MAT 307, Multivariable Calculus with Linear Algebra – Fall 2024
Supplemental Material

1 Limits

Definition 1. Let $P \in \mathbb{R}^n$ be a point, the **open ball of radius** $\epsilon > 0$ **about** P is the set

$$B_\epsilon(P) = \{Q \in \mathbb{R}^n \mid \|\overrightarrow{PQ}\| < \epsilon\}.$$

The **closed ball of radius** $\epsilon > 0$ **about** P is the set

$$\overline{B_\epsilon(P)} = \{Q \in \mathbb{R}^n \mid \|\overrightarrow{PQ}\| \leq \epsilon\}.$$

Definition 2. A subset $A \subset \mathbb{R}^n$ is called **open** if for every $P \in A$, there is an $\epsilon > 0$ such that the open ball of radius ϵ about P is entirely contained in A, i.e. $B_\epsilon(P) \subset A$. We say B is **closed** if the **complement** of B, B^c, is open. A **neighborhood** of P is an open set containing P.

An open set is a union of open balls. Open balls are open and closed balls are closed.

Example 1. $(0, 1), [0, 1], [0, 1) \subset \mathbb{R}$ are respectively open, close, and neither.

Definition 3. Let $B \subset \mathbb{R}^n$, we say $P \in \mathbb{R}^n$ is a **boundary point** (a.k.a. **limit point**) if for every $\epsilon > 0$, the intersection

$$B_\epsilon(P) \cap B \neq \emptyset, \qquad \text{and} \qquad B_\epsilon(P) \cap B^c \neq \emptyset$$

Boundary point of B has no neighborhood entirely in or out of B.

Example 2. 0 is a boundary point of $\{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{R}$.

Lemma 1. *$B \subset \mathbb{R}^n$ is closed if and only if B contains all of its boundary points.*

Proof. First assume B is closed, then B^c is open. Suppose there is a boundary point P of B such that P is not in B, so $P \in B^c$. Since B^c is open, there is an $\epsilon > 0$ such that $B_\epsilon(P) \subset B^c$, i.e. $B_\epsilon(P) \cap B = \emptyset$, therefore, P is not a boundary point of B. The contradiction shows that boundary points of B must lie in B.

Conversely, assume B is not closed, i.e. B^c is not open, so there is a point $P \in B^c$, such that for all $\epsilon > 0$, $B_\epsilon(P) \cap B \neq \emptyset$. This P is therefore a boundary point of B, which is in B^c. Therefore, B does not contain all its boundary points. □

Definition 4. Let $A \subset \mathbb{R}^n$, and let P be a point in A or a boundary point of A. Suppose that $f : A \to \mathbb{R}^m$ is a function, we say that f approaches L as Q approaches P and write

$$\lim_{Q \to P} f(Q) = L,$$

if for every $\epsilon > 0$, we can find $\delta > 0$ such that for all $Q \in B_\delta(P) \cap A$, $Q \neq P$, $f(Q) \in B_\epsilon(L)$. We call L the **limit**.

The smaller ϵ is, the smaller δ has to be.

Proposition 2. *Let f and g be two functions from A to $\mathbb{R}^m$. Let $\lambda \in \mathbb{R}$ be a scalar. If P is a limit point of A or $P \in A$ and*

$$\lim_{Q\to P} f(Q) = L \qquad \textit{and} \qquad \lim_{Q\to P} g(Q) = M,$$

then

1. $\lim_{Q\to P}(f+g)(Q) = L + M$.
2. $\lim_{Q\to P}(\lambda f)(Q) = \lambda L$.
3. *If $m = 1$, then* $\lim_{Q\to P}(fg)(Q) = LM$.
4. *If $m = 1$ and $M \neq 0$, then* $\lim_{Q\to P}(f/g)(Q) = L/M$.

Proof. We show 1 and leave the rest to reader. Suppose $\epsilon > 0$, since L and M are limits, there exist δ_1 and δ_2, such that for $0 < \|Q - P\| < \delta_1$, $Q \in A$, $\|f(Q) - L\| < \epsilon/2$, and if $0 < \|Q - P\| < \delta_2$, and $Q \in A$, $\|g(Q) - L\| < \epsilon/2$.

