Geometric and Dynamical aspects of fluid motion

Outline : Lecture 1 : The Euler equations

Lecture 2 : Long time dynamics of two-dimensional inviscid fluids

Lecture 3 : Transition to turbulence and a problem of Kolmogorov

Lecture 4 : Phenomenology and Mathematics of three-dimensional turbulence

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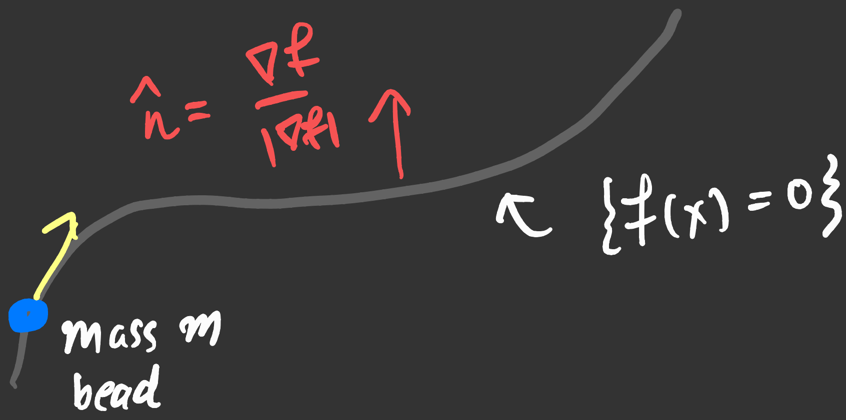
Problem: find equations of motion of a bead on a wire



Kinetic energy of bead: $K = \frac{1}{2} m \dot{x}^2$

Action: $S(\{x\}_{t \in [0,T]}) = \int_0^T \frac{m}{2} |\dot{x}(t)|^2 \, dt$

Equations of motion:
Hamilton's Principle

$\delta S = 0$

variations along paths on the wire
$\delta x(0) = \delta x(T) = 0$

$$0 = \delta S = m \int_0^T \dot{x}(t) \cdot \delta \dot{x}(t) \, dt = -m \int_0^T \ddot{x}(t) \cdot \delta x(t) \, dt$$

must be tangent

Newton's equations:

$$\boxed{\begin{aligned} \ddot{x}(t) &= \lambda \nabla f(x(t)) \\ f(x(t)) &= 0 \end{aligned}}$$

D'Alembert's principle for constrained motion

$(T_x S)^\perp$ = normal space $\qquad \mathbb{H}, \quad (\cdot,\cdot)_{\mathbb{H}}$

inner product

Consider a surface S in some Euclidean space $\mathbb{H}$. Than the trajectory of a frictionless mechanical particle satisfies the Newton equation

$$\boxed{\begin{array}{l} \ddot{X}(t) \in (T_x S)^\perp \\ X(t) \in S \\ X(0) = X_0 \in S, \quad \dot{X}(0) = V_0 \in T_{X_0} S. \end{array}}$$

Geodesic motion on Submanifold

Example: suppose $\mathbb{H} = \mathbb{R}^n$, $(\cdot,\cdot)_{\mathbb{H}}$ is Euclidean inner product, and $S = \{f(x) = 0\}$ for $f : \mathbb{R}^n \to \mathbb{R}^k$. Than

$$\ddot{X}(t) = \sum_{i=1}^{k} \lambda_i(t) \nabla f_k(x(t))$$

$$f_i(x(t)) = 0 \qquad i = 1, \dots, k$$

How to find the "Lagrange multipliers" λ?

<u>Codimension 1:</u>

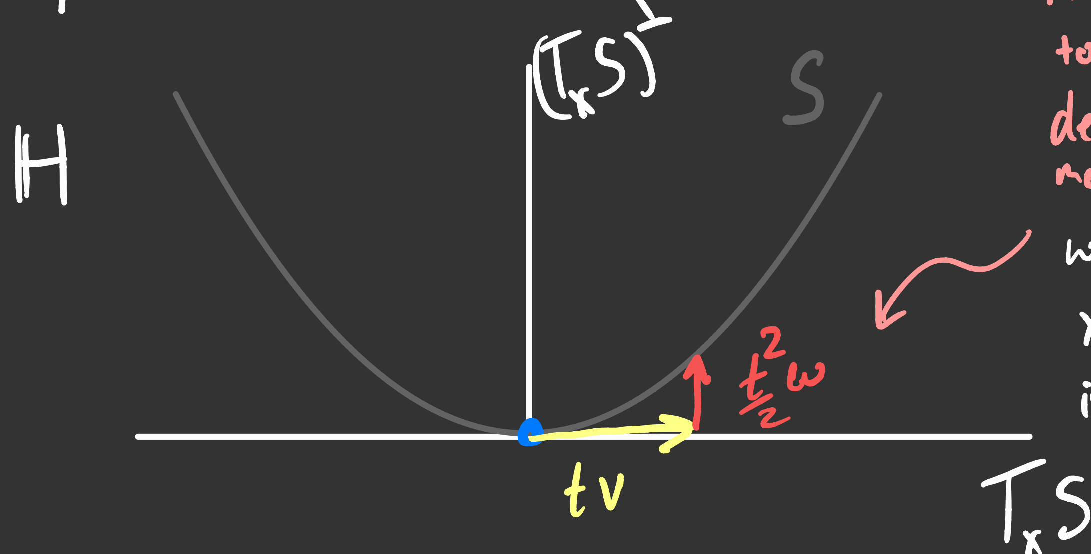
$$\ddot{x}(t) = \lambda(t)\nabla f(x(t))$$
$$f(x(t)) = 0$$

Differentiating the constraint:

$\dot{x}(t)\cdot\nabla f(x(t)) = 0$ ← velocity is tangent

$$\ddot{x}(t)\cdot\nabla f(x(t)) = -\mathrm{Hess} f_x(\dot{x},\dot{x}) \Rightarrow \boxed{\lambda(t) = -\frac{\mathrm{Hess} f_x(\dot{x},\dot{x})}{|\nabla f(x)|^2}}$$

Interpretation of the force $\lambda\nabla f$:



Must add $\frac{t^2}{2}w$ to suppress the deviation of free motion from S.

$w\in(T_xS)^\perp$ is so $x \mapsto x + tv + \frac{t^2}{2}w$ is on S up to $\mathcal{O}(t^3)$.

