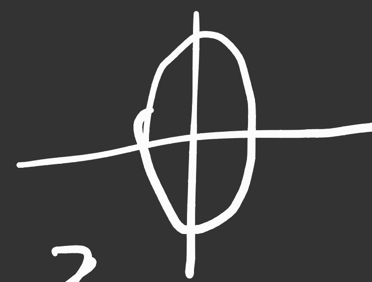
Given a simple curve

$$x(t) = \cos t$$
$$y(t) = 2 \sin t$$



ellipse.

Already, it's impossible to express the length of the curve in terms of elementary functions! So we must adopt a more robust approach.

Consider a generic curve

$$\vec{r}(t) = (x(t), y(t), z(t))$$

In principle, we can define the length

$$s(t) = \int_{t_0}^{t} \|\vec{r}\,'(\tau)\| \, d\tau .$$

difficult to compute

$$\frac{ds}{dt} = \|\vec{r}\,'(t)\| = \|\vec{v}(t)\| =: v(t).$$

this is known.

$$\vec{T} = \frac{d\vec{r}}{ds} = \frac{dt}{ds}\frac{d\vec{r}}{dt} \qquad \text{(chain rule)}$$

$$= \frac{1}{v(t)}\frac{d\vec{r}}{dt} \qquad \left(\text{since } \frac{ds}{dt} = v(t)\right)$$

Thus we can express unit tangent vector by velocity divided by abs. value of velocity

$$\frac{d\vec{T}}{ds} = \frac{1}{ds/dt}\frac{d\vec{T}}{dt} = \frac{1}{v(t)}\frac{d}{dt}\left(\frac{1}{v(t)}\frac{d\vec{r}}{dt}\right)$$

$$= \frac{1}{v}\left[-\frac{v'(t)}{|v(t)|^2}\frac{dr}{dt} + \frac{1}{v(t)}\frac{d^2}{dt^2}\vec{r}(t)\right] \qquad \text{(Leibnitz rule)}$$

$$= -\frac{v'}{v^3}\vec{v} + \frac{1}{v^2}\vec{a} \qquad \text{on one hand...}$$

on other hand...

$$\frac{d\vec{T}}{ds} = \kappa \vec{N}.$$

Thus we found

$$-\frac{v'}{v^3}\vec{v} + \frac{1}{v^2}\vec{a} = \kappa \vec{N}$$

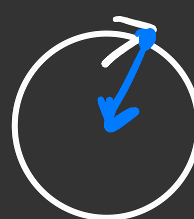
Isolating $\vec{a}$:

$$\frac{1}{v^2}\vec{a} = \frac{v'}{v^3}\vec{v} + \kappa \vec{N}$$

or

$$\vec{a} = \frac{v'}{v}\vec{v} + v^2 \kappa \vec{N}$$

Note that $\frac{\vec{v}}{v} = \frac{\vec{v}}{\|\vec{v}\|} = \vec{T}$. Thus

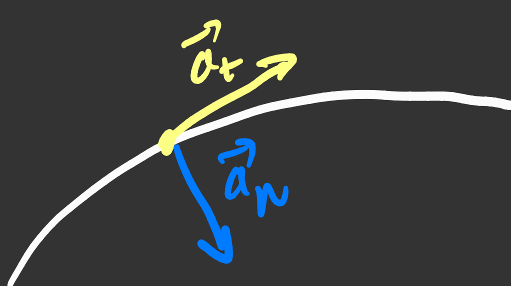


looks like centripetal force.

$$\boxed{\vec{a} = \frac{dv}{dt}\vec{T} + \|\vec{v}\|^2 \kappa \vec{N}}$$

$(\text{speed})^2$ (inverse radius). (unit normal)

scalar acceleration (comp of acceleration along the road)



$$\vec{a} = \vec{a}_t + \vec{a}_N.$$

We must extract from here κ and $\vec{N}$.