Let $\delta = \min(\delta_1, \delta_2)$. For all $Q \in A$ where $\|Q - P\| < \delta$,

$$\|(f+g)(Q)-L-M\| = \|f(Q)-L+(g(Q)-M)\| \leq \|f(Q)-L\|+\|g(Q)-M\| < \epsilon,$$

where we used triangle inequality to obtain the first inequaility. □

Definition 5. Let $A \subset \mathbb{R}^n$, and $P \in A$. If $f : A \to \mathbb{R}^m$ is a function, then we say that f is **continuous** at P if $\lim_{Q\to P} f(Q)$ exists and further more

$$\lim_{Q\to P} f(Q) = f(P).$$

f is said to be continuous if it is continuous at every point in A.

Intuitively, if $n = 1$, f is continuous when the graph of f can be plotted without lifting your pen.

As a consequence of Proposition 2, we have

Theorem 3. *Let f and g be two functions from $A \subset \mathbb{R}^n$ to $\mathbb{R}^m$. Let $\lambda \in \mathbb{R}$ be a scalar. Suppose f and g are both continuous at $P \in A$. then*

1. *$f + g$ is continuous at P, $(f + g)(P) = f(P) + g(P)$.*
2. *λf is continuous at P, $(\lambda f)(P) = \lambda f(P)$.*
3. *If $m = 1$, then fg is continous at P, $(fg)(P) = f(P)g(P)$.*
4. *If $m = 1$ and $g(P) \neq 0$, then $1/g$ is continuous at P, $(1/g)(P) = 1/g(P)$.*

5. *If* $f(x) = (f_1(x), \dots, f_m(x))$, *then* f *is continuous at* P *if and only if each of the real-valued functions* $f_1, \dots f_m$ *is continuous at* P.

Proposition 4. *Let* $A \subset \mathbb{R}^n$ *and* $B \subset \mathbb{R}^m$, $f : A \to B$ *and* $g : B \to \mathbb{R}^p$. *Suppose*

$$\lim_{Q \to P} f(Q) = L \qquad \text{and} \lim_{M \to L} g(M) = K.$$

then

$$\lim_{Q \to P} (g \circ f)(Q) = K.$$

Hence, composition of continuous functions is continuous.

Proof. Let $\epsilon > 0$, there exists $\delta > 0$ such that if $\|M - L\| < \delta$, and $M \in B$, then $\|g(M) - K\| < \epsilon$. Given $\delta > 0$, there exist $\eta > 0$, such that $\|Q - P\| < \eta$ and $Q \in A$, $\|f(Q) - L\| < \delta$. Hence, for $\|Q - P\| < \eta$ and $Q \in A$, we have $M = f(Q) \in B$ and $\|M - L\| < \delta$, so

$$\|(g \circ f)(Q) - K\| = \|g(f(Q)) - K\| = \|g(M) - K\| < \epsilon.$$

□

Example 3. If $f : \mathbb{R}^n \to \mathbb{R}$ is a polynomial function, then f is continuous.

Example 4. $\sin(1/x)$ has no limit at $x = 0$, so it is not continuous.

Example 5. Does the limit

$$\lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x - y}$$

exist? Here the domain of f is

$$A = \{(x, y) \in \mathbb{R}^2 \mid x \neq y\}.$$

A is open. $(0, 0)$ is a limit point of A. For $(x, y) \in A$,

$$\frac{x^2 - y^2}{x - y} = x + y, \qquad \lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x - y} = \lim_{(x,y) \to (0,0)} x + y = 0.$$

The limit does exist since both x and y are continuous functions in $\mathbb{R}^2$.

Example 6. Does

$$\lim_{(x,y) \to (0,0)} \frac{y}{x^2 + y}$$

exist?
Let (x, y) approach $(0, 0)$ along the line $y = kx$, $k \neq 1$,

$$\lim_{x \to 0} \frac{kx}{x^2 + kx} = \lim_{x \to 0} \frac{k}{x + k} = 1.$$

Along the line $y = 0$,

$$\lim_{x \to 0} \frac{0}{x^2} = 0 \neq 1.$$

So the limit does not exist.

