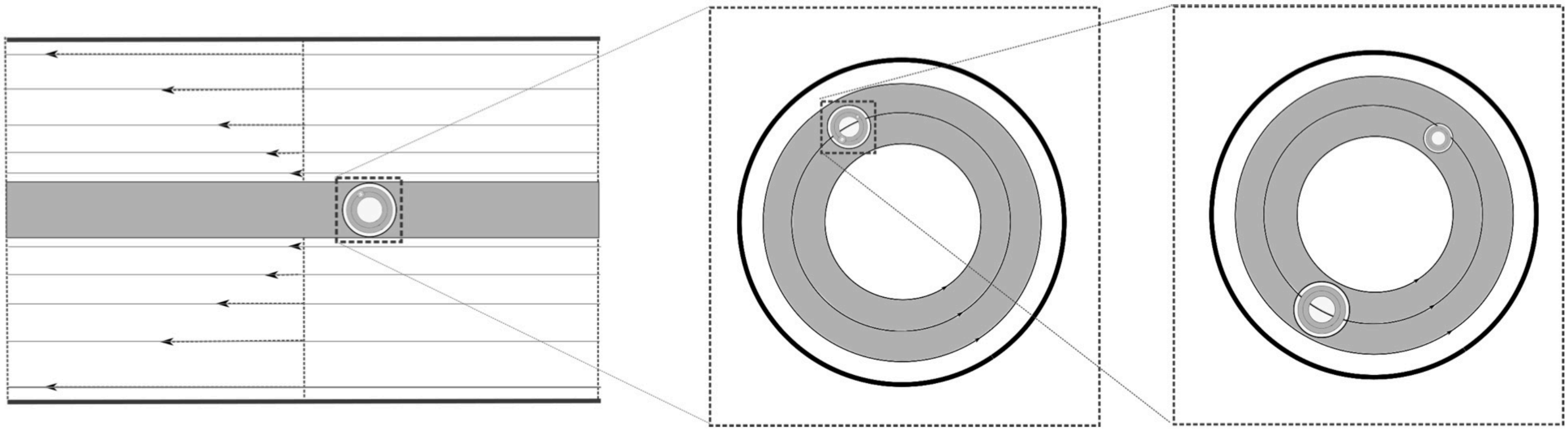
$$\Longrightarrow \quad |\nabla \theta|^2 = 0 \quad \Longrightarrow \quad u \parallel e_1 .$$

Any non-stagnant velocity is shear!

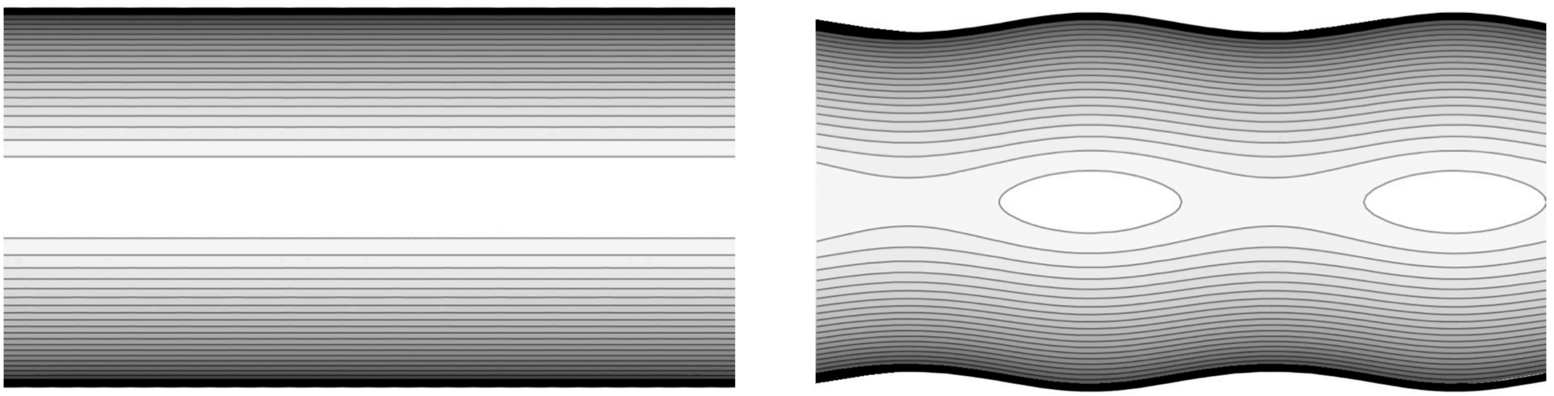
Generalization:
Laminar steady states are shear (D.- Nualart)

Requires only all streamlines be non-contractible loops.

If one drops the conditions of laminar, you can have many non-shear steady & dynamical states. See e.g. (D- Nualart):



Also, on perturbed domains, islands are always present in a wide class of steady states:



(D- Ginsberg)

Steady states come in infinite dimensional families

THEOREM: (Choffrut–Šverák, '10) Let Ω be an annular domain and consider a non-degenerate Arnol'd stable steady state. Then each vorticity distribution function in its neighborhood corresponds to a unique stationary state.

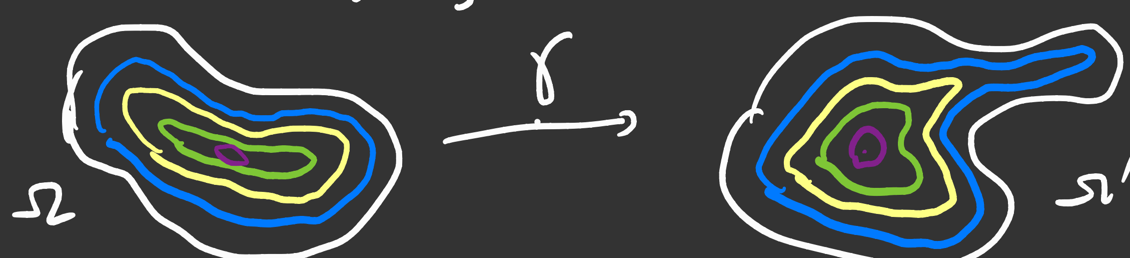
$$\mathcal{O}_{\omega_0} = \{\omega : \omega = \omega_0 \circ \phi, \ \phi \in \mathrm{Diff}_\mu(A)\}$$

THEOREM: (Constantin–D–Ginsberg, '20) Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with smooth boundary and let u_0 be an Arnol'd stable steady state on Ω with

- u_0 has a single stagnation point in Ω
- $\mu(c) = \oint_{\{\psi_0 = c\}} \frac{d\ell}{|\nabla \psi_0|} < \infty$

Then, there exists $\varepsilon = \varepsilon(u_0, \Omega)$ such that for all nearby domains $|\Omega' - \Omega| \leq \varepsilon$ and all functions $|\rho - 1| \leq \varepsilon$ there exists a diffeomorphism $\gamma : \Omega \to \Omega'$ s.t.

$$\det \nabla \gamma = \rho \qquad \int_\Omega \rho = \int_{\Omega'}$$



and $\psi = \psi_0 \circ \gamma^{-1}$ defines Euler solution on Ω' nearby to u_t.

$$\mathcal{O}_{\psi_0} = \{\psi : \psi = \psi_0 \circ \phi, \ \phi \in \mathrm{Diff}_\mu(\Omega)\}$$

Some isovortical leaves have no steady states

In fact, for any $\omega_0 \in L^\infty$ and any $\varepsilon > 0$,

$\exists \quad \omega_1 \in B_\varepsilon^{L^\infty}(\omega_0)$ such that

$$\mathcal{O}_{\omega_1} = \{ \omega : \omega = \omega_1 \circ \varphi \quad \varphi \in \mathcal{D}_\mu \}.$$

contains no steady state!

This follows from a very nice work: Ginzburg–Khesin.

Conditions

- ω has 1 max, 1 min, 1 saddle.
- $\omega > 0$.

contradiction

$u = 0 = \partial_n \psi$

$\Delta\psi > 0$

ω would have to take same value on $\{f = c\}$, which is multiply connected, unless the figure ∞ is a zero set of $\nabla\psi$.

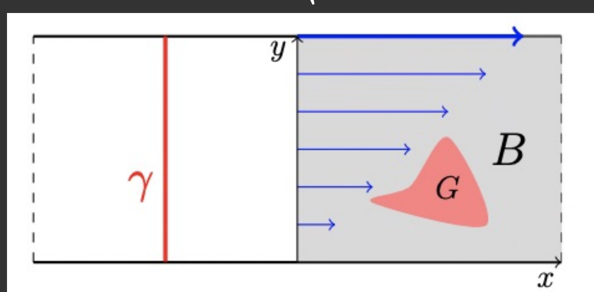
$$\omega = F(\psi) \qquad 0 < \omega = \Delta\psi$$

Wandering and infinite time blowup

THEOREM (Nadirashvili, 91). Let Ω be the periodic channel. There exists a vorticity $\xi \in L^\infty$ and numbers $\varepsilon > 0$, $T > 0$ such that for any $\omega \in L^\infty$ with $t > T$

$$|\omega - \xi|_{L^\infty} \leq \varepsilon \quad \text{while} \quad \| S_t(\omega) - \xi \|_{L^\infty} > \varepsilon$$

Let v be an Arnold stable steady state with the property that $v|_{top} \neq v|_{bott.}$ For simplicity take Couette $v = \binom{y}{0}$



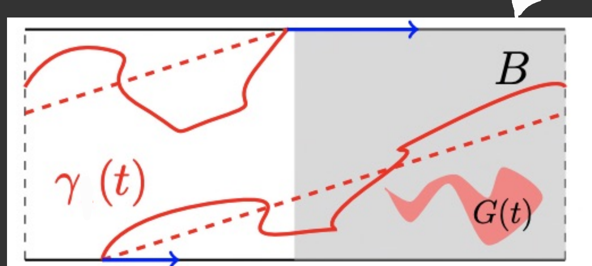
Take h s.t. $h \in C_0^\infty$

$$h|_\ell \geq 1/2 \quad \& \quad |h|_{L^\infty} \leq 1.$$

Set $\xi = -1 + \delta h$ for $\delta \ll 1$. Take $\varepsilon \ll 1$ and $g \in L^\infty$ with $|g|_{L^\infty} \leq 1$ and set $\omega_0 = \xi + \varepsilon g$

$u_t|_{top} > 3/4$

$u_t|_{bott} < 1/4$



$\gamma(t) := X_t(\ell)$

Thus $\text{length}(\gamma(t)) \gtrsim t/2$. Thus, $\ell(t) \cap \text{supp}(h)^c \neq \emptyset$, $\forall\, t > 2\pi$

Similar construction shows that $|u(t)|_{C^{1,\alpha}} \gtrsim t^\alpha$.

Infinite time blowup! Kiselev–Šverák, Denisov, Zlatoš, Elgindi–Jeong.