<u>The second fundamental form</u> II_x of the embedded surface is a quadratic map $\mathrm{II}_x : T_xS\times T_xS \to (T_xS)^\perp$. The value $\mathrm{II}_x(v,v)$ is the acceleration of a point moving on S.

$$f(x+tv+t^2w) = f(x) + tv\cdot\nabla f(x) + \frac{t^2}{2}\left(\mathrm{Hess} f_x(v,v) + w\cdot\nabla f\right) + \mathcal{O}(t^3).$$

0 if $x\in S$ 0 if $v\in T_xS$

0 if $w = -\frac{\mathrm{Hess} f_x(v,v)}{|\nabla f|^2}\nabla f(x)$

Codimension k: $\ddot{x} = \sum_{i=1}^{k} \lambda_i \, \mathrm{grad} f_i(x)$

$$f_j(x) = 0$$

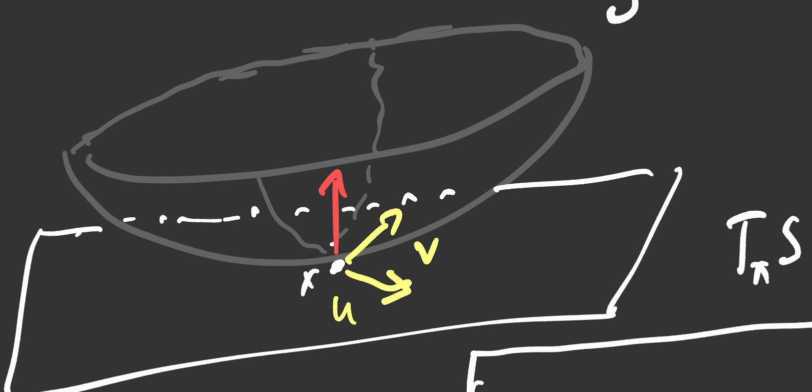
Then
$$\dot{x} \cdot \mathrm{grad} f_j(x) = 0$$
$$\ddot{x} \cdot \mathrm{grad} f_j(x) = -\mathrm{Hess} f_{j,x}(\dot{x}, \dot{x})$$
$$\sum_{i=1}^{k} \lambda_i \, \mathrm{grad} f_i(x) \cdot \mathrm{grad} f_j(x) = -\mathrm{Hess} f_{j,x}(\dot{x}, \dot{x})$$

Let $(J_f(x))_{ij} = \mathrm{grad} f_i(x) \cdot \mathrm{grad} f_j(x)$. Then

$$J_f(x)\lambda = \mathrm{Hess} f_x(\dot{x}, \dot{x}) \implies \lambda = -(J_f(x))^{-1} \mathrm{Hess} f_x(\dot{x}, \dot{x})$$

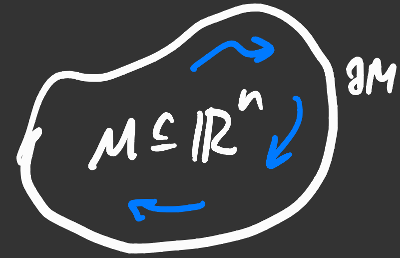
The second fundamental form of the submanifold $S = \{ f(x) = 0 \}$

$f: \mathbb{R}^n \to \mathbb{R}^k$



$$\boxed{\mathrm{II}_x(u,v) = -(J_f(x))^{-1} \mathrm{Hess} f_x(u,v) \cdot \nabla f(x) \qquad u, v \in T_x S}$$

Example: motion of an ideal fluid



$$\mathbb{H} = L^2(M;\mathbb{R}^n)$$

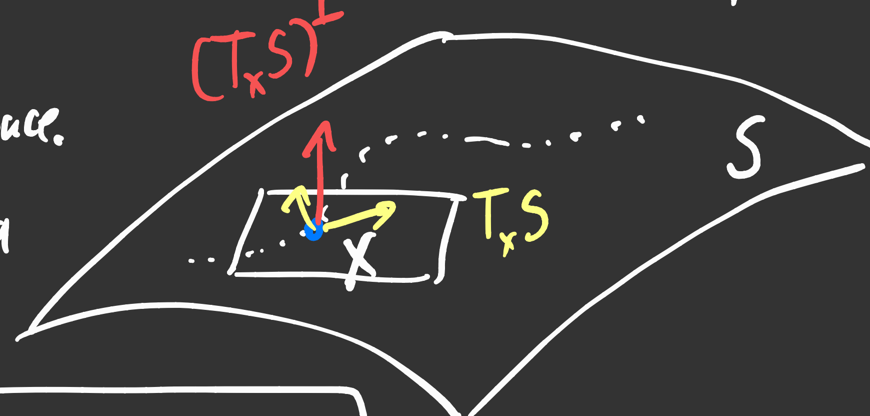
$$(u,v)_{\mathbb{H}} = \int_M u\cdot v\,dx$$

u, v is vector field on M tangent to ∂M, e.g. $\mathbb{H}$ since $\mathbb{H}$ is linear

$$S = \{\varphi \in \mathrm{Diff}(M) : \det \nabla\varphi(x) = 1 \text{ for all } x \in M\}$$

Volume preserving diffeos (infinite dimension and codimension)

$\mathbb{H}$



Let us find the tangent space.
We require Jacobi's formula

$$\boxed{\frac{d}{d\varepsilon}\det A(\varepsilon) = \mathrm{tr}\left(A^{-1}(\varepsilon)\,\frac{d}{d\varepsilon}A(\varepsilon)\right)\det A(\varepsilon)}$$

Apply this for a deformation of X, X^ε, which is a vector field $\frac{d}{d\varepsilon}X^\varepsilon = u^\varepsilon(X^\varepsilon)$, $X^\varepsilon\big|_{\varepsilon=0} = X$. Then

$$0 = \frac{d}{d\varepsilon}\det \nabla X^\varepsilon\Big|_{\varepsilon=0} = \mathrm{tr}\left((\nabla X)^{-1}\nabla(u^0\circ X)\right) = \nabla\cdot u^0(x)$$

Thus $\boxed{T_X S = \{u\circ X : \mathrm{div}\, u = 0 \text{ and } u\cdot n|_{\partial M} = 0\}}$

By Hodge decomposition, the L^2 orthogonal complement is

a vector field η is expressed uniquely as
$\eta = v + \mathrm{grad}\,\phi$ where $\mathrm{div}\, v = 0$, $v\cdot n|_{\partial M} = 0$.

$$\boxed{(T_X S)^\perp = \{\mathrm{grad}\, p\}}$$

Now, what are the equations of motion?
By D'Alembert's principle, they will be

$$\ddot{X}(t) \in (T_X S)^{\perp}, \qquad X(t) \in S$$

Let us compute more directly by considering S as a "submanifold" of H. Recall:

$$S := \{ X \in \mathrm{Diff}(M) : \det \nabla X(a) = 1 \text{ for all } a \in M \}$$

$$= \bigcap_{a \in M} S_a, \qquad S_a := \{ X \in \mathrm{Diff}(M) : \det \nabla X(a) = 1 \}$$

Thus we have $f_a(X) = \det \nabla X(a)$ are the components of the function $f(X)$ whose level set defines S.
By analogy with finite dimensions, we have

$$\ddot{X} = \int_M \lambda(a) \, \mathrm{grad}\, f_a(X) \, da$$

$$\lambda(a) = -\int_M (J_f)^{-1}_{a,a'} \, \mathrm{Hess}\, f_{a'}(\dot{X}, \dot{X}) \, da'$$

Who are $\mathrm{grad}\, f_a(X)$, $(J_f)^{-1}_{a,a'}$ and $\mathrm{Hess}\, f_{a'}$?