Example 7. Does

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$$

exist?

Let (x, y) approach $(0, 0)$ along the line $y = kx$,

$$\lim_{x\to 0} \frac{x^3}{x^2+k^2x^2} = \lim_{x\to 0} \frac{x}{1+k^2} = 0.$$

To prove that the limit exist, we need to show that for any $\epsilon > 0$, there exists $\delta = \epsilon$, such that for all $(x, y) \in B_\delta(0, 0)$, $\mid \frac{x^3}{x^2+y^2} \mid < \epsilon$.

$$\left| \frac{x^3}{x^2+y^2} - 0 \right| \leq \left| \frac{x^3}{x^2} \right| = \mid x \mid < \delta = \epsilon.$$

So the limit exists and is 0.

Alternatively, use the **squeezing method**: if $g \leq f \leq h$, g and h are both have the same limit at P, then f has the same limit at P.

$$0 \leq \left| \frac{x^3}{x^2+y^2} \right| \leq \left| \frac{x^3}{x^2} \right| = |x| \to 0 \text{ as } (x, y) \to (0, 0).$$

Alternatively, can use **polar coordinates**:

$$\frac{x^3}{x^2+y^2} = \frac{r^3\cos^3\theta}{r^2} = r\cos^3\theta \to 0, \qquad r \to 0$$

To show that a function f does not have a limit at P, you need to find two paths toward P that have different limits, or a path where denominator = 0. To show that a function f has a limit at P, you can use **Proposition 2, 4**, the squeeze method, polar coordinates, or $\epsilon - \delta$ argument.

Example 8. Does

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

exist?

Along lines $y = kx$,

$$\frac{x^2y}{x^4+y^2} = \frac{kx^3}{x^4+k^2x^2} = \frac{kx}{x^2+k^2} \to 0, \qquad x \to 0$$

But along the curve $y = kx^2$,

$$\frac{x^2y}{x^4+y^2} = \frac{kx^4}{x^4+k^2x^4} = \frac{k}{1+k^2}$$

Limit does not exist!

2 Differentiability

Recall that the derivative of a function in single-variable calculus represents the best linear approximation of the function, it is the slope of the tangent line to the graph of the function. Intuitively, a differentiable function $f : \mathbb{R} \to \mathbb{R}$ is smooth.

Definition 6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $a \in \mathbb{R}$ be a real number. f is **differentiable at** a, with derivative $f'(a) \in \mathbb{R}$, if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a), \qquad \lim_{x \to a} \frac{f(x) - f(a) - f'(a)(x - a)}{x - a} = 0.$$

Definition 7. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function where

$$f(x_1, \cdots, x_n) = (y_1, \cdots, y_m), \qquad y_i = f_i(x_1, \cdots, x_n), \qquad f_i : \mathbb{R}^n \to \mathbb{R}$$

Let $P \in \mathbb{R}^n$ be a point. f is **differentiable at** P, with derivative the $m \times n$ matrix T if

$$\lim_{Q \to P} \frac{f(Q) - f(P) - T\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = 0,$$

we write $Df(P) = T$.

Along the line determined by the standard basis element $\hat{e}_j$ (a column vector), let $\overrightarrow{PQ} = h\hat{e}_j$, where $h > 0$, then

$$\begin{aligned} \frac{f(Q) - f(P) - T(h\hat{e}_j)}{\|\overrightarrow{PQ}\|} &= \frac{f(Q) - f(P) - T(h\hat{e}_j)}{h} \\ &= \frac{f(Q) - f(P) - hT\hat{e}_j}{h} \\ &= \frac{f(Q) - f(P)}{h} - T\hat{e}_j \end{aligned}$$