Instability in strong norms

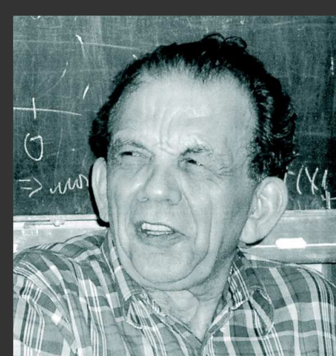
THEOREM: (Koch, 02; Morgulis-Shnirelman-Yudovich, 08)
Every stationary solution $\bar{w} \in C^{1,\alpha}$ of 2D Euler whose Lagrangian flowmap is not time-periodic (isochronal) is nonlinearly unstable in C^α. Specifically, $\forall M, \varepsilon > 0$ there exists $T = T(M,\varepsilon)$ and a solution $w(t) = S_t(w_0)$ s.t.

$$\|w_0 - \bar{w}\|_{C^\alpha} \leq \varepsilon \quad \text{while} \quad \|w(T) - \bar{w}\|_{C^\alpha} \geq M.$$

To prove this result, one exploits shearing. Denote

$$\mu(t) = \|\nabla X_t^{-1}\|_\infty, \qquad \dot{X}_t = v(X_t)$$

LEMMA 1: If $\mu(t) \leq C$ for all $t > 0$, then v is isochronal.



Example: (Elliptical vortex)

$$v = \nabla^\perp \psi, \qquad \psi = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$$

Remark: characterized in neighborhood of the disk.

LEMMA 2: There exist $w \in C(0,T; C^\alpha)$ such that

$$\|w_0 - \bar{w}\|_{C^\alpha} \leq \varepsilon \quad \text{while} \quad \|w(T) - \bar{w}\|_{C^\alpha} \geq C\,\mu(T)^\alpha\,\varepsilon$$

Detour on Isochronal Flows

$w = F(\psi)$

Conj: (Yudovich): The only constant vorticity isochronal flows are elliptical ($F \to$ domain).

Conj: (D-Elgindi): $\forall$ simple connected domains, $\exists$ a unique isochronal flow. (domain $\to F$)

Theorem (D.-Elgindi): For slight deformations D' of a disk domain D, $\exists$ a unique isochronal flow.

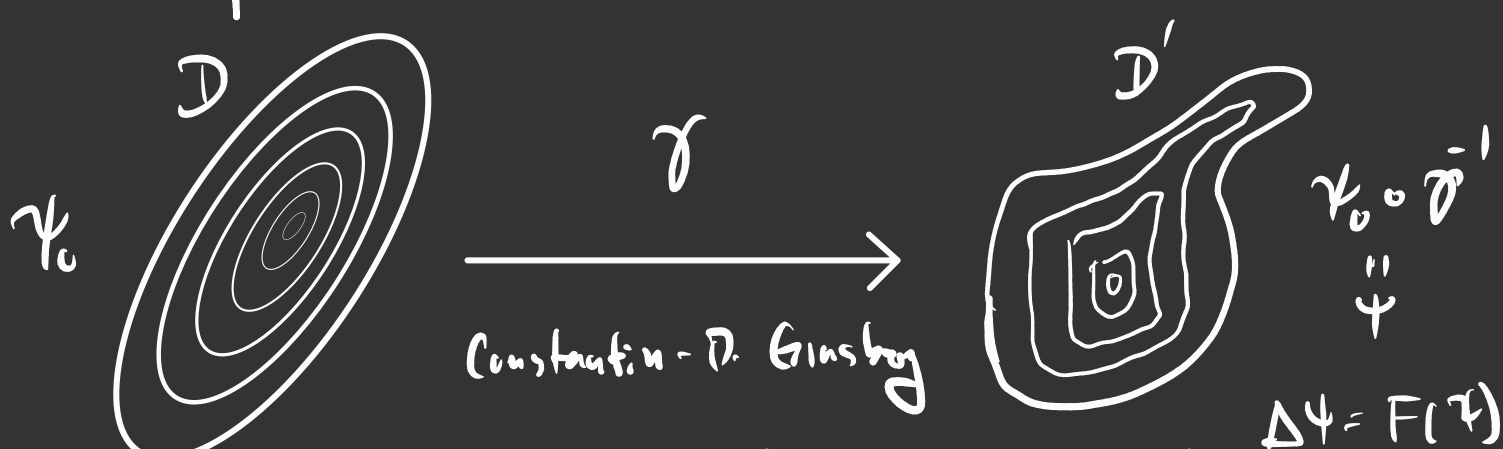
Proof: CoArea formula:

$$\mu(c) = \oint_{\{\psi = c\}} \frac{dl}{|\nabla\psi|} = \frac{d}{dc} A(\{\psi \le c\})$$

Let ψ_0 be isochronal (e.g. elliptical). Then all $\psi \in \mathcal{O}_{\psi_0}$ have equal enclosed area

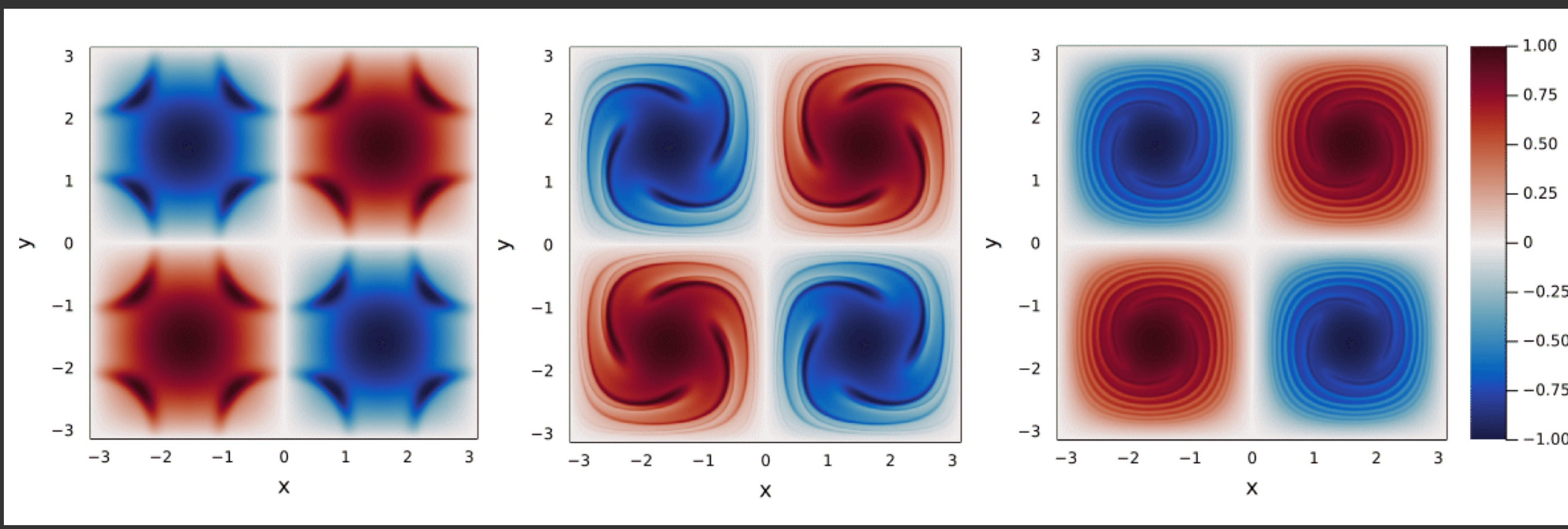
$$A(\{\psi \le c\}) = A(\{\psi_0 \le c\}) = \mu_0 c$$

Since elliptical isochronal flow is stable, $\Delta\psi = \text{const}$,

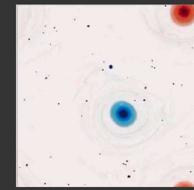


Near ellipses, all steady states are unstable!

Conjecture 1 (Yudovich (1974), [34, 35], quoted from [23]). *There is a 'substantial set' of inviscid incompressible flows whose vorticity gradients grow without bound. At least this set is dense enough to provide the loss of smoothness for some arbitrarily small disturbance of every steady flow.*



Stability of twisting.



(D-Elgindi-Jeong)

Theorem: Let ψ_* be non-isochronal. There exists $\varepsilon = \varepsilon(\psi_0) > 0$ s.t. for all $u = \nabla^\perp \psi$ satisfying

$$\frac{1}{T}\int_0^T \|\psi - \psi_0\|_{L^1}\, ds < \varepsilon \quad \& \quad \frac{1}{T}\int_0^T \|\nabla^\perp \psi_0 \cdot \nabla \psi\|_{L^1}\, ds < \varepsilon$$

the corresponding flowmap $\dot{\Phi}_t = u(\Phi_t, t)$ satisfies

$$\|\nabla \Phi_t\|_{L^1} \geq c|t|$$

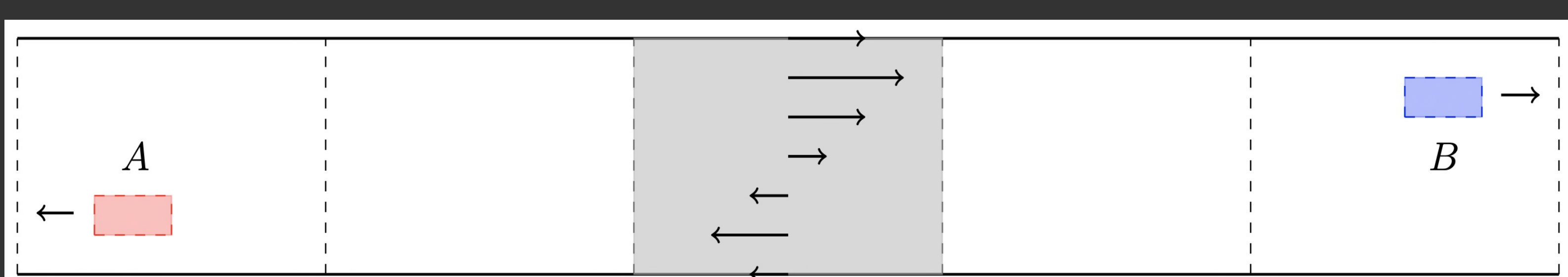
<u>Remark:</u> $u^\varepsilon = (\sin(y), \varepsilon)$ defines a time periodic flow!

<u>Proof:</u> Aim to show

$$\frac{d}{dt}\int \Phi_1(x,t)\, F_1(\Phi_2(x,t)) > V_*(y_1) - \varepsilon f(t)$$

$$\frac{d}{dt}\int \Phi_1(x,t)\, F_2(\Phi_2(x,t)) < V_*(y_2) - \varepsilon g(t)$$

where f, g satisfy $\frac{1}{T}\int_0^T f \leq 1$, $\frac{1}{T}\int_0^T g \leq 1$.



Conjecture 1 (Yudovich (1974), [34, 35], quoted from [23]). *There is a 'substantial set' of inviscid incompressible flows whose vorticity gradients grow without bound. At least this set is dense enough to provide the loss of smoothness for some arbitrarily small disturbance of every steady flow.*

THEOREM (D-Elgindi-Jeong): Let ω_* be a L^2 non-isochronal stable steady state. Then $\exists\ \varepsilon > 0$ s.t.

$$\left\{ \omega_0 \in B_\varepsilon(\omega_*) \ : \ \text{s.t.} \ \sup_t \frac{\|\omega(t)\|_{C^\alpha}}{t^{\alpha-}} = +\infty \right\}$$

contains a dense in $B_\varepsilon(\omega_*)$.