First, let's discuss "gradient": V is a functional

$V: \mathbb{H} \to \mathbb{R}$ mapping H to $\mathbb{R}$

We can introduce its derivative

$V'(x)$ is again a functional, maybe depending on x.

$H \ni x$ fixed ← functional derivative

It is an abstract object. But recall there is a scalar product in H, so we can use Riesz representation theorem (general form of a linear continuous functional on Hilbert sp)

If a linear, cont functional $\ell: H \to \mathbb{R}$, then $\forall x \in H$ $\ell x = (x, h)$ where $h = h_\ell \in H$

The definition of V' is

$$V'(x)y = \frac{d}{d\varepsilon} V(x + \varepsilon y)\Big|_{\varepsilon=0}$$ "Gateau derivative"

This is a linear functional, & cont. By Riesz:

$$V'(x)\,y = (y, h) = (y, \operatorname{grad} V)_{L^2}$$

h ← df

action of linear functional on any point y

definition of gradient.

where $h = h(x) := \operatorname{grad} V(x)$

Distinguish between differential of functional and gradient:

Notion of gradient depends on choice of scalar product.

But we might like to use other scalar product.

Lemma: $\operatorname{grad} f_a(X) = -(\nabla X)^{-1}\nabla\delta_a$

Proof: $f_a'(X)\,\xi := \frac{d}{d\varepsilon} f_a(X^\varepsilon)\Big|_{\varepsilon=0}$ where $\frac{d}{d\varepsilon}X^\varepsilon\Big|_{\varepsilon=0} = \xi(a)$

$$= \operatorname{tr}\left((\nabla X)^{-1}_{(a)}\nabla\xi(a)\right) \quad \text{Jacobi's formula}$$

$$= (\operatorname{div}\tilde{\xi})(X(a)) \quad \text{where } \xi := \tilde{\xi}\circ X$$

$$= \int_M (\operatorname{div}\tilde{\xi})(X(y))\,\delta_a(y)\,dy$$

$$= \int_M \operatorname{div}\tilde{\xi}(y)\,\delta_a(X^{-1}(y))\,dy$$

$$= -\int_M (\xi\circ X^{-1})(y)\cdot\nabla X^{-1}(y)\cdot\nabla\delta_a(X^{-1}(y))\,dy$$

$$= -\left(\xi, (\nabla X^{-1})\circ X\cdot\nabla\delta_a\right)_{L^2}$$

$$= -\left(\xi, (\nabla X)^{-1}\nabla\delta_a\right)_{L^2}$$

Note $(\operatorname{grad} f_a, \xi) = \operatorname{div}(\xi\circ X^{-1})\circ X(a)$

$$\ddot{X} = \int_M \lambda(a)\operatorname{grad} f_a(X)\,da = \int_M \lambda(a)(\nabla X)^{-1}\nabla_a\delta_a\,da$$

$$= -(\nabla X)^{-1}\nabla\lambda = (\nabla p)\circ X \quad \text{where } \lambda = p\circ X.$$

Hodge! vectors orthogonal to S are gradients!

Recall the definition of the Hessian

$$\operatorname{Hess} f_a(X)(\xi,\eta) = \left(\nabla_\eta \operatorname{grad} f_a, \xi \right)_{L^2}$$

Lemma: $\operatorname{Hess} f_a(X)(\xi,\eta) = \operatorname{tr}\left(\nabla\tilde{\eta}\, \nabla\tilde{\xi} \right) \circ X(a)$

where $\tilde{\eta} = \eta \circ X^{-1}$ and $\tilde{\xi} = \xi \circ X^{-1}$.

Proof: By definition, we have

$$\eta \cdot \operatorname{grad}(\operatorname{grad} f_a(X)) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \operatorname{grad} f_a(X^\varepsilon)$$

with $\frac{d}{d\varepsilon} X^\varepsilon \Big|_{\varepsilon=0} = \eta$

$(\operatorname{grad} f_a(X))' \eta$

$$= -\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} (\nabla X^\varepsilon)^{-1} \nabla \delta_a$$

$$= (\nabla X)^{-1} \nabla \eta (\nabla X)^{-1} \nabla \delta_a$$

Then

$$\operatorname{Hess} f_a(X)(\xi,\eta) = \left((\nabla X)^{-1} \nabla\eta (\nabla X)^{-1} \nabla\delta_a, \xi \right)_{L^2}$$

$$= (\nabla X)^{-1} \nabla\eta (\nabla X)^{-1} \nabla\xi$$

Let $\eta = \tilde{\eta} \circ X$ and $\xi = \tilde{\xi} \circ X$

$$= \operatorname{tr}\left(\nabla\tilde{\eta}\ \nabla\tilde{\xi} \right) \circ X$$

Now, finally, the Jacobian matrix

$$\left(J_f(x)\right)_{a,a'} := \left(\operatorname{grad} f_a(x),\ \operatorname{grad} f_{a'}(x)\right)_{L^2}$$

Recall, $\left(\operatorname{grad} f_a, \xi\right)_{L^2} = \operatorname{div}(\xi \circ X^{-1}) \circ X(a)$.

Lemma: $$\left(J_f(x)\right)_{a,a'} = (\Delta \tilde{\delta}_a)(X(a'))$$

where $\tilde{\delta}_a = \delta_a \circ X^{-1}$.