Taking the limit of $h \to 0$, $T\hat{e}_j$ is the j-th column of T, where

$$T\hat{e}_j = \lim_{h \to 0} \frac{f(P + h\hat{e}_j) - f(P)}{h}.$$

$P = (p_1, p_2, \cdots, p_n)$, $f(P + h\hat{e}_j) - f(P)$ is a column vector whose entry in the i-th row is given by

$$f_i(P + h\hat{e}_j) - f_i(P) = f_i(p_1, p_2, \cdots, p_{j-1}, p_j + h, p_{j+1}, \cdots, p_n) - f_i(p_1, p_2, \cdots, p_n).$$

Definition 8. The **partial derivatives** of f_i at P with respective to x_j is

$$T_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_P = \lim_{h \to 0} \frac{f_i(P + h\hat{e}_j) - f_i(P)}{h},$$

and this is also the i-th entry of $T\hat{e}_j$.

Example 9. Let $f : U \subset \mathbb{R}^3 \to \mathbb{R}^2$ be the function

$$f(x, y, z) = (xy + z\log(xy), x\sin(yz)).$$

$U \subset \mathbb{R}^3$ is the region where x and y are both positive. If f is differentiable at P, then the derivative of f at P is given by the 2×3 matrix of partial derivatives,

$$Df(P) = \begin{bmatrix} y + \frac{z}{x} & x + \frac{z}{y} & \log(xy) \\ \sin(yz) & xz\cos(yz) & xy\cos(yz) \end{bmatrix}.$$

Theorem 5. *Let $f : U \to \mathbb{R}^m$ be a function where $U \subset \mathbb{R}^n$ is open. If f is differentiable at P, then f is continuous at P.*

Proof. Suppose f is differentiable at P, then there is an $m \times n$ matrix T such that $Df(P) = T$.

$$\lim_{Q \to P} \frac{\|f(Q) - f(P) - T\overrightarrow{PQ}\|}{\|\overrightarrow{PQ}\|} = 0,$$

and *a fortiori*,

$$\lim_{Q \to P} \|f(Q) - f(P) - T\overrightarrow{PQ}\| = 0.$$

$$\begin{aligned} \|f(Q) - f(P)\| &= \|f(Q) - f(P) - T\overrightarrow{PQ} + T\overrightarrow{PQ}\| \\ &\leq \|f(Q) - f(P) - T\overrightarrow{PQ}\| + \|T\overrightarrow{PQ}\| \\ &\leq \|f(Q) - f(P) - T\overrightarrow{PQ}\| + K\|\overrightarrow{PQ}\| \end{aligned}$$

where if $\vec{t_i}$ is the i-th row of T, then K is defined as follows:

$$\begin{aligned} \|T\overrightarrow{PQ}\|^2 &= \sum_{i=1}^{m} (\vec{t_i} \cdot \overrightarrow{PQ})^2 \\ &\leq \sum_{i=1}^{m} \left(\|\vec{t_i}\|^2 \|\overrightarrow{PQ}\|^2\right) \text{ by Cauchy-Schwarz} \\ &= \left(\sum_{i=1}^{m} \|\vec{t_i}\|^2\right) \|\overrightarrow{PQ}\|^2 \\ &= K^2 \|\overrightarrow{PQ}\|^2 \end{aligned}$$

As $Q \to P$, $\|f(Q) - f(P) - T\overrightarrow{PQ}\| + K\|\overrightarrow{PQ}\| \to 0$, so $f(Q) \to f(P)$. f is continuous at P. □

Definition 9. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. Then the derivative of f at P, $Df(P)$ is a row vector, called the **gradient** of f, denoted by $(\nabla f)|_P$ or *grad*(f),

$$(\nabla f)|_P = \left(\frac{\partial f}{\partial x_1}\bigg|_P, \frac{\partial f}{\partial x_2}\bigg|_P, \cdots, \frac{\partial f}{\partial x_n}\bigg|_P \right).$$

If $f : \mathbb{R}^n \to \mathbb{R}^m$, each row of the derivative matrix $Df(P)$ is a gradient at P.