$\omega = \omega_0 \circ \Phi_t^{-1}$ $\quad \nabla\omega = \nabla\Phi_{-t}^{-1} \cdot \nabla\omega_0 \circ \Phi_t^{-1}$

IDEA: Assume $|\nabla\Phi|_{L^\infty} \geq c(t)$ $\forall\ \omega_0 \in B_\varepsilon(\omega_*)$.

Fix $\delta(t)$ s.t. $\delta(t)\, c(t)^\alpha \to \infty$. Then $\exists$ a dense set of ω_0 in B_ε s.t. $\sup_t \frac{|\omega(t)|_{C^\alpha}}{\delta(t)\, c(t)^\alpha} = +\infty$. Let

$$U_N = \left\{ \omega_0 \in B_\varepsilon \ : \ \sup_t \frac{|\omega(t)|_{C^\alpha}}{\delta(t)\, c(t)^\alpha} > N \right\}$$

By lowersemicontinuity of $S_t : C^\alpha \to C^\alpha$, U_N is open in C^α_{top}.

Koch: given $\omega_0 \in B_\varepsilon$, and any $T, \kappa > 0$, $\exists\ \xi$ s.t. $|\xi|_{C^\alpha} < \kappa$ and $|S_t(\omega_0 + \xi)| \geq \frac{1}{2} \kappa\, c(T)^\alpha$. Take κ small and T large so $\frac{1}{\delta(T)} \geq 4\kappa^{-1} N$. Then $\frac{|S_t(\omega_0 + \xi)|_{C^\alpha}}{\delta(T)\, c(T)^\alpha} \geq 2N \Rightarrow \omega_0 + \xi \in U_N$.

Thus U_N dense in C^α. Let B be any ball with $\overline{B} \subseteq B_\varepsilon$. $\overline{B}$ is a complete metric space, $U_N \cap \overline{B}$ are open and dense. $\bigcap_N^\infty U_N \cap \overline{B}$ is dense in $\overline{B}$. by Baire category thm.

Application: Aging of the fluid

Let $\mathcal{D}_\mu(M)$ be group of area preserving diffeos.

If $\gamma : [0,1] \mapsto \mathcal{D}_\mu(M)$, its length is

$$\mathcal{L}[\gamma] = \int_0^1 \|\dot{\gamma}_\tau(\cdot)\|_{L^2} \, d\tau$$

Critical point are Euler flows! (Arnold). Geodesic distance

$$\mathrm{dist}_{\mathcal{D}_\mu(M)}(id, \phi) = \inf_{\substack{\gamma : [0,1] \mapsto \mathcal{D}_\mu(M) \\ \gamma(0) = id \\ \gamma(1) = \phi}} \mathcal{L}[\gamma]$$

Eliashberg & Ratiu: $\mathrm{diam}(\mathcal{D}_\mu(M)) = \infty$ ← infinite capacity for information

Given $\varphi \in \mathcal{D}_\mu(M)$ and $E > 0$, define

$$t_{age}(\varphi; E) := \inf \left\{ T > 0 : \gamma_\cdot : [0,T] \to \mathcal{D}_\mu,\ \gamma_0 = id,\ \gamma_T = \varphi,\ \frac{1}{T}\int_0^T |\dot{\gamma}|_{L^2} ds \leq E \right\}$$

Note: $t_{age}(\varphi, E) \geq \mathrm{dist}(\varphi, id)/\sqrt{E}$

Theorem: Nearby stable steady states, $t_{age}(\Phi_t, E_0) \xrightarrow{t \to \infty} \infty$

Some form of irreversibility in a reversible system!

The limit of $t \to \infty$.

$$\partial_t \omega + u \cdot \nabla \omega = 0$$

Since $|\omega(t)|_{L^\infty} = |\omega_0|_{L^\infty}$, the vorticity weak-* converges at long times: $\omega(t_i) \to \bar{\omega}$.

$$\boxed{\Omega_+(\omega_0) = \bigcap_{t \geq 0} \text{weak-* closure of } \{ S_\tau(\omega_0),\ \tau \geq t \}}$$

END STATES.

GOAL: Understand the structure of $\Omega_+(\omega_0)$!

<u>Lemma:</u> The kinetic energy is weak-* continuous, i.e.

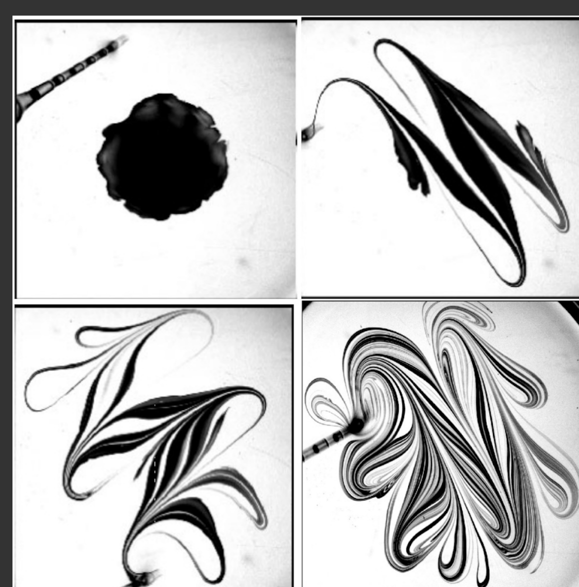
$$\omega_j \to \bar{\omega} \implies E(\omega_j) \to E(\bar{\omega})$$

Thus, energy is a <u>robust invariant</u>, $E(\bar{\omega}) = E(\omega_0)$

On other hand, the casimirs

$$I_f(\omega) = \int_\Omega f(\omega(t))\, dx$$

are not weak-* continuous unless f affine.



In general we cannot expect

$$I_f(\bar{\omega}) = \lim_{t \to \infty} I_f(\omega(t)) = I_f(\omega_0).$$

If f convex, we have

$$I_f(\bar{\omega}) \leq \liminf_{t \to \infty} I_f(\omega(t)) = I_f(\omega_0)$$

"Enstrophy" is lost due to fine scale mixing.

$$\boxed{\text{CONJECTURE: (Šverák 2013): Generically, orbits } \{\omega(t)\}_{t \geq 0} \text{ are not precompact.}}$$

Some Naïve Predictions

NAIVE CONJECTURE 1: The vorticity is everywhere mixed, i.e. $\omega(t) \to \int_\Omega \omega_0 := \overline{\omega_0}$ as $t \to \infty$.
i.e. $\Omega_+(\omega_0) = \{\overline{\omega_0}\}$.

FALSE: On $\mathbb{T}^2$, we must take $\int_{\mathbb{T}^2} \omega_0 = 0$.
but $E(\bar{\omega}) = E_0 \neq 0$. Thus $\bar{\omega} \neq 0$.

NAIVE CONJECTURE 2: The end state minimizes enstrophy subject to fixed energy.

$$I(\omega) = \frac{1}{2}\int \omega^2 dx \qquad E(\omega) = E_0 := E(\omega_0)$$

Both I and E are quadratic in ω. Minimizing $I(\omega) - \lambda E(\omega)$ yields

$$\omega = \Delta\psi = -\lambda_1 \psi$$

large scale "vortex"

where λ_1 is the first eigenvalue of Δ.

Mix (defined by reducing enstrophy) to max degree consistent with fixed energy.

Why choose minimum enstrophy? Theory of selective decay
(Bretherton & Haidvogel, 1976)

Note: for decaying Navier-Stokes, NC2 appears true.
(Foias - Saut, Schneider - Farge, Matthaeus et al ...)

Inverse Cascade of Energy

If $\omega_t \to \bar{\omega}$ then

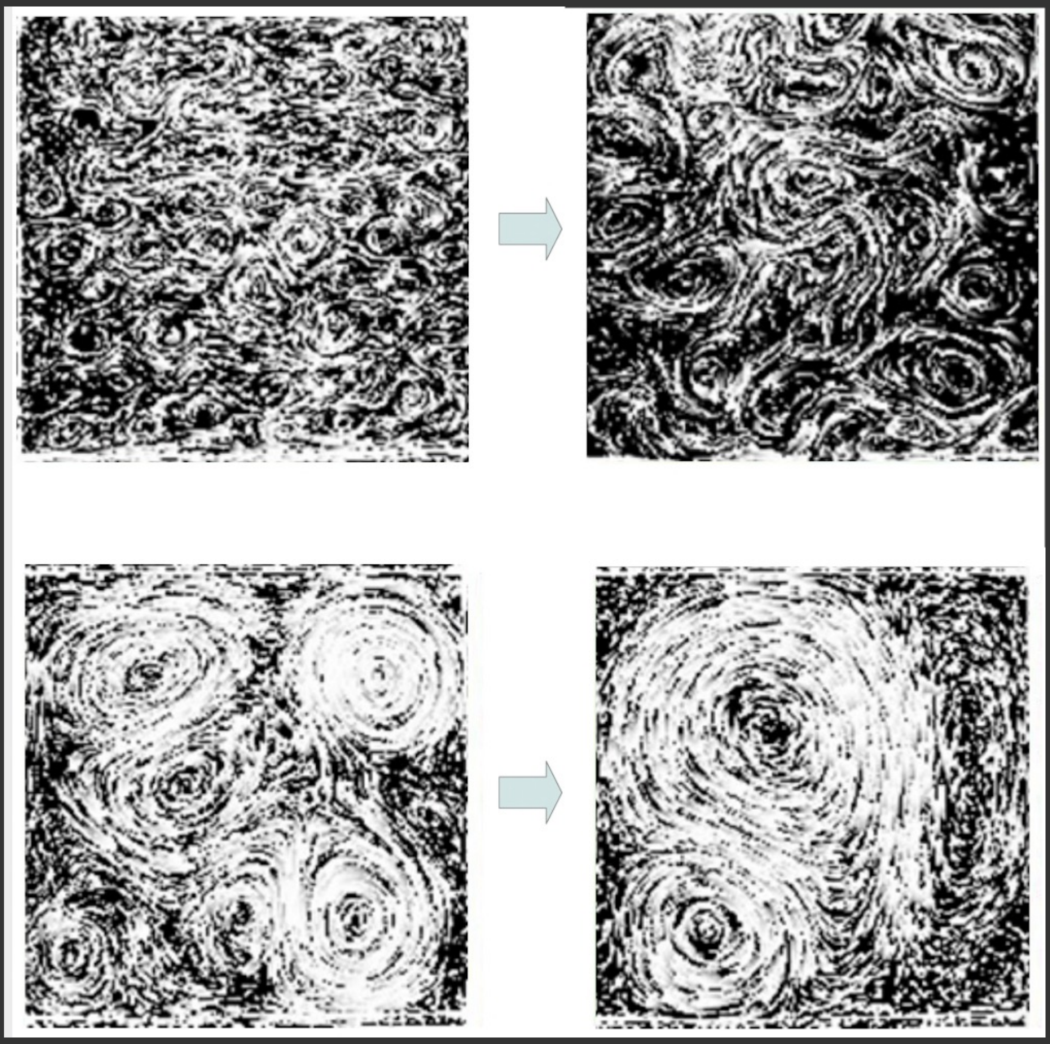
$$E(\omega_0) = E(\bar{\omega}), \qquad |\omega_0|_{L^2} \geq |\bar{\omega}|_{L^2}.$$