$$\vec{a} = \frac{dv}{dt}\vec{T} + v^2 \kappa \vec{N}.$$

Crossing with $\vec{T}$, we find

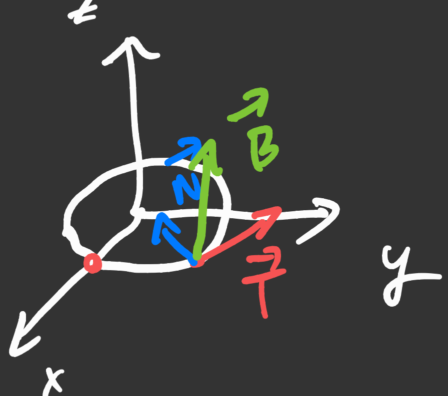
$$\vec{T} \times \vec{a} = \frac{dv}{dt}\cancelto{0}{\vec{T} \times \vec{T}} + v^2 \kappa \, \vec{T} \times \vec{N}$$

$$= v^2 \kappa \, \vec{T} \times \vec{N}$$

As $\vec{T}$ is a unit tangent, and $\vec{N}$ is unit normal, orthogonal to $\vec{T}$, the vector

$$\vec{B} = \vec{T} \times \vec{N} \qquad \text{(binormal)}$$

is orthogonal to both $\vec{T}$ and $\vec{N}$ and length 1.



$\vec{B} = (0,0,1) = \vec{k}$ for the unit circle

It is the same for all points on the circle.

Thus we found

$$\vec{T} \times \vec{a} = v^2 \kappa \vec{B}.$$

Now we find κ. Recall $\vec{v} = v\vec{T}$.

$$\vec{v} \times \vec{a} = v\vec{T} \times \vec{a} = v^3 \kappa \vec{B}.$$

$$\|\vec{v} \times \vec{a}\| = v^3 \kappa \|\vec{B}\| = v^3 \kappa.$$

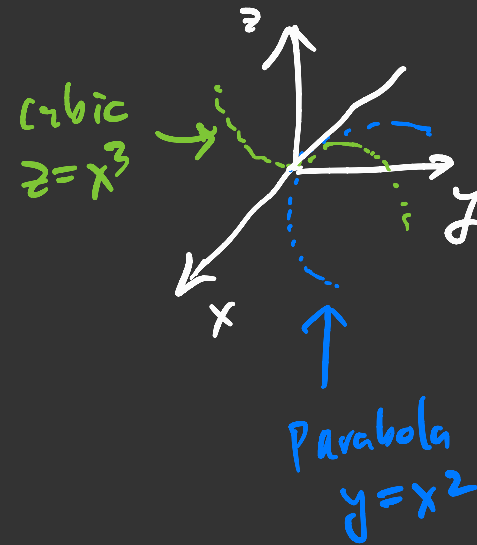
Since $\|\vec{B}\| = 1$. Thus

$$\boxed{\kappa = \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|^3}} = \frac{\|\vec{v} \times \vec{a}\|}{(\vec{v} \cdot \vec{v})^{3/2}}$$

Now we can take any parametric curve, $\vec{r}(t)$, find $\vec{v}(t) = \vec{r}\,'(t)$ and $\vec{a}(t) = \vec{r}\,''(t)$ and thereby find the curvature.

Example:

$$\vec{r}(t) = (t, t^2, t^3).$$



$$\vec{v}(t) = \vec{r}\,'(t) = (1, 2t, 3t^2)$$

$$\vec{a}(t) = \vec{r}\,''(t) = (0, 2, 6t)$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = i\left(12t^2 - 6t^2\right) - j(6t) + k(2)$$

$$= (6t^2, -6t, 2)$$

$$\vec{v} \cdot \vec{v} = 1 + 4t^2 + 9t^4$$

$$K(t) = \frac{\sqrt{36t^4 + 36t^2 + 4}}{(1 + 4t^2 + 9t^4)^{3/2}}.$$

Note $\quad K(0) = 2.$

How to find the principal normal $\vec{N}$?