The point $(x_1, x_2, \cdots, x_n, x_{n+1})$ lies on the graph of $f : \mathbb{R}^n \to \mathbb{R}$ if and only if $x_{n+1} = f(x_1, x_2, \cdots, x_n)$. By definition, $\nabla f(P) = Df(P)$, so

$$f(P) + \nabla f(P)(\vec{x} - P)$$

is a good linear approximation of f near P, where

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad P = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

In particular, if f is differentiable at P, then the **tangent hyperplane** to the graph of f, $(P, f(P))$, is given by the equation

$$x_{n+1} = f(P) + \nabla f(P)(\vec{x} - P) \qquad \text{or} \qquad \nabla f(P) \cdot (\vec{x} - P) - (x_{n+1} - f(P)) = 0.$$

The normal vector to the tangent hyperplane is given by $(\nabla f(P), -1)$, and $(P, f(P))$ itself is on the tangent hyperplane.

Example 10. Find the tangent hyperplane to the the surface defined by $f(x, y) = x^3 - xy + y^2$ at the point $(2, 1)$.
$f(2, 1) = 7$, and $Df(x, y) = \begin{bmatrix} 3x^2 - y & -x + 2y \end{bmatrix}$, so $\nabla f(2, 1) = (11, 0)$. The tangent hyperplane has normal vector $(11, 0, -1)$ and passes through the point $(2, 1, 7)$, so it has equation

$$(11, 0, -1) \cdot (x - 2, y - 1, z - 7) = 0, \qquad 11(x - 2) - (z - 7) = 0.$$

Theorem 6. *Let $A \subset \mathbb{R}^n$ be an open subset and $f : U \to \mathbb{R}^m$. If the partial derivatives $\frac{\partial f_i}{\partial x_j}$ exist and are continuous in an open neighborhood of $P \in A$, then f is differentiable at P.*

Proof. We will assume that $m = 1$, and prove the case for $n = 2$. The general case is similar. Take $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose $P = (a, b)$, and $\overrightarrow{PQ} = h_1\hat{\imath} + h_2\hat{\jmath}$. Let

$$P_0 = (a, b) = P, \qquad P_1 = (a + h_1, b), \qquad P_2 = (a + h_1, b + h_2) = Q.$$

$$f(Q) - f(P) = [f(P_2) - f(P_1)] + [f(P_1) - f(P_0)]$$

Recalled the Mean Value Theorem:

Theorem 7 (Mean Value Theorem)**.** *Let $g : [a, b] \to \mathbb{R}$ be continuous and differentiable everywhere on (a, b), then there exists $c \in (a, b)$, such that*

$$f(b) - f(a) = f'(c)(b - a).$$

So we can find Q_1 somewhere on the segment P_0P_1 and Q_2 somewhere on the segment P_1P_2 such that

$$f(P_1) - f(P_0) = \frac{\partial f}{\partial x}(Q_1)h_1, \qquad f(P_2) - f(P_1) = \frac{\partial f}{\partial y}(Q_2)h_2.$$

Hence,

$$f(Q) - f(P) = \frac{\partial f}{\partial x}(Q_1)h_1 + \frac{\partial f}{\partial y}(Q_2)h_2.$$

$$\begin{aligned}
\frac{|f(Q) - f(P) - T\overrightarrow{PQ}|}{\|\overrightarrow{PQ}\|} &= \frac{|(\frac{\partial f}{\partial x}(Q_1) - \frac{\partial f}{\partial x}(P))h_1 + (\frac{\partial f}{\partial y}(Q_2) - \frac{\partial f}{\partial y}(P))h_2|}{\|\overrightarrow{PQ}\|} \\
&\leq \frac{|(\frac{\partial f}{\partial x}(Q_1) - \frac{\partial f}{\partial x}(P))h_1|}{\|\overrightarrow{PQ}\|} + \frac{|(\frac{\partial f}{\partial y}(Q_2) - \frac{\partial f}{\partial y}(P))h_2|}{\|\overrightarrow{PQ}\|} \\
&\leq \frac{|(\frac{\partial f}{\partial x}(Q_1) - \frac{\partial f}{\partial x}(P))h_1|}{|h_1|} + \frac{|(\frac{\partial f}{\partial y}(Q_2) - \frac{\partial f}{\partial y}(P))h_2|}{|h_2|} \\
&= \big|(\frac{\partial f}{\partial x}(Q_1) - \frac{\partial f}{\partial x}(P)\big| + \big|\frac{\partial f}{\partial y}(Q_2) - \frac{\partial f}{\partial y}(P)\big|
\end{aligned}$$