Can the energy go to small scales? (Fjørtoft, 1953)

$$E_K(\bar{\omega}) := \frac{1}{2}\sum_{n \geq K} |\hat{u}(n)|^2 \leq \frac{1}{K^2}\|\bar{\omega}\|_{L^2}^2 \leq \frac{1}{K^2}\|\omega_0\|_{L^2}^2$$

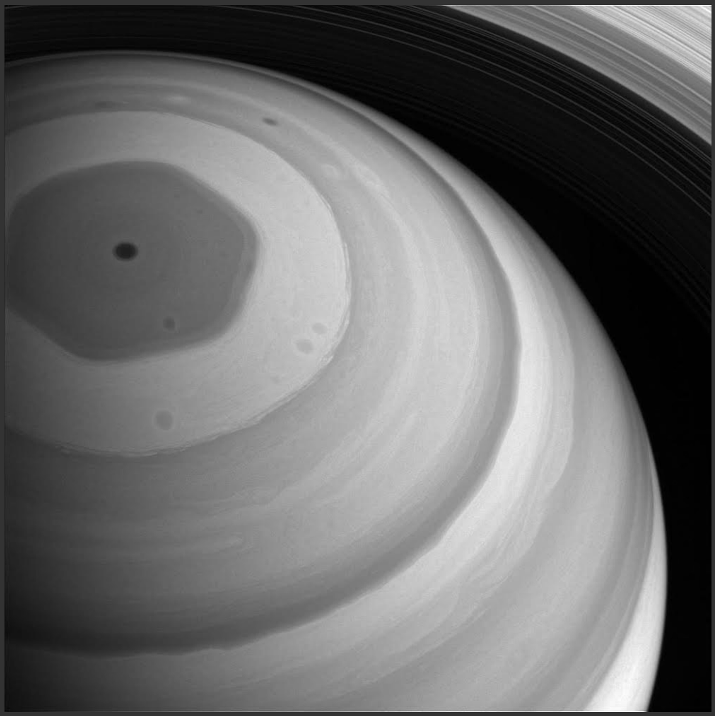
"Spectral blocking": energy cannot pile up at small scales.

In fact, energy is observed to cascade to large scales

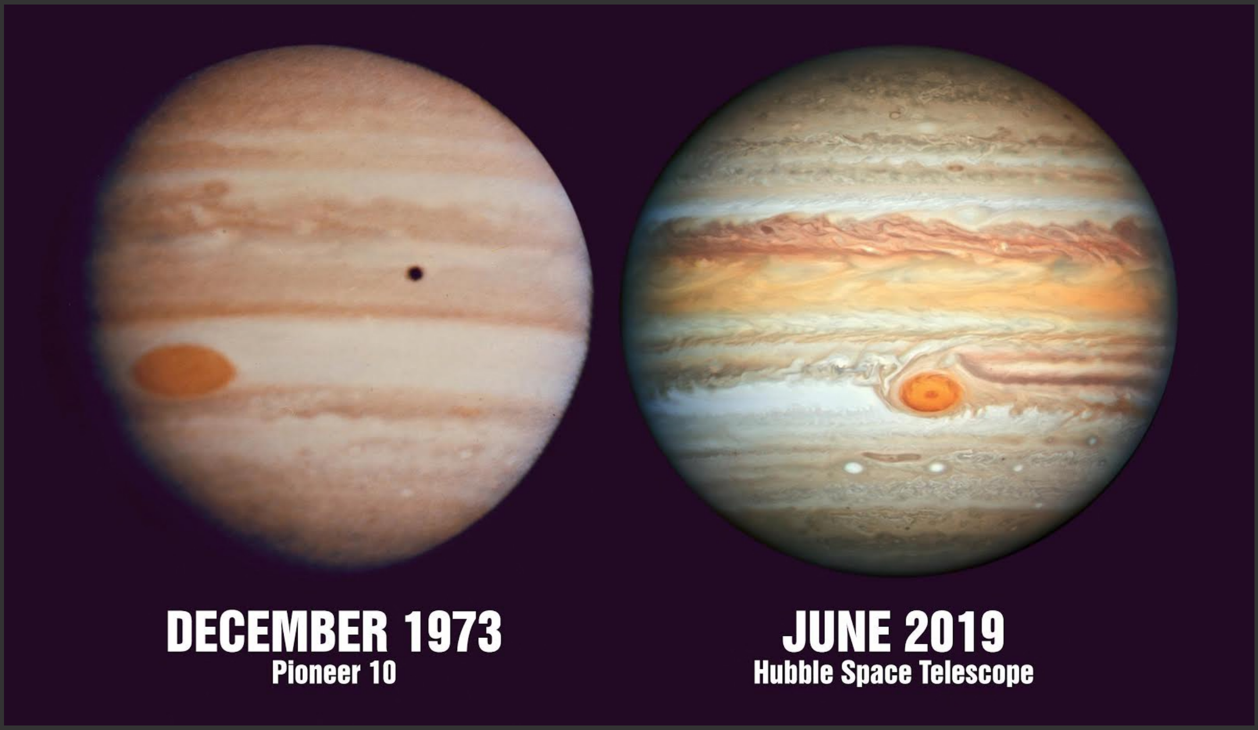


(Maassen, Clercx, van Heijst, 1999)

Saturn's Hexagon



Jupiter's Great Red Spot



observed first by Gallileo in 1665

$t_{exist} \geq$ 356 earth years!!!

Statistical Hydrodynamics

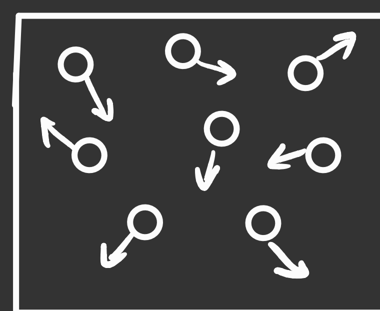
The formation of large, isolated vortices is an extremely common, yet spectacular phenomenon in unsteady flow. Its ubiquity suggests an explanation on statistical grounds.

L. Onsager, Statistical Hydrodynamics, 1949

Joyce-Montgomery Considered point vortices

$$\omega(t) = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i(t)} - \frac{1}{N}\sum_{j=1}^{N} \delta_{\tilde{x}_j(t)} \equiv \rho_+ - \rho_-$$

Under assumption of ergodicity of point vortex dynamics on energy surface and hypothesis that entropy



$$S = -\int_\Omega \rho_+ \ln \rho_+ - \int_\Omega \rho_- \ln \rho_-$$

be maximized (property of thermo. equilibrium) predicted

$$-\omega = -\Delta\psi = e^{-\beta(\psi-\mu_+)} - e^{\beta(\psi+\mu_-)}$$

where β and $\mu_\pm$ enforce energy and net +/- circulation.

When $\beta<0$, describes agrigation of vortices

Negative temperature states

However, if $1/\Theta < 0$, then vortices of the same sign will tend to cluster, — preferably the strongest ones —, so as to use up excess energy at the least possible cost in terms of degrees of freedom. It stands to reason that the large compound vortices formed in this manner will remain as the only conspicuous features of the motion; because the weaker vortices, free to roam practically at random, will yield rather erratic and disorganised contributions to the flow.

See Eyink - Sreenivasan, 2006

The little vortices who wanted to play

Once upon a time there were n vortices of strengths $K_1, \ldots, K_n$ in a two-dimensional frictionless incompressible fluid. They were enclosed by a boundary but could play ring-around-the-rosy otherwise. The rule of that game was [1)]

$$K_i\, dx_i/dt = -\partial W/\partial y_i \quad ; \quad K_i\, dy_i/dt = \partial W/\partial x_i$$

where $-\rho \underline{W}(x_1, y_1, \ldots, x_n, y_n)$ equals the energy apart from an additive constant (which is infinite on account of the self-energies). The function $\underline{W}$ is something like this:

$$W = \frac{1}{2\pi}\sum_{i,j}{}' K_i K_j \log(r_{ij}) + \text{(potential of image forces)}$$

and the image forces are finite except near the boundary, --- Now the vortices were very playful like I said and they liked to distribute themselves in completely random fashion but they could not do that because they had too much energy. You see they were not like molecules which have more room in momentum-space the more energy they have. The vortices had only a finite configuration-space. So when they had more energy than the average over that space, they could not play quite the way they wanted to. --- You can describe the ergodic distribution approximately by a canonical distribution

$$f(x_1, y_1, \ldots, x_n, y_n) = \exp((\psi + \rho W)/\Theta)$$

with an antitemperature $-\Theta > 0$. You will note that the phase-integral converges for one pair of vortices if and only if $(\rho/2\pi\Theta)K_i K_j > -2$

For a set of vortices there are further necessary conditions. You can figure out that there is no way to take care of much energy unless you let at least one pair of vortices of the same sign get close together. --- And now you know how the little vortices arranged it so that most of them could play just the way they wanted to. They just pushed the biggest vortices together until the big vortices had all the energy the little ones did not want, and then the little vortices played ring-around-the-rosy until you could not tell which was where, and it make no difference anyway.

Note to L. Pauling (1945)

The case $W > \overline{W}$ is quite different. We now need a negative temperature to get the required energy. The appropriate statistical methods have analogs not in the theory of electrolytes, but in the statistics of stars. In a general way we can foresee what will happen. Vortices of the same sign will tend to move together, more so the stronger the repulsion between them. After this aggregation of the stronger vortices has disposed of the excess energy, the weaker vortices are free to roam at will.

(?)

These predicted effects carry some resemblance to familiar habits of vortex sheets. If the rolling up of vortices is to be explained thus on a statistical basis, we may describe it as a process of crystallization, which occurs in response to a prevailing negative "market price" for energy.

Letter to C.C. Lin (1945)

Galerkin Trucation

$$\partial_t \omega_N + \mathbb{P}_{\leq N}(u_N \cdot \nabla \omega_N) = 0$$

Invariants: $E_N = \frac{1}{2}\|u_N\|_{L^2}^2$, $\Omega_N = \frac{1}{2}\|\omega_N\|_{L^2}^2$.

Writing $\omega_N = \sum_i a_i(t) f_i(x)$, we have ODEs

$$\dot{a}_i = v_i(a), \quad v_i(a) = B_{ijk} a_j a_k, \quad B_{ijk} = \frac{1}{\lambda_j}\left(\{f_j, f_k\}, f_i\right)_{L^2}$$

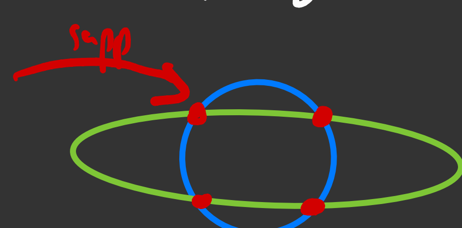
- v_i is divergence free (Liouville theorem)
- v_i is tangent to energy ellipsoids & enstrophy spheres.