Distributional representative of Laplacian

Proof: Letting $\tilde{\delta}_a = \delta_a \circ X^{-1}$, we have $\left(J_f(x)\right)_{a,a'}$ equals

$$\left(\operatorname{grad} f_a(x),\ \operatorname{grad} f_{a'}(x)\right)_{L^2} = \int_M \nabla \tilde{\delta}_a(y) \cdot \nabla \tilde{\delta}_{a'}(y)\, dy$$

$$= (\operatorname{div} \nabla \tilde{\delta}_{a'})(X(a))$$

$$= (\Delta \tilde{\delta}_{a'})(X(a))$$

Since

$$\langle \nabla \tilde{\delta}_a, \xi \rangle = \langle (\delta \circ X^{-1})_a, \operatorname{div} \xi \rangle = \operatorname{div} \xi (X(a)).$$

$$\boxed{\begin{aligned}&\underline{\text{Lemma:}} \quad \left(J_f(x)\right)_{a,a'} = (\Delta \tilde{\delta}_a)(x(a')) \\ &\text{where } \tilde{\delta}_a = \delta_a \circ x^{-1}.\end{aligned}}$$

$\delta_a(y) = \delta(y-a)$, $\tilde{\delta}_a(y) = \delta(x^{-1}(y) - a)$

We then have

$$\int_M \left(J_f(x)\right)_{a,a'} \phi(a')\, da'$$

$$= \int_M \Delta \tilde{\delta}_a(a')\, \phi(x^{-1}(a'))\, da'$$

$$= \int_M \tilde{\delta}_a(a')\, \Delta(\phi \circ x^{-1})(a')\, da'$$

$$= \Delta(\phi \circ x^{-1}) \circ x(a).$$

$$\boxed{\int_M \left(J_f(x)\right)_{a,a'} \phi(a')\, da' = \Delta(\phi \circ x^{-1}) \circ x(a).}$$

What about the inverse Jacobian matrix? Defined by (12)

$$\int (J_f(x))_{a,a'} (J_f^{-1}(x))_{a',b} \, da' = \delta_b(a)$$

$\uparrow$ Kronecker delta in function space: identity operator

From before, we have

$$\int (J_f(x))_{a,a'} (J_f^{-1}(x))_{a',b} \, da' = \Delta\left(J_f^{-1}(x)_{X^{-1}(\cdot), b} \right) \circ X(a)$$

Let $G_M(x\, x')$, be the Green's function for the Laplacian: $\Delta_x G(x, x') = \delta_{x'}(x)$. Now

$$\Delta\left(J_f^{-1}(x)_{X^{-1}(\cdot), b} \right) \circ X(a) = \delta_b(a)$$

$$\Delta\left(J_f^{-1}(x)_{X^{-1}(a), b} \right) = \delta_b\left(X^{-1}(a)\right) = \delta_{X(b)}(a)$$

Thus we deduce that $J_f^{-1}(x)_{X^{-1}(a), b} = G(a, X(b))$. Thus

$$\boxed{(J_f^{-1}(x))_{a,b} = G_M(X(a), X(b))}$$

Finally, returning to λ, we have

$$\lambda(a) = -\int_M (J_f)^{-1}_{a,a'} \operatorname{Hess} f_{a'}(\dot{x}, \dot{x})\, da'$$

$$= -\int_M (J_f)^{-1}_{a,a'} \operatorname{tr}(\nabla u)^2 (X(a'))\, da' \qquad u = \dot{x} \circ x^{-1}$$

$$= -\int_M G_M(x(a), x(a'))\, \operatorname{tr}(\nabla u)^2 (x(a'))\, da'$$

$$= \left(\Delta^{-1} \operatorname{tr}(\nabla u)^2\right) \circ X(a)$$

Thus $\quad \lambda \circ X^{-1} =: P = \Delta^{-1} \operatorname{tr}(\nabla u)^2.$

This recovers the pressure poisson equation!

We finally have our geodesic equations, eliminating the constraint:

$$\ddot{X} = \int_M \lambda(a) \operatorname{grad} f_a(x)\, da = \nabla P \circ X$$

$$= \left(\nabla \Delta^{-1} \operatorname{tr}(\nabla u)^2\right) \circ X$$

$$= \left(\nabla \Delta^{-1} \operatorname{div}(u \cdot \nabla u)\right) \circ X$$

$$\ddot{X} = (\partial_t u + u \cdot \nabla u) \circ X$$

Euler equations

$$\boxed{\mathbb{P}(\partial_t u + u \cdot \nabla u) = 0, \qquad \mathbb{P} = I - \nabla \Delta^{-1} \operatorname{div}}$$

Second fundamental form of SDiff

Recall the second fundamental form of a submanifold

$$\boxed{\mathrm{II}_x(u,v) = -(J_f(x))^{-1}\, \mathrm{Hess} f_x(u,v) \cdot \nabla f(x)}$$
$u, v \in T_x S$

Introducing $\mathbb{P}_x$, the orthogonal projection onto $T_x S$ and $\mathbb{Q}_x$, the projection onto $(T_x S)^\perp$, we found

$$\mathbb{Q}_x f = (\nabla \Delta^{-1} \operatorname{div} f) \circ X, \quad \mathbb{P}_x = id_x - \mathbb{Q}_x$$

We have, by analogy

$$\mathrm{II}_x(u,v) = \mathbb{Q}(\nabla_u v) \circ X, \quad u, v \text{ div-free v.f.}$$

Note that since $\nabla_u v - \nabla_v u = [u,v]$ is divergence free provided u, v are, we have that $\mathrm{II}_x(u,v)$ is symmetric and u, v.

general geometric fact: if two vector fields are tangent to a submanifold, so is their commutator.

Exersize: Show that Euler can be said as

Motion which preserves volume to second order.

Solution: Consider the linear flow

$$\psi_t(a) = a + t\,u(a) - \frac{t^2}{2}\omega$$

$$\nabla\psi_t(a) = I + t\,\nabla u(a) - \frac{t^2}{2}\nabla\omega$$

$$\det(\nabla\psi_t(a)) = \det\left(I + t\nabla u - \frac{t^2}{2}\omega\right)$$

$$= 1 + t\frac{d}{dt}\det\left(I + t\nabla u - \frac{t^2}{2}\nabla\omega\right)\Big|_{t=0} + \frac{t^2}{2}\frac{d^2}{dt^2}\det\left(I + t\nabla u - \frac{t^2}{2}\nabla\omega\right)\Big|_{t=0} + \mathcal{O}(t^3)$$

$$= 1 + t\,\mathrm{div}\,u - \frac{t^2}{2}\left(\mathrm{div}\,\omega - (\mathrm{div}\,u)^2 + \mathrm{tr}(\nabla u)^2\right) + \mathcal{O}(t^3)$$

If $\mathrm{div}\,u = 0$ and $\omega = \nabla p$ where $-\Delta p = \mathrm{tr}(\nabla u)^2$, volume is preserved to second order.