Return to our formula:

$$\vec{a} = \frac{dv}{dt}\vec{T} + v^2 k \vec{N}.$$

We know that

$$\vec{v} \times \vec{a} = v^3 k \vec{B} \qquad (\text{since } \vec{v} \times \vec{T} = 0)$$

Now

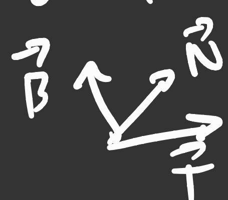
$$(\vec{T}, \vec{N}, \vec{B})$$

form a right triple of orthogonal unit vectors

$$\|\vec{T}\| = \|\vec{N}\| = \|\vec{B}\| = 1$$

$$\vec{T} \cdot \vec{N} = 0 \quad \vec{T} \cdot \vec{B} = 0 \quad \vec{N} \cdot \vec{B} = 0$$

$$\vec{T} \times \vec{N} = \vec{B}$$

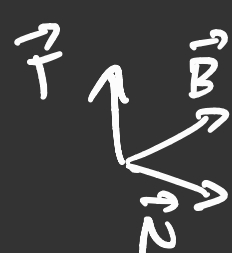


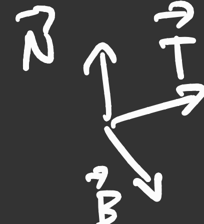
but also

since we took and turned them

$$\vec{B} \times \vec{T} = \vec{N}$$

$$\vec{N} \times \vec{B} = \vec{T}$$





Thus to find $\vec{N}$,

$$\vec{N} = \vec{B} \times \vec{T}$$

$$= \vec{B} \times \frac{\vec{v}}{\|\vec{v}\|}$$

$$= \left(\frac{\vec{v} \times \vec{a}}{k\,v^3}\right) \times \frac{\vec{v}}{v} \qquad \left(\text{since } \vec{B} = \frac{1}{kv^3}\vec{v} \times \vec{a}\right)$$

$$= \frac{1}{kv^4}\left(\vec{v} \times \vec{a}\right) \times \vec{v}$$

$$= \frac{1}{v}\,\frac{\vec{v} \times \vec{a}}{\|\vec{v} \times \vec{a}\|} \times \vec{v} \qquad \left(\text{using } k = \frac{\|\vec{v} \times \vec{a}\|}{v^3}\right)$$

Thus we found the formula

$$\boxed{\vec{N} = \frac{(\vec{v} \times \vec{a}) \times \vec{v}}{\|(\vec{v} \times \vec{a}) \times \vec{v}\|}}$$

$$\left(\begin{array}{l} \text{since } \vec{v} \perp \vec{v} \times \vec{a} \\ \text{so } \|\vec{v} \times (\vec{v} \times \vec{a})\| \\ \quad = \|\vec{v}\|\,\|\vec{v} \times \vec{a}\| \end{array}\right)$$

Plane curves

$$\vec{r}(t) = (x(t), y(t), 0)$$

Keep since some objects are visualized in 3D.

Let us restrict further to consider our curve to be the graph of a function

$$y = f(x)$$

$$\vec{r}(t) = (t, f(t), 0)$$

$$\vec{v}(t) = \vec{r}\,'(t) = (1, f'(t), 0)$$

$$\vec{a}(t) = \vec{r}\,''(t) = (0, f''(t), 0)$$

$$v(t) = \|\vec{v}(t)\| = \sqrt{1 + (f'(t))^2}$$

Unit tangent vector:

$$\vec{T}(t) = \frac{\vec{v}(t)}{v(t)} = \frac{1}{\sqrt{1+(f')^2}}(1, f', 0)$$

Curvature

$$\kappa(t) = \frac{\|\vec{v}\times\vec{a}\|}{v^3} = \frac{|f''(t)|}{(1+(f')^2)^{3/2}}$$