Moreover by coarea formula, if $f : \mathbb{R}^N \to \mathbb{R}^{N-k}$ are so that $v \cdot \nabla f_i = 0$ $\forall i$, then

$$\left.\frac{\operatorname{div}(v)}{J}\right|_{S_h} = \operatorname{div}_{S_h}\left(\left.\frac{v}{J}\right|_{S_h}\right)$$

when $J = \det(\nabla f \, \nabla f^T)$ and $S_h = \{x : f(x) = h\}$.

Invariant measure: $\frac{1}{J} \, d\mathcal{H}^{(d-2)}$



$d \to \infty$, concentrates on $\Delta\psi = -\lambda_1 \psi$! Kraichnan Theory.

Statistical Hydrodynamics cont.

Miller, Weichman, Cross, Robert, Sommeria, Turkington

STEP 1: (Approximation)

Infinite dim phase space $\mathcal{H}$ $\longmapsto$ finite dim phase space $\mathcal{H}_N$

e.g. point vortices, Galerkin approx, discrete cells.

STEP 2: (Dynamics)

dynamics are defined which preserve energy and phase space volume (Liouville thm)

STEP 3: (Entropy maximization)

let $\mathcal{H}_0$ be space of states with same energy as u_0

A microcanonical ensemble (invariant measure on $\mathcal{H}_0$) is predicted as a stationary state u_f by demanding available phasespace for given state be maximized.

Ergodic hypothesis says finite dim system spends most of its time near u_f.

<u>Critisisms</u>: illegal transposition of limits $T \to \infty$, $N \to \infty$.

Works in systems near-equilibrium, but fluid is far.

- Lyapunov functional (Shnirelman, 1997)
- Wandering domains (Nadirashvili, 1991)
- integrable n-vortex motion (Khanin, 1982)

infinite dimensional fluid is an "open system"

- Mixing: $\omega(t) \rightharpoonup \bar{\omega}$ but not strongly. (Bedrossian, Masmoudi)
- configuration space is unbounded $\dim(\mathrm{Diff}_\mu(\mathbb{T}^2)) = \infty$

Shnirelman's Mixing Theory

Mixing operators on $L^2(\Omega)$

$$Kf(x) = \int_\Omega K(x,y) f(y)\, dy$$

where
i) $K(x,y) \geq 0$
ii) $\int_\Omega K(x,y)\, dy = 1$ iii) $\int_\Omega K(x,y)\, dx = 1$

Example: (a) $K(x,y) = \delta(y - \varphi^{-1}(x))$, $\phi \in \mathrm{Diff}_\mu(\Omega)$
(b) $K(x,y) = 1$

Set of all mixing operators on $L^2(\Omega)$ called $\mathcal{K}$ is a convex, weakly compact semigroup of contractions in $L^2(\Omega)$. Thus it defines partial order:

$$f \prec g \quad \text{if} \quad f = Kg$$
$$f \sim g \quad \text{if} \quad f \prec g \text{ and } g \prec f.$$

For vector fields, say $u \prec v$ if $\nabla^\perp \cdot u \prec \nabla^\perp \cdot v$.

$$\Omega_{u_0} = \{u \mid u \prec u_0\} \cap \{E(u) = E(u_0)\}.$$

Note that solutions of Euler $u(t) \in \Omega_{u_0}$ and moreover any limit $u(t) \to \bar{u} \in \Omega_{u_0}$.

A <u>minimal</u> element $v \in \Omega_{u_0}$ is such that for all $w \in \Omega_{u_0}$ with $w \prec v$, we have $v \sim w$.

Mixing is "reversible" on minimal elements!

Lemma: There exists a minimal element $\omega \in \Omega_{u_0}$.
Consequence of Zorn's Lemma

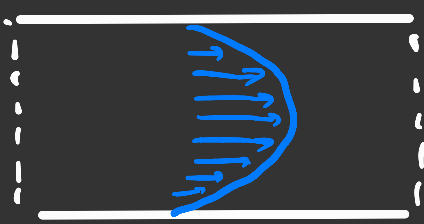
THEOREM: (Shnirelman, 93)
(i) any minimal element of Ω_{u_0} is a stationary solution of Euler equations
(ii) the minimal elements have monotonic $\omega = F(\Psi)$

Namely, all minimal elements satisfy
$$\Delta \Psi = F(\Psi)$$
for a univalent function F satisfying Arnold stability.

If Euler is "maximally mixing" then long time limits are steady stable states.

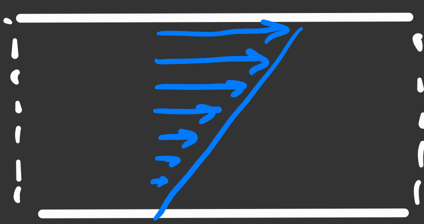
Three types:

Energy-excessive u



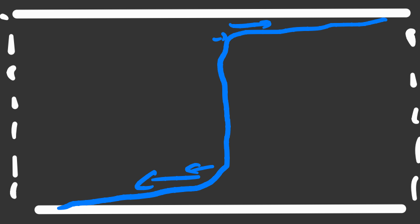
if $v \prec u$ then
$E(v) \leq E(u)$

Energy-neutral u



if $v \prec u$ then
$E(v) = E(u)$

Energy-deficient u



if $v \prec u$ then
$E(v) \geq E(u)$

Variational Approach & consequences (with Michele Dolce)

Let $\mathcal{O}_{\omega_0} = \{ \omega_0 \circ \phi, \ \phi \in \mathcal{D}_\mu(M) \}$

$$\mathcal{O}_{\omega_0, E_0} = \mathcal{O}_{\omega_0} \cap \{ E = E_0 \}$$

clearly: $\{ S_t(\omega_0) \}_{t \geq 0} \subseteq \mathcal{O}_{\omega_0, E_0}$

$$\Omega_+(\omega_0) \subseteq \overline{\mathcal{O}_{\omega_0 E_0}}^* = \overline{\mathcal{O}_{\omega_0}}^* \cap \{ E = E_0 \}$$

Remark: if ω_0 is Arnold stable then $\overline{\mathcal{O}_{\omega_0, E_0}}^* = \{ \omega_0 \}$.

Theorem: (Dolce-D) Let $f : L^\infty \to \mathbb{R}$ be strictly convex. For any $\omega_0 \in L^\infty$ with E_0, there exists $\omega^* \in L^\infty$

$$\min_{\omega \in \overline{\mathcal{O}_{\omega_0, E_0}}^*} \int_M f(\omega) \, dx$$

These are minimal flows. Moreover, $\exists$ convex Φ s.t.

$$\omega_* = \operatorname*{arg\,min}_{\substack{\omega \in L^\infty \\ |\omega|_{L^\infty} \leq 1}} \left(\int_M \Phi(\omega) + \lambda (E - E_0) \right)$$

No datum $\omega_0 \in X$ is isolated from equilibrium.

Theorem (Bouchet): extremizers correspond to Miller–Robert equilibria.

Proof relies on the following characterization

if $X = \{ f : |f|_{L^\infty} \le 1 \}$

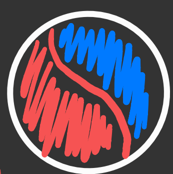
Shnirelman, Brenier-Gangbo

Sverak

$$\overline{\mathcal{O}_{\omega_0,E_0}}^{*} = \{ \omega \in X : \omega = k\omega_0 \text{ for } k \in \mathcal{K} \}$$

$$= \{ \omega \in X : \int_M \omega = \int_M \omega_0 , \int_M (\omega - c)_+ \le \int (\omega_0 - c)_+ \ \forall c \in \mathbb{R} \}$$

$$= \{ \omega \in X : \int_M \omega = \int_M \omega_0 , \int_M f(\omega) \le \int_M f(\omega_0) \ \forall \text{convex } f \}$$

E.g: $\omega_0 =$ 

, then $\overline{\mathcal{O}_{\omega_0,E_0}}^{*} = \{ \omega \in X : \int_M \omega = \int_M \omega_0 \}$

Minimal elements exist by weak-* compactness. They are stationary since for any permutation of cubes Q_1 & Q_2, the mixing $k^\varepsilon \omega = (1-\varepsilon)\omega + \varepsilon \omega\circ\phi$

$$\frac{d}{d\varepsilon} E\left(k^\phi_\varepsilon \omega^* \right) = \int_{Q_1} \left(\omega^*(x) - \omega^*(\phi(x)) \right) \left(\psi^*(x) - \psi^*(\phi(x)) \right)$$

≤ 0 or ≥ 0 $\forall$ permutations and Q_1.

$$\Rightarrow \quad (\omega^*(x) - \omega^*(y))(\psi^*(x) - \psi^*(y)) \begin{matrix} \ge 0 \\ \le 0 \end{matrix} \quad \text{a.e. } x, y$$

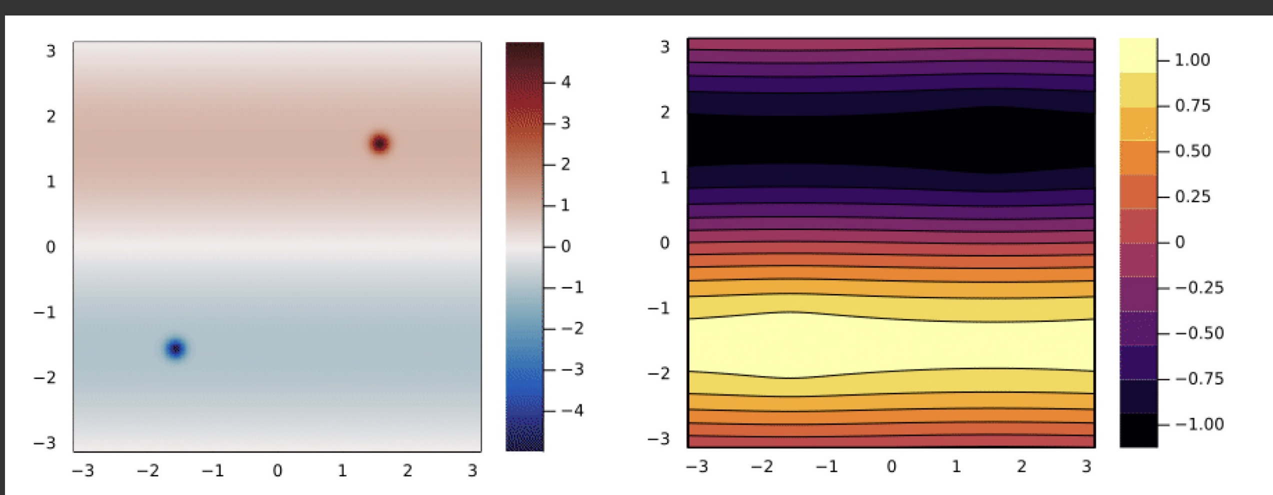
Monotonic $\omega = F(\psi)$ since

Lem: Let $\omega \in X$, $k \in \mathcal{K}$, $\omega_1 = k\omega$. $\exists\, \tilde{k} \in \mathcal{K}$ s.t. $\omega = \tilde{k}\omega_1$ iff $I_f(\omega_1) = I_f(\omega)$ for a strict. conv. f.