As $Q \to P$, $Q_1, Q_2 \to P$. Since the partial derivatives of f are continuous, we have

$$\lim_{Q\to P} \frac{|f(Q) - f(P) - T\overrightarrow{PQ}|}{\|\overrightarrow{PQ}\|} \leq \lim_{Q\to P} \left(\big|(\frac{\partial f}{\partial x}(Q_1) - \frac{\partial f}{\partial x}(P))\big| + \big|\frac{\partial f}{\partial y}(Q_2) - \frac{\partial f}{\partial y}(P))\big|\right) = 0.$$

Therefore, f is differentiable at P with derivative T. □

Remark. The existence of derivative at P is much stronger than the existence of partial derivatives at P. Just because function behave nicely along the x and y axes directions, it doesn't mean that the function behaves nicely along every path approaching P. See the following example.

Example 11.

$$f(x, y) = \begin{cases} \frac{x^2y}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Check that f is continuous at the origin, and by definition of partial derivative,

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{0 - 0}{h} = 0$$

we have $\frac{\partial f}{\partial x}(0, 0) = \frac{\partial f}{\partial y}(0, 0) = 0$, so $z = 0$ will be the tangent plane at the origin if f were differentiable, i.e. $Df(0, 0) = (0, 0)$. But along the line $y = x$,

$$\lim_{h\to 0} \frac{f(h, h) - f(0, 0)}{\|(h, h)\|} = \frac{1}{\sqrt{2}} \neq 0.$$

So even though the partial derivatives exist at $(0,0)$, f is not differentiable at $(0,0)$. $z=0$ is certainly not tangent to a path in the plane along the direction $y=x$. One can check that when $(x,y)\neq(0,0)$

$$\frac{\partial f}{\partial x}=\frac{2xy}{x^2+y^2}-\frac{2x^3y}{(x^2+y^2)^2},\qquad \frac{\partial f}{\partial y}=\frac{x^2}{x^2+y^2}-\frac{2x^2y^2}{(x^2+y^2)^2}.$$

Neither of which has limit or is continuous at $(0,0)$. In fact, let

$$g(x,y)=\begin{cases}\frac{xy}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

$\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ are both 0 at $(0,0)$. But $g(x,y)$ is not even continuous at $(0,0)$, and hence certainly not differentiable.

Definition 10. If f is differentiable and Df is continuous, then f is of class $\mathcal{C}^1$.

Remark. The converse to Theorem 6 is not true: f can be differentiable with non-continuous partial derivatives. For example,

$$f(x,y)=\begin{cases}(x^2+y^2)\sin\frac{1}{(x^2+y^2)^{1/2}} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases},\qquad \lim_{(x,y)\to(0,0)} f(x,y)=0$$

so f is continuous.

$$\frac{\partial f}{\partial x}(0,0)=\lim_{h\to0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to0} h\sin\frac{1}{h}=0.$$
$$\frac{\partial f}{\partial y}(0,0)=\lim_{h\to0}\frac{f(0,h)-f(0,0)}{h}=\lim_{h\to0} h\sin\frac{1}{h}=0.$$

f is differentiable at $(0,0)$ since

$$\lim_{(h,k)\to(0,0)}\frac{f(h,k)-f(0,0)-\begin{bmatrix}0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}}{\sqrt{h^2+k^2}}=\sqrt{h^2+k^2}\sin\frac{1}{(h^2+k^2)^{1/2}}=0.$$