Here, we used:

$$\frac{d}{dt}A^{-1} = -A^{-1}\frac{d}{dt}A\,A^{-1}$$

$$\frac{d}{dt}\det A = \mathrm{tr}\left(A^{-1}\frac{d}{dt}A\right)\det A$$

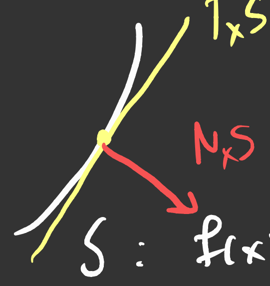
$$\frac{d^2}{dt^2}\det A = -\mathrm{tr}\left(A^{-1}\frac{d}{dt}A\,A^{-1}\frac{d}{dt}A\right)\det A + \mathrm{tr}\left(A^{-1}\frac{d^2}{dt^2}A\right)\det A$$
$$+\left(\mathrm{tr}\left(A^{-1}\frac{dA}{dt}\right)\right)^2\det A$$

Sectional Curvatures for codimension one surfaces

$$\ddot{x} = -\lambda \operatorname{grad} f$$

$$f(x) = 0$$

$$\nabla f(x) \neq 0 \quad \forall x \in S$$



implicit function theorem gives a surface

$x = \bar{x} + \xi, \quad \lambda = \bar{\lambda} + \mu$

$$\ddot{\xi} = -\bar{\lambda} \operatorname{Hess} f \cdot \xi - \mu \operatorname{grad} f(\bar{x})$$

$$\xi \cdot \operatorname{grad} f(\bar{x}) = 0$$

Now, we convert our linearized system into the invariant form.
It is sufficient to calculate the second covariant derivative along our geodesic.

$$\frac{\nabla}{dt}\frac{\nabla}{dt}\xi = \frac{\nabla}{dt} P_{\bar{x}} \dot{\xi}$$

$$= P_{\bar{x}} \frac{d}{dt}\left(\dot{\xi} - \frac{\dot{\xi} \cdot \nabla f_{\bar{x}}}{|\nabla f|_{\bar{x}}} \frac{\nabla f_{\bar{x}}}{|\nabla f|_{\bar{x}}} \right)$$

$$= P_{\bar{x}} \ddot{\xi} - P_{\bar{x}}\left(\frac{\dot{\xi} \cdot \nabla f_{\bar{x}}}{|\nabla f|_{\bar{x}}} \frac{\operatorname{Hess} f_{\bar{x}}(\dot{\bar{x}}, \cdot)}{|\nabla f|_{\bar{x}}} \right)$$

since $P_x(\text{anything } \nabla f) = 0$

$$\frac{\nabla}{dt}\frac{\nabla}{dt}\xi = P_{\bar{x}}\left(\ddot{\xi} - \frac{\dot{\xi}\cdot\nabla f_{\bar{x}}}{|\nabla f_{\bar{x}}|}\frac{\mathrm{Hess}\, f_{\bar{x}}(\dot{\bar{x}},\cdot)}{|\nabla f_{\bar{x}}|}\right)$$

$\ddot{\xi} = -\bar{\lambda}\,\mathrm{Hess}\, f\cdot\xi$
$- \mu \,\mathrm{grad} f(\bar{x})$

$$= -P_{\bar{x}}\left(\bar{\lambda}\,\mathrm{Hess}\, f_{\bar{x}}(\xi,\cdot) + \frac{\dot{\xi}\cdot\nabla f_{\bar{x}}}{|\nabla f_{\bar{x}}|}\frac{\mathrm{Hess}\, f_{\bar{x}}(\dot{\bar{x}},\cdot)}{|\nabla f_{\bar{x}}|}\right)$$

$$= -\bar{\lambda}\, P_{\bar{x}}\,\mathrm{Hess} f_{\bar{x}}(\xi,\cdot) - \frac{\dot{\xi}\cdot\nabla f}{|\nabla f|^2}\, P_{\bar{x}}\,\mathrm{Hess} f_{\bar{x}}(\dot{\bar{x}},\cdot)$$

where, recall

$$\bar{\lambda} = \frac{\mathrm{Hess} f(\dot{\bar{x}},\dot{\bar{x}})}{|\nabla f(\bar{x})|^2}$$

This is it! we found the curvature operator:

$$R(\xi,\dot{\bar{x}})\dot{\bar{x}} := -\frac{\nabla}{dt}\frac{\nabla}{dt}\xi$$

Let us compute the sectional curvatures

since $\xi \in T_x S$
we can omit P_x

$$C_{\xi,\eta} = \left(R(\xi,\eta)\eta,\xi\right)$$

$$= \bar{\lambda}\,\mathrm{Hess} f_{\bar{x}}(\xi,\xi) + \frac{\dot{\xi}\cdot\nabla f(\bar{x})}{|\nabla f(\bar{x})|^2}\,\mathrm{Hess} f_{\bar{x}}(\dot{\eta},\xi)$$

What about $\dot{\xi} \cdot \nabla f(\bar{x})$?

Recall that $\xi \cdot \nabla f(\bar{x}) = 0$. Differentiating, we find

$$\dot{\xi} \cdot \nabla f(\bar{x}) = -\mathrm{Hess} f_{\bar{x}}(\dot{x}, \xi).$$

Thus, using this and our $\bar{\lambda}$ expression, we have.

$$C_{\xi,\eta} = (R(\xi,\eta)\eta, \xi)$$

$$= \bar{\lambda}\, \mathrm{Hess} f_{\bar{x}}(\xi,\xi) + \frac{\dot{\xi}\cdot\nabla f(\bar{x})}{|\nabla f(\bar{x})|^2} \mathrm{Hess} f_{\bar{x}}(\dot{\eta}, \xi)$$

$$= \frac{\mathrm{Hess} f_{\bar{x}}(\eta,\eta)\, \mathrm{Hess} f_{\bar{x}}(\xi,\xi) - |\mathrm{Hess} f_{\bar{x}}(\eta,\xi)|^2}{|\nabla f(\bar{x})|^2}$$

Gauss equation!