Excluding Symmetric equilibria

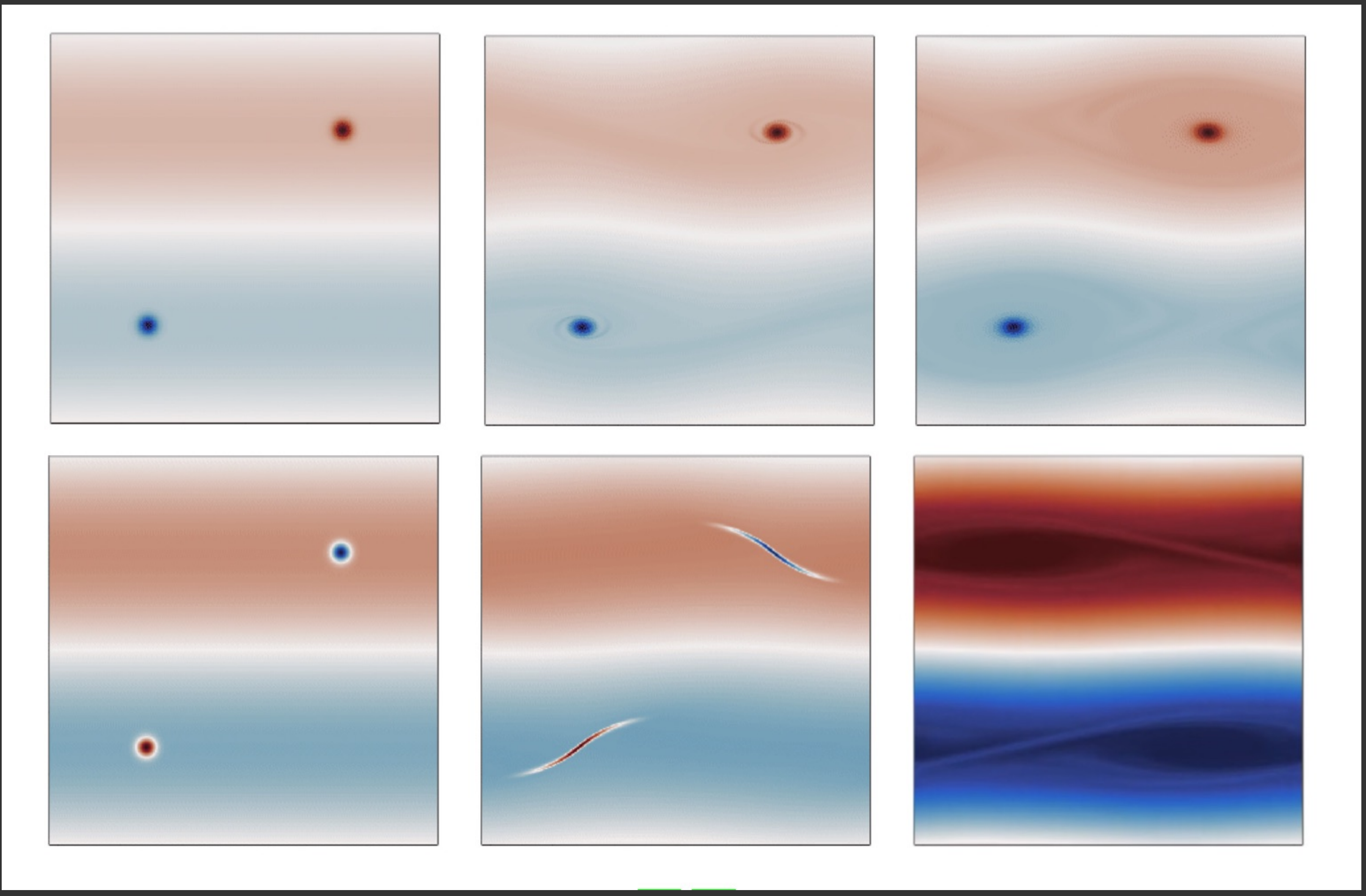
Theorem (Dolce - D.) Let M be $\mathbb{T}^2$ or $\mathbb{T}\times[0,1]$.
For any shear flow $\eta = -v'(y)$ and any $\varepsilon > 0$, $\delta > 0$,
$\exists\, \xi \in C_0^\infty$ (Gevrey-2) such that $|\xi - \eta|_{H^{-1-\delta}} \leq \varepsilon$
and $\overline{\mathcal{O}_{\xi, E_\xi, M_\xi}}^{*}$ contains no shear flows,
where E_ξ & M_ξ are energy and momentum of ξ.

Remark: In particular Euler cannot inviscid damp for data of the form:



Idea, have ε-approx point vortices, Energy $\sim \log 1/\varepsilon$
but any shearflow on its orbit has energy $= \mathcal{O}(1)$
since max value ε^{-2} distributed to a set at most area ε^2.

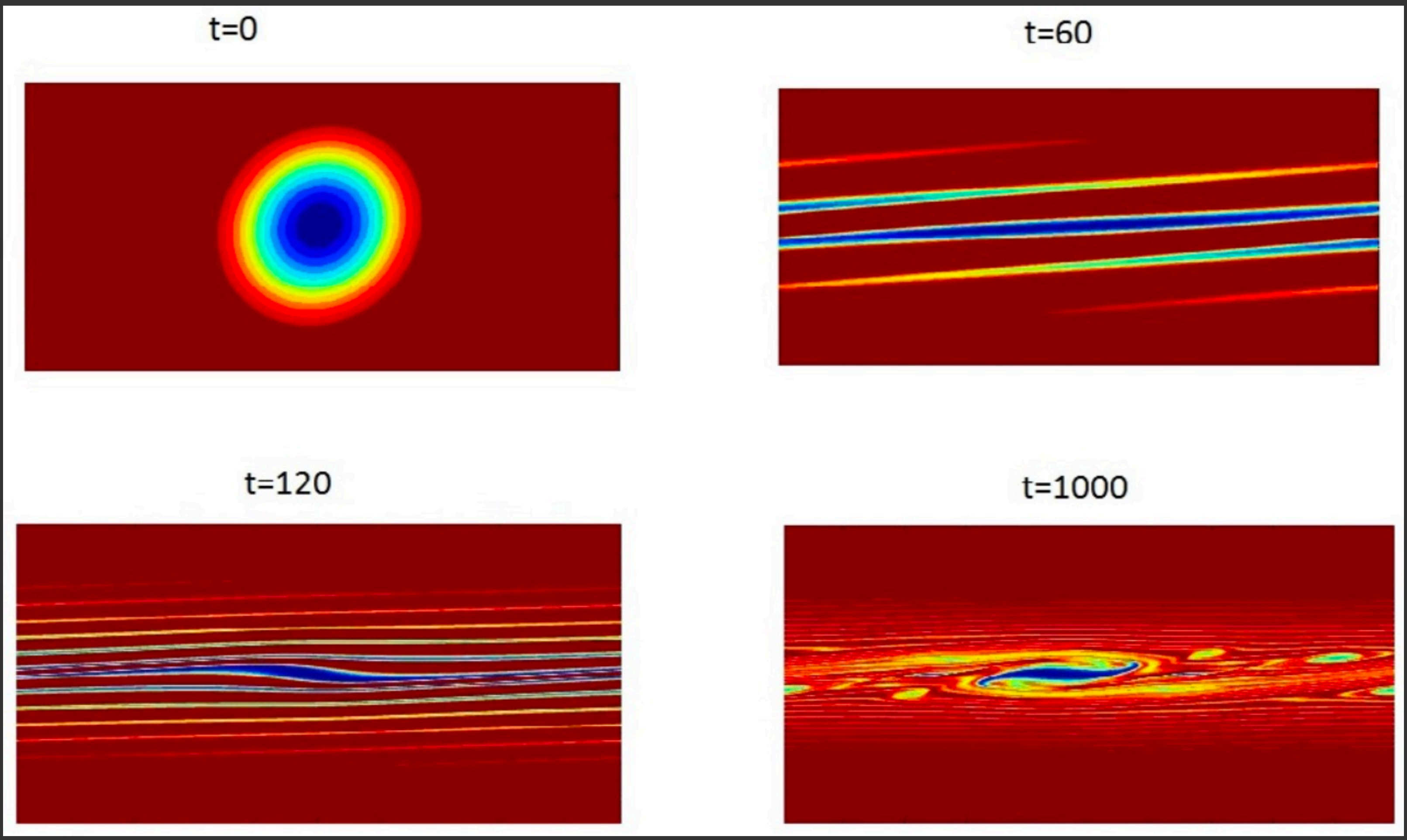
Instead, this is what happens:



Quasisteady, but may have undying periodic or quasiperiodic structures.

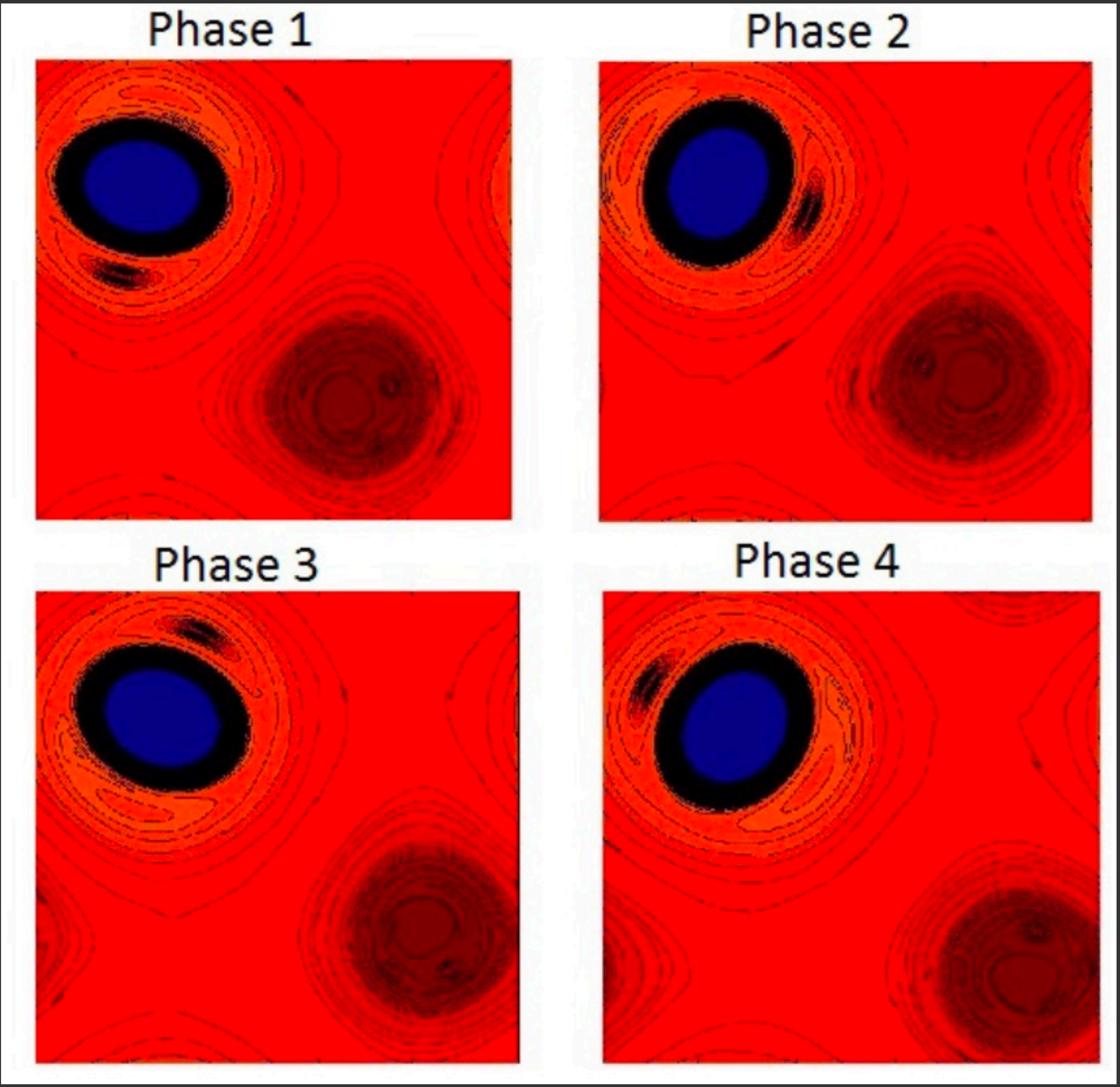
Returning to the Mystery...