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist everywhere away from the origin,

$$\frac{\partial f}{\partial x}=2x\sin\frac{1}{(x^2+y^2)^{1/2}}-\frac{x}{(x^2+y^2)^{1/2}}\cos\frac{1}{(x^2+y^2)^{1/2}},\qquad (x,y)\neq(0,0)$$
$$\frac{\partial f}{\partial y}=2y\sin\frac{1}{(x^2+y^2)^{1/2}}-\frac{y}{(x^2+y^2)^{1/2}}\cos\frac{1}{(x^2+y^2)^{1/2}},\qquad (x,y)\neq(0,0)$$

but they are not continuous at $(0,0)$ because their limits do not exist at $(0,0)$:

$$\frac{x}{(x^2+y^2)^{1/2}}\cos\frac{1}{(x^2+y^2)^{1/2}}=\frac{r\cos\theta}{r}\cos\frac{1}{r}=\cos\theta\cos\frac{1}{r}$$

Let $r=\sqrt{x^2+y^2}$, $f(x,y)=g(r)=r^2\sin\frac{1}{r}$ is differentiable at $r=0$, but

$$g'(r)=2r\sin\frac{1}{r}-\cos\frac{1}{r}$$

has no limit at $r=0$ and a fortiori is not continuous at $r=0$.

3 Review on Limits and Differentiability

1. How to show limit of a function exists at $(0,0)$:

   - $\epsilon - \delta$ argument.
   - The function can be decomposed as sums, products, compositions of functions with limits.
   - Squeezing method: usually applies when denominator is a sum of even powers, usually can reduce a function of two variables to a function of one variable. Eg: $\frac{x^4y^4}{x^6+y^2}$, $\frac{x^4y^2}{x^8+y^2}$.
   - Polar coordinates method: Eg: $\frac{x^2y}{x^2+y^2}$, $\frac{x^4y^4}{x^6+y^6}$.
   - L'ôpital's Rule can be useful when evaluating limit of a function of a single variable. Eg: $\frac{e^{xy}-1}{y} = \frac{e^{xy}-1}{xy} \cdot x = \frac{e^t-1}{t} \cdot x$.

2. How to show limit of a function doesn't exist at $(0,0)$:

   - If the denominator can be made to vanish. Eg: $\frac{x^{10}y^5}{x^5+y^{10}}$ set $x = -y^2$.
   - If you can find two approaches to the origin with different limits. Usually try set $x = 0$, $y = 0$, or $y = kx$, to see if two different limits can be obtained. Eg: $\frac{xy}{x^2+y^2}$. Sometimes, you may need to find a more obscure path to get a different limit. Eg: $\frac{x^2y}{x^4+y^2}$.
   - Polar coordinates method: $\frac{x^2y^3}{x^6+y^6}$, $\frac{x^2y^4}{x^6+y^6}$.

3. How to show a function is continuous at $(0,0)$:

   - The function can be decomposed as sums, products, compositions of functions continuous at $(0,0)$.
   - Show limit of the function exists and coincides with the value of the function at $(0,0)$.

4. How to show a function is not continuous at $(0,0)$:

   - Show limit of the function doesn't exist.
   - If the limit exists, show it's different from the value of the function at $(0,0)$.

5. How to show a function is differentiable at $(0,0)$:

   - The function can be decomposed as sums, products, compositions of functions differentiable at $(0,0)$.

- Computer partial derivatives of the function at $(0,0)$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}, \qquad f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k}$$

  and use it to show that

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-\nabla f(0,0)\cdot(h,k)}{\|(h,k)\|} = 0$$

- Compute the partial derivatives of the function $f_x(x,y)$ and $f_y(x,y)$ show they are continuous at $(0,0)$.

6. How to show a function is not differentiable at $(0,0)$:

   - Show function is not continuous at $(0,0)$.
   - If function is continuous at $(0,0)$, show partial derivatives do not exist at $(0,0)$. Eg: $f(x,y) = \sqrt{x^2+y^2}$.
   - If partial derivatives exist at $(0,0)$, show

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-\nabla f(0,0)\cdot(h,k)}{\|(h,k)\|}$$

     doesn't exist or is not 0. Eg: $f(x,y) = (xy)^{1/3}$.