$$\mathrm{II}(u,v) = -\frac{\mathrm{Hess} f(u,v)}{\|\nabla f\|^2} \nabla f(x)$$

We proved

Lemma: The sectional curvatures $C_{\xi,\eta}$ at $x \in S$ is

$$C_{\xi,\eta} = \frac{\overbrace{\mathrm{Hess} f_x(\eta,\eta)\, \mathrm{Hess}_x f(\xi,\xi)}^{\mathrm{II}(\eta,\eta)\cdot \mathrm{II}(\xi,\xi)} - \overbrace{\mathrm{Hess} f_x(\eta,\xi)^2}^{|\mathrm{II}(\eta,\xi)|^2}}{\|\xi \wedge \eta\|^2\, |\nabla f(x)|^2}$$

for any $\xi, \eta \in T_x S$ provided $\|\xi \wedge \eta\| \neq 0$

Codimension-k submanifold

scalar product in $\mathbb{R}^k$

$$C_{\xi\eta} = \frac{\Lambda_x(\xi,\xi)\cdot\Lambda_x(\eta,\eta) - \Lambda_x(\xi,\eta)\cdot\Lambda_x(\xi,\eta)}{\|\xi\wedge\eta\|^2}$$

where Λ is connected to the reaction forces: it is a vector-valued bilinear form $\Lambda : (\xi;\eta) \mapsto \mathbb{R}^k$

$$\Lambda_x(\xi,\eta) = J_f^{-1}\,\mathrm{Hess}\,f_x(\xi,\eta)$$

where $(J_f)_{ij} = \nabla f_i \cdot \nabla f_j \; : \mathbb{R}^k \to \mathbb{R}^k$

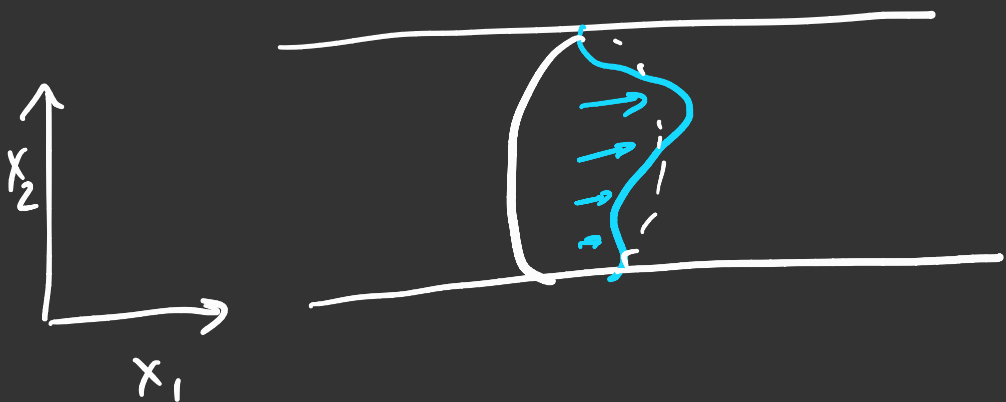
Volume preserving diffeomorphisms: By analogy, the sectional curvatures of SDiff are

$$\boxed{C_{\xi,\eta} = \int_M \left[\mathbb{Q}(\xi\cdot\nabla\xi)\,\mathbb{Q}(\eta\cdot\nabla\eta) - |\mathbb{Q}(\xi\cdot\nabla\eta)|^2\right] dx}$$

Jacobi equation $\frac{\nabla}{dt}\frac{\nabla}{dt}\xi + R(\xi,\dot{x})\dot{x} = 0$

Curvature determines (Lagrangian) stability!

Arnold considered shear flows



duct
$\eta(x) = (v(x_2), 0, 0)$

or

pipe
$\eta(x) = (v(x_2,x_3), 0, 0)$

This is an exact solution of Euler with pressure zero. Indeed:

$$\eta\cdot\nabla\eta = v(x_2)\partial_1\begin{pmatrix} v(x_2) \\ 0 \\ 0 \end{pmatrix} = 0.$$

Why? The motion of all particles are geodesics in enveloping space. Such are totally geodesic.

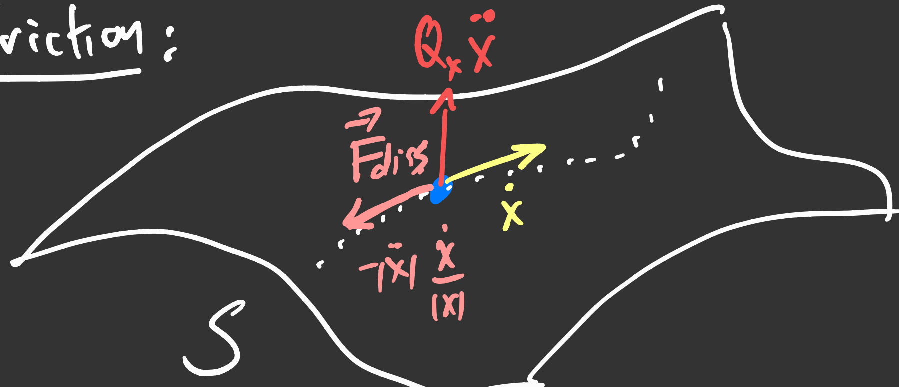
Thus $Q(\eta\cdot\nabla\eta) = 0$, so

$$C_{\xi,\eta} = -\int_M |Q(\eta\cdot\nabla\xi)|^2\,dx \leq 0$$

So, all curvatures non-positive (negative in most directions).
zero only if ξ also parallel flow

Lagrangian Unstable!

Incorporating Friction:

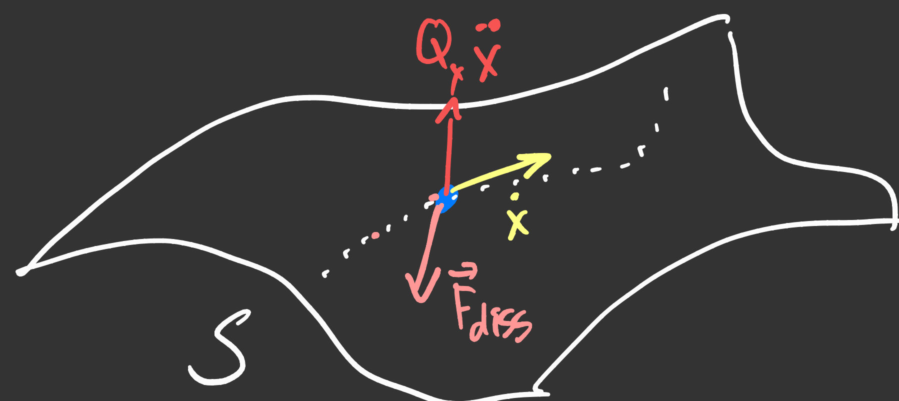


Mechanical friction: proportional to the normal (constraining) force, directed opposite to motion:

Euler with mechanical friction:

$$\ddot{X} = -\nabla p \circ X - \|\nabla p\|_{L^2} \frac{\dot{X}}{\|\dot{X}\|_{L^2}}$$

Viscous friction



Navier-Stokes equation: fix a number $\nu > 0$. Then

$$\ddot{X} = -\nabla p \circ X - \nu \int_M (J_f(x))_{\circ, a'} \dot{X}(a')\, da$$

where

$$\int (J_f(x))_{\circ, a'} \dot{X}(a')\, da' = \Delta(\dot{X} \circ X^{-1}) \circ X$$

Note: It is friction since

$$(\dot{X}, \Delta(\dot{X} \circ X^{-1}) \circ X) = -|\nabla(\dot{X} \circ X^{-1})|^2_{L^2}.$$

Conservation laws: Noether's theorem.