Formation of catseye vortices
("plasma echos")

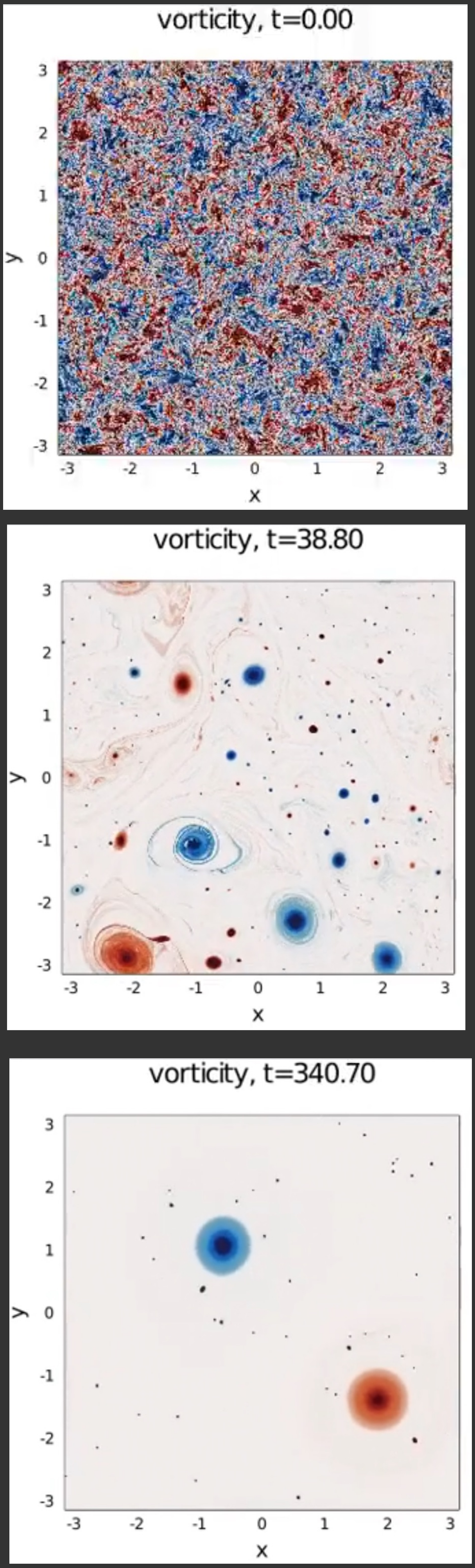


numerical simulation of A. Shnirelman

Quasiperiodical States of the fluid



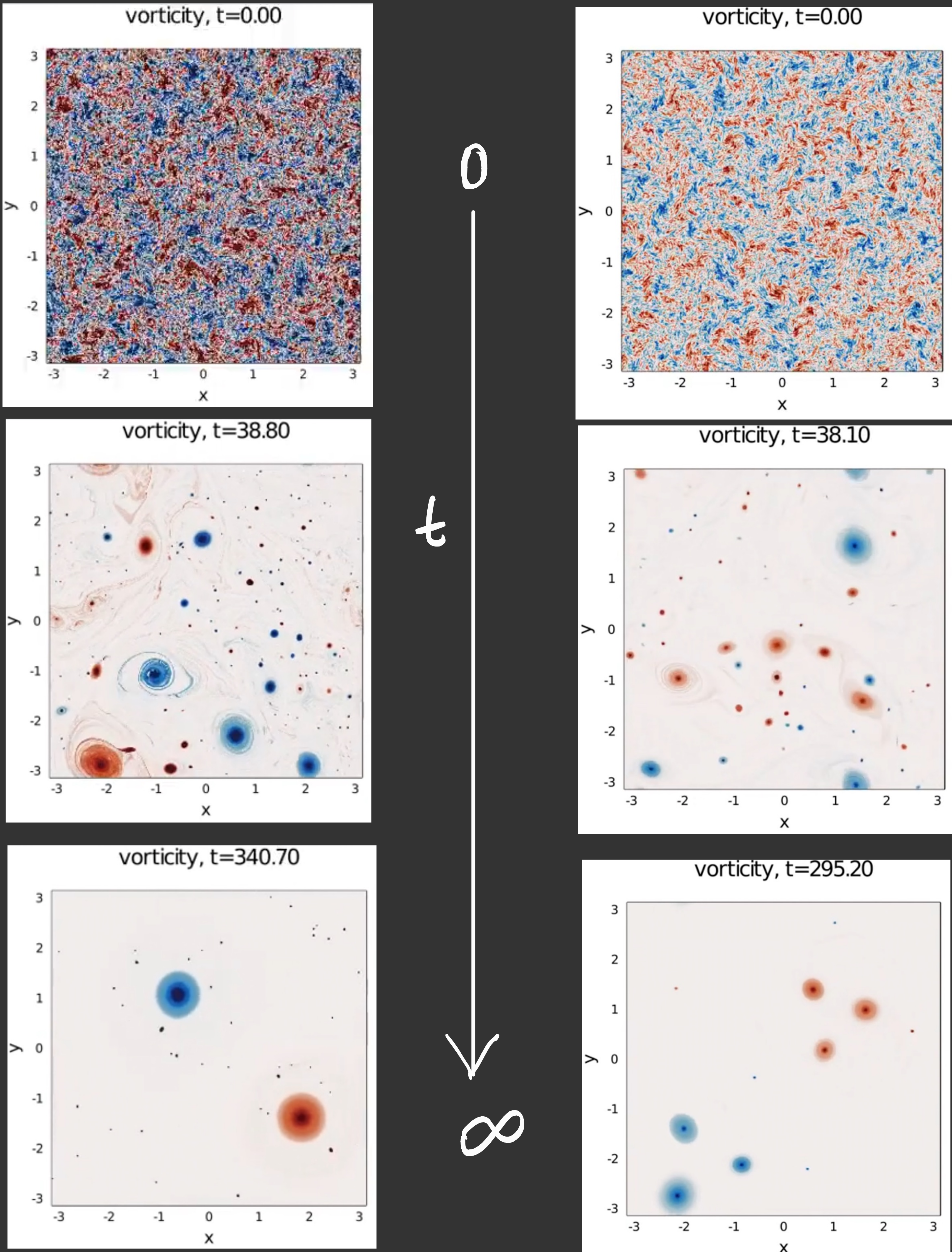
numerical simulation of A. Shnirelman

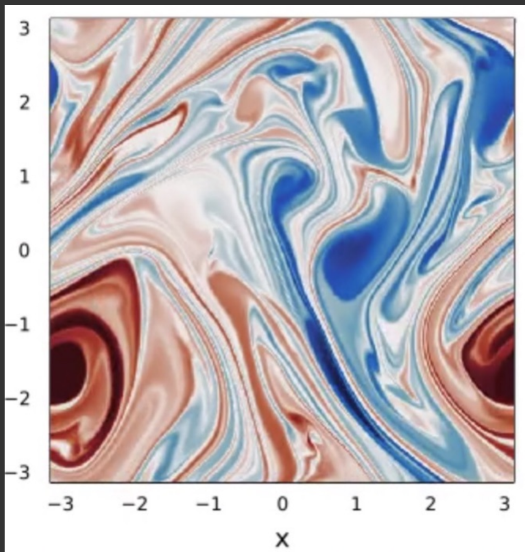


0

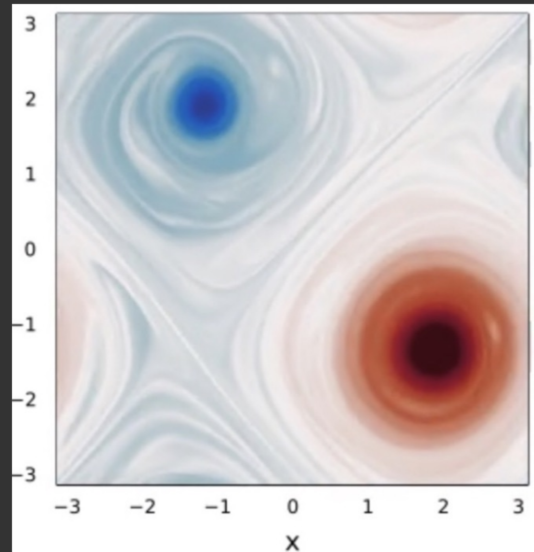
t

∞


vorticity, t=0.00
vorticity, t=0.00
0
vorticity, t=38.80
vorticity, t=38.10
t
vorticity, t=340.70
vorticity, t=295.20
∞
x
y



$t \to \infty$

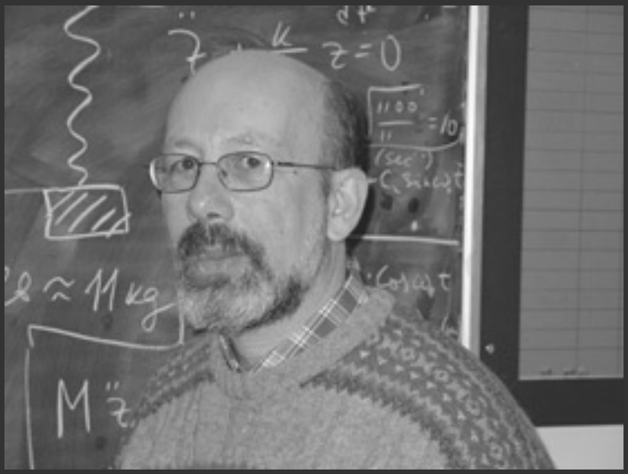


MYSTERY: How can one understand this apparent decrease of entropy* for the Euler equations at long times, at least if one looks at velocity fields.

* loosely interpreted, some measure of the diversity of a set in the space of all possible velocities.