Recall the action $S(\{X\}_{t\in[0,T]}) = \frac{1}{2}\int_0^T \int_M |\dot{X}(t,a)|^2 \, da\, dt$

Principle: symmetry of action corresponds to conserved quantities for extremal trajectories

Energy (time translation invariance):

$$S(\{X\}_{t\in[0,T]}) = \frac{1}{2}\int_\varepsilon^{T+\varepsilon} \int_M |\dot{X}(t-\varepsilon, a)|^2 \, da\, dt$$

$$0 = \frac{d}{d\varepsilon} S\Big|_{\varepsilon=0} = \frac{1}{2}\int_M |\dot{X}(T,a)|^2 da - \frac{1}{2}\int_M |\dot{X}(0,a)|^2 da$$

$$+ \int_0^T \int_M \cancel{\underbrace{\ddot{X}(t,a)}_{\in (T_X S)^\perp} \cdot \underbrace{\dot{X}(t,a)}_{\in T_X S}}^{\,0} \, da\, dt$$

Momentum (space translation invariance, provided $M - \varepsilon V = M$):

$$S(\{X\}_{t\in[0,T]}) = \frac{1}{2}\int_0^T \int_{M-\varepsilon V} |\dot{X}(t, a+\varepsilon V)|^2 \, da\, dt$$

$$0 = \frac{d}{d\varepsilon} S\Big|_{\varepsilon=0} = \int_0^T \int_M \dot{X}(t,a) \cdot \nabla_V \dot{X}(t,a)\, da\, dt$$

Kelvin theorem (particle relabelling symmetry)

$$S\left(\{X\}_{t\in[0,T]}\right) = \frac{1}{2}\int_0^T\int_M |\dot{x}(t,\phi^\varepsilon(a))|^2\, da\, dt$$

div $\xi = 0$

$$0 = \frac{d}{d\varepsilon}S\Big|_{\varepsilon=0} = \int_0^T\int_M \dot{x}(t,a)\cdot \xi(a)\cdot\nabla\dot{x}(t,a)\, da\, dt$$

$$= \int_M \dot{X}_T\circ X_T^{-1}\cdot X_T^*\xi\, da - \int_M \dot{X}_0\circ X_0^{-1}\cdot X_0^*\xi\, da$$

$$+ \int_0^T\int_M \underbrace{\cancel{\ddot{X}_t\circ X_t^{-1}}}_{\in (T_xS)^\perp}\cdot \underbrace{X_t^*\xi}_{\in T_xS}\, da\, dt$$

0

since $X_t^*\xi$ is divergence-free provided ξ is

Thus

$$\int_M u_{t_1}\cdot w_{t_1}\, da = \int_M u_{t_2}\cdot w_{t_2}\, da$$

for any $w_t = X_t^*\xi$, e.g. any w solving

$$\partial_t w + [u,w] = 0$$
$$w|_{t=0} = \xi$$

one conservation law per parameter. parameters are vp. diffeos, in correspondence with div free vector fields. So you have as many cns. as those.

<u>Remark</u>: (Pushforward of a divergence free vector is div-free):

One way to see this is to note that the flow of the pushforward v.f. is related by conjugation

$$\Phi_{\Phi_* \xi} = \Phi \circ \Phi_\xi \circ \Phi^{-1}$$

Thus, if ξ is divergence-free, its flow Φ_ξ is volume preserving so $\Phi \circ \Phi_\xi \circ \Phi$ is also. As such, $\Phi_* \xi$ must be divergence-free by Liouville theorem.

Another way to see, by direct computation, is:

$$\operatorname{div}(\phi_* \xi) = \operatorname{div}\left(\xi \circ \phi^{-1} \cdot (\nabla \phi) \circ \phi^{-1}\right)$$

$$= \nabla \xi \circ \phi^{-1} \overbrace{\nabla \phi^{-1} (\nabla \phi) \circ \phi^{-1}}^{I} + \xi \circ \phi^{-1} \cdot \nabla \phi^{-1} (\nabla^2 \phi) \circ \phi^{-1}$$

$$= (\cancelto{0}{\operatorname{div} \xi}) \circ \phi^{-1} + \xi \circ \phi^{-1} \cdot \operatorname{tr}\left(\cancelto{0}{\nabla \phi^{-1} (\nabla^2 \phi) \circ \phi^{-1}}\right)$$

as

$$= 0$$

$$0 = \nabla \det \nabla \phi = \operatorname{tr}\left((\nabla \phi)^{-1} \nabla^2 \phi\right) \det \nabla \phi$$

as $(\nabla \phi)^{-1} = \nabla \phi^{-1} \circ \phi$.

$$\Longrightarrow \operatorname{tr}\left((\nabla \phi)^{-1} \nabla^2 \phi\right) = 0$$

$$\Rightarrow \operatorname{tr}\left(\nabla \phi^{-1} \nabla^2 \phi \circ \phi^{-1}\right) = 0$$

Kelvin theorem for loops.