* while diversity in the space of velocities appears to decrease, the entropy of the corresponding Lagrangian flows are likely increasing (Shnirelman, 1997)

Shnirelman's Conjecture



- Numerical simulations indicate that the long time behavior is <u>not</u> typically stationary, but rather some time dependent solution. - periodic, quasiperiodic etc..

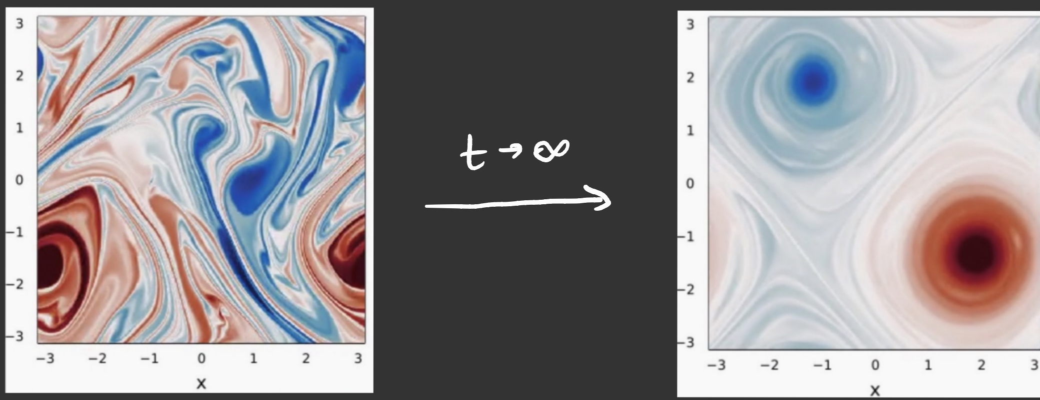
- This indicates that Euler is not a completely effective mixer, leaving solutions "trapped" in time dependent regimes.

CONJECTURE: (Shnirelman, 2013) The space of L^2-compact (under Euler evolution) vorticity orbits is the weak-$*$ attractor for the Euler dynamics

- Note; the space of compact orbits is at least not the entire phase space (Inviscid damping) (Bedrossian - Masmoudi, Ionescu - Jia)

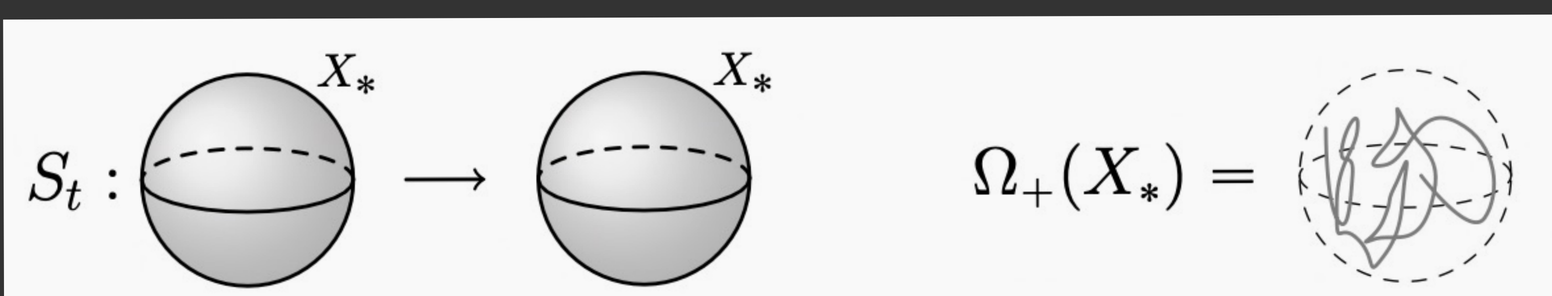
- For any $\omega_0 \in L^\infty$, at least one long-time limit corresponds to a compact orbit (Šverák)

Decrease of "Entropy" for 2D Euler



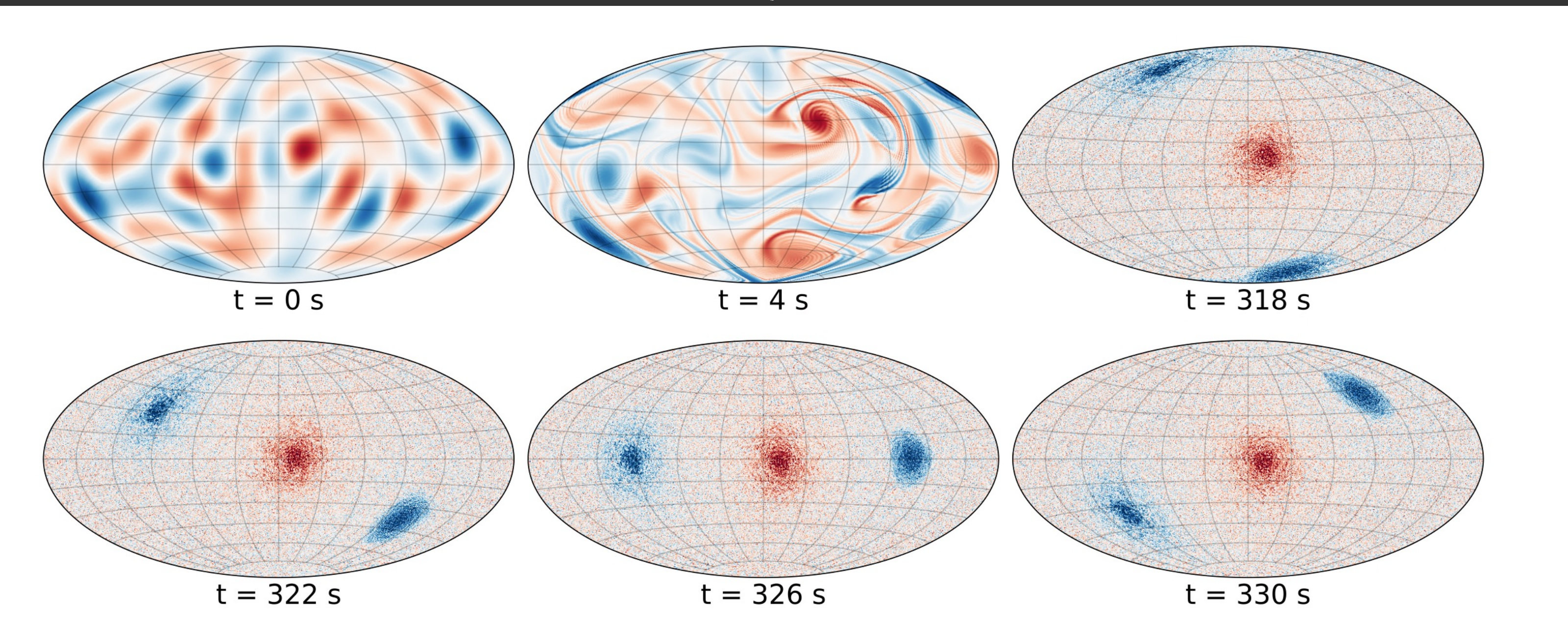
CONJECTURE: (Sverak, 2013) For generic $\omega_0 \in L^\infty$, the orbits $\{\omega(t)\}_{t \geq 0}$ are not precompact in L^2.

CONJECTURE: (Shnirelman, 2013) For any $\omega_0 \in L^\infty$, the collection of weak-* limits of the orbit $\{\omega(t)\}_{t \geq 0}$ consist of vorticities which generate a L^2-compact orbit.
e.g. steady, periodic, quasiperiodic ...

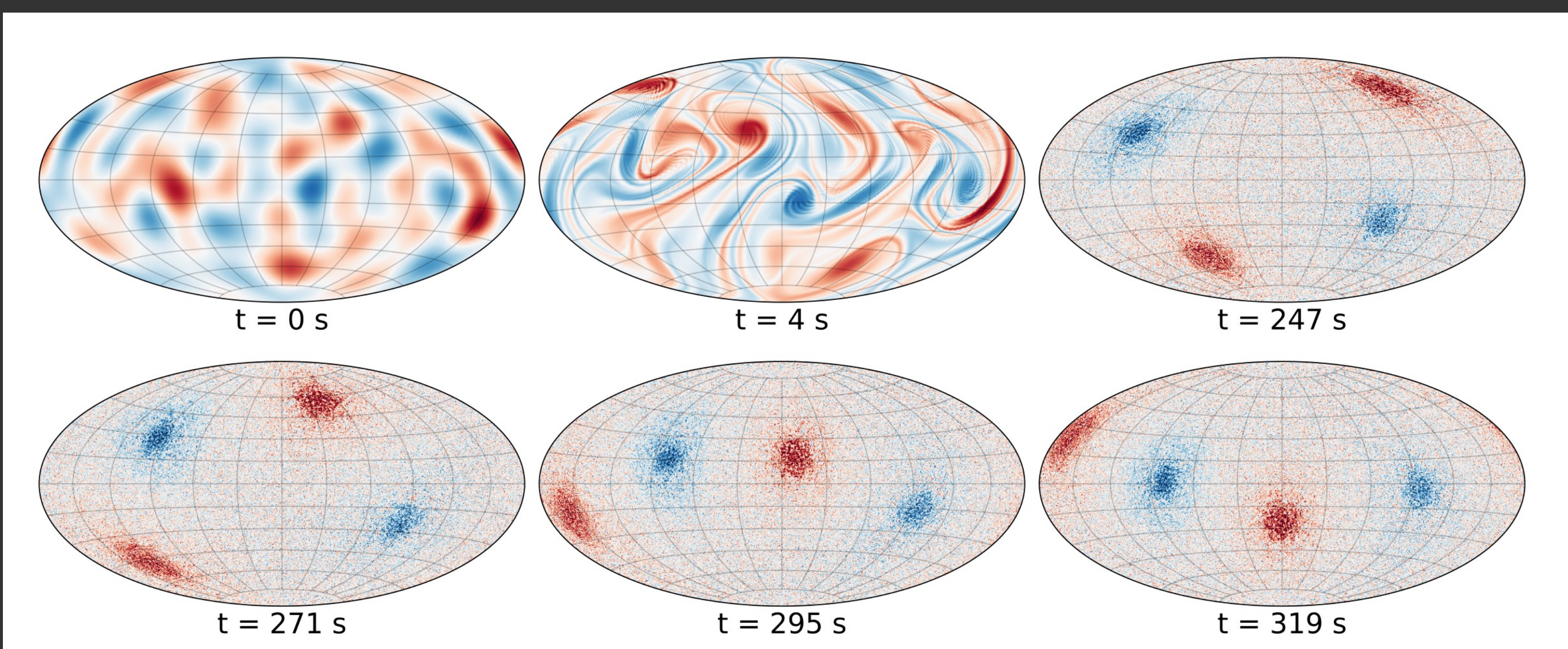


- Theorem: (Sverak) For any $\omega_0 \in L^\infty$, $\Omega_+(\omega_0)$ contains an L^2-compact orbit.
- Both conjectures are "true" in a neighborhood of special equilibria (Bedrossian & Masmoudi) Ionescu & Jia

Non-Zero net angular Momentum.



Zero net angular momentum



Conjecture: (Modin, Viviani): Long time behavior tracks integrable point vortex motions!

Thank-you!