We proved that

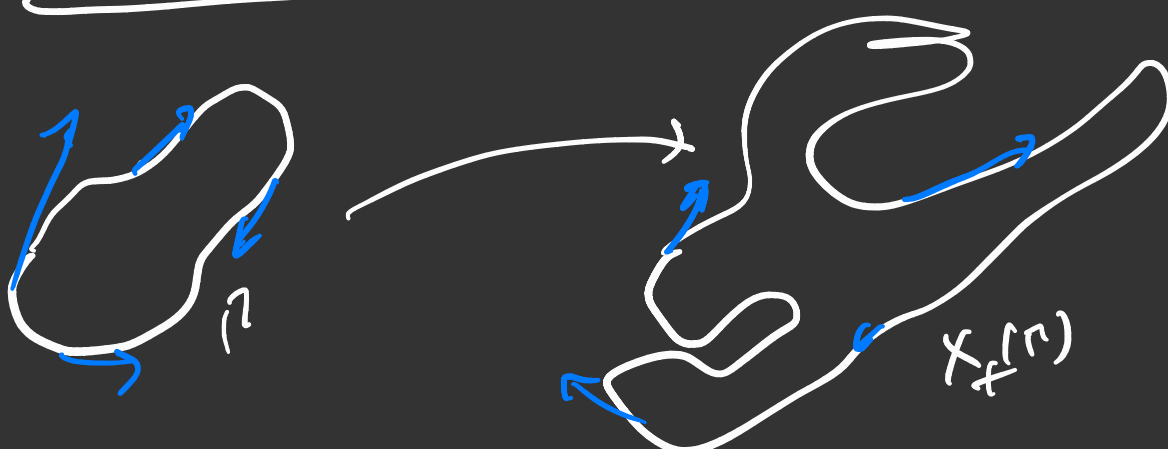
$$\int_M u_{t_1}\cdot w_{t_1}\,da = \int_M u_{t_2}\cdot w_{t_2}\,da \qquad \text{for any } w_t = X_t^*\xi$$

Taking $\xi(a) := \oint_\Gamma \delta_{a'}(a)\cdot dl_\Gamma(a')$, we have

$$(X_t^*\xi)(a) = \oint_{X_t(\Gamma)} \delta_{a'}(a)\cdot dl_{X_t(\Gamma)}(a')$$

Thus we find that

$$\boxed{\oint_{X_t(\Gamma)} u_t\cdot dl_{X_t(\Gamma)} = \oint_\Gamma u_0\cdot dl_\Gamma}$$



<u>Vorticity</u>: Recall Kelvin's theorem:

$$\oint_{X_t(\Gamma)} u_t \cdot dl_{X_t(\Gamma)} = \oint_{\Gamma} u_0 \cdot dl_{\Gamma}$$

We find, by Stoke's theorem, the flux of vorticity through a comoving bounding surface is preserved.

$$\boxed{\oint_{X_t(S)} \omega \cdot \hat{n} \, d\sigma_{X_t(S)} = \int_S \omega_0 \cdot \hat{n} \, d\sigma_S}$$

Taking an infinitesimal loop, we arive at the vorticity transport law $\omega_t = X_t^* \omega_0$

E.g.

$$\partial_t \omega + u \cdot \nabla \omega = 0 \quad \text{in 2d} \quad \omega = \nabla^\perp \cdot u$$

$$\partial_t \omega + [u, \omega] = 0 \quad \text{in 3d} \quad \omega = \operatorname{curl} u.$$

$$\partial_t \omega + u \cdot \nabla \omega + \omega \nabla u^T + \nabla u \, \omega = 0 \quad \text{in n-d} \quad \omega = \frac{\nabla u - \nabla u^T}{2}$$

Kelvin's theorem then shows helicity $H = \int_M u_t \cdot \omega_t \, dx$ is conserved for Euler.

Well-posedness

Consider the evolution of ∇u

$$\partial_t \nabla u + (u\cdot\nabla)\nabla u + (\nabla u)^2 = -\nabla^2 p$$
$$\Delta p = \mathrm{tr}(\nabla u)^2$$

Observe that ∇u should satisfy

$$\frac{d}{dt}|\nabla u|_Y \leq C|\nabla u|_Y^2$$

when Y encodes enough regularity. Namely

1) Y is an algebra, e.g. $f,g \in Y \Rightarrow fg \in Y$

(true for $Y = C^{k,\alpha}$ $k \geq 0$ or $Y = H^s$ $s > d/2$)

2) Y is compatible with the Neumann problem

$$|\nabla^2 p|_Y \leq C|\nabla u|_Y^2.$$

This means avoid endpoint space like $W^{k,p}$ $p=1$, $p=\infty$

3) The solution to transport $(\partial_t + u\cdot\nabla)$ with $\nabla u \in Y$ is wellposed in Y. Want u to be Lipshitz. Generally not an issue at this regularity.

One can take $Y = C^{1,\alpha}$, $\alpha \in (0,1)$. Requires

Lemma: Let $f \in C^\alpha(\bar{M})$, $\int_M f\,dx = 0$. Then the unique solution to

$$\begin{aligned} \Delta g &= f \quad &\text{in } M \\ \partial_n g &= 0 \quad &\text{on } \partial M \end{aligned}$$

satisfies

$$|g|_{C^{\alpha+2}} \leq C(\alpha, d)\, |f|_{C^\alpha}$$

Proof: by potential theory estimates

Application: $-\Delta p = \mathrm{tr}(\nabla u)^2 \Rightarrow |D^2 p|_{C^\alpha} \leq C\, |\nabla u|_{L^\infty} |\nabla u|_{C^\alpha}$
by the algebra property

Then we can obtain

$$\|\nabla u(t)\|_{C^\alpha} \leq \exp\left(C(\alpha, d) \int_0^t |\nabla u(s)|_{L^\infty} ds \right) \|\nabla u_0\|_{C^\alpha}$$

Let $T_* > 0$ be the maximal existence time. Either

- $T_* > 0$
- $\int_0^{T_*} |\nabla u|_{L^\infty} dt = \infty$

Construction by a fixed point

$$\partial_t u_n + u_{n-1} \cdot \nabla u_n = \nabla P_n$$
$$\nabla \cdot u_n = 0$$

Obtain $\nabla u \in C_t^0 C_x^\alpha$ and also $u \in C_t^1(\bar{M} \times [0,T])$.

<u>Uniqueness</u>. Let $w = u - v$. Then

$$\partial_t w + u \cdot \nabla w + \nabla u \cdot w = \nabla p - w \cdot \nabla w$$

Thus

$$\frac{d}{dt} \frac{1}{2} |w|_{L^2}^2 = \int_M w \cdot \nabla u \cdot w \, dx \leq |w|_{L^2}^2 |\nabla u|_{L^\infty}$$

Thus, if u is $L^1_t \mathrm{Lip}_x$ then $w_0 = 0 \Rightarrow w(t) = 0$.

Thus $u(t) = S_t(u_0)$, $S_t : C^{1,\alpha}(M) \to C^{1,\alpha}(M)$ defines an infinite dimensional dynamical system.

<u>Blowup</u>: In 2d, the equations are globally wellposed in $C^{1,\alpha}$ spaces (or Sobolev)

Elgindi (2019) showed that 3d Euler blows up in finite time with C^α vorticity, α small.