Since

$$\vec{v}\times\vec{a} = \begin{vmatrix} i & j & k \\ 1 & f' & 0 \\ 0 & f'' & 0 \end{vmatrix} = f''\,\vec{k}.$$
$$= (0, 0, f'')$$

Binormal vector

remember $\vec{a}(t) = \frac{dv}{dt}\vec{T} + \kappa v^2 \vec{N}$

Thus

$$\vec{v}\times\vec{a} = \kappa v^3 \vec{B} \implies \vec{B} = \frac{\vec{v}\times\vec{a}}{\|\vec{v}\times\vec{a}\|}$$

Thus $\vec{B} = \frac{\vec{v}\times\vec{a}}{\|\vec{v}\times\vec{a}\|} = \frac{f''(t)}{|f''(t)|}\hat{k}$

Normal vector

$$\vec{N} = \vec{B} \times \vec{T}$$

$$= \frac{f''}{|f''|} \hat{k} \times \vec{T} \qquad = \mathrm{sgn}(f'')\, \vec{k} \times \frac{(1, f', 0)}{\sqrt{1+(f')^2}}$$

$$= \frac{\mathrm{sgn}(f'')}{\sqrt{1+(f')^2}} \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ 1 & f' & 0 \end{vmatrix}$$

$$= \frac{\mathrm{sgn}(f'')}{\sqrt{1+(f')^2}} (-f'(t), 1, 0)$$

What is this?

$y = f(x)$

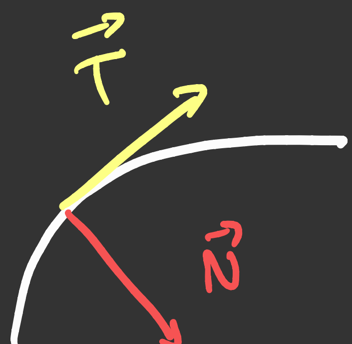
$$\vec{T} = \frac{(1, f', 0)}{\sqrt{1+(f')^2}}$$

rotation ccw $90°$

$$\vec{N} = \mathrm{sgn}(f'') \frac{(-f', 1, 0)}{\sqrt{1+(f')^2}}$$

directed in concave direction $f'' > 0$

In opposite case



$f'' < 0$

Thus we found:

$$\vec{r}(t) = (t, f(t), 0)$$

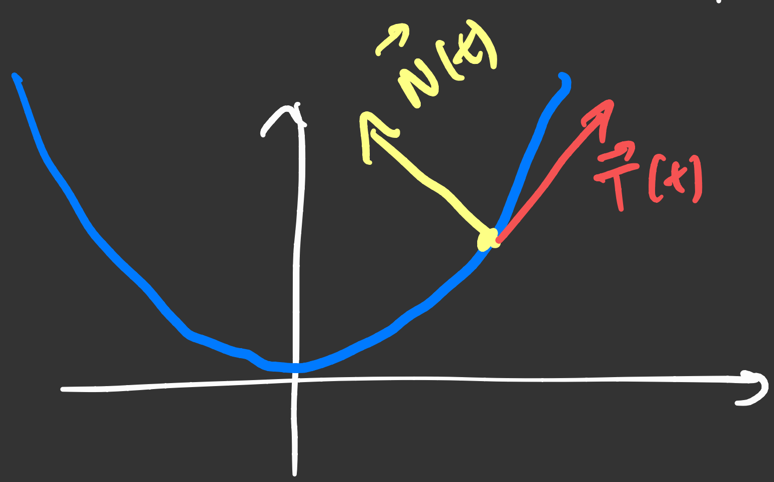
$$\kappa(t) = \frac{|f''(t)|}{\left(1 + (f'(t))^2\right)^{3/2}}$$

$$\vec{N}(t) = \frac{\operatorname{sgn}(f''(t))}{\sqrt{1 + (f')^2}} \left(-f'(t), 1, 0\right)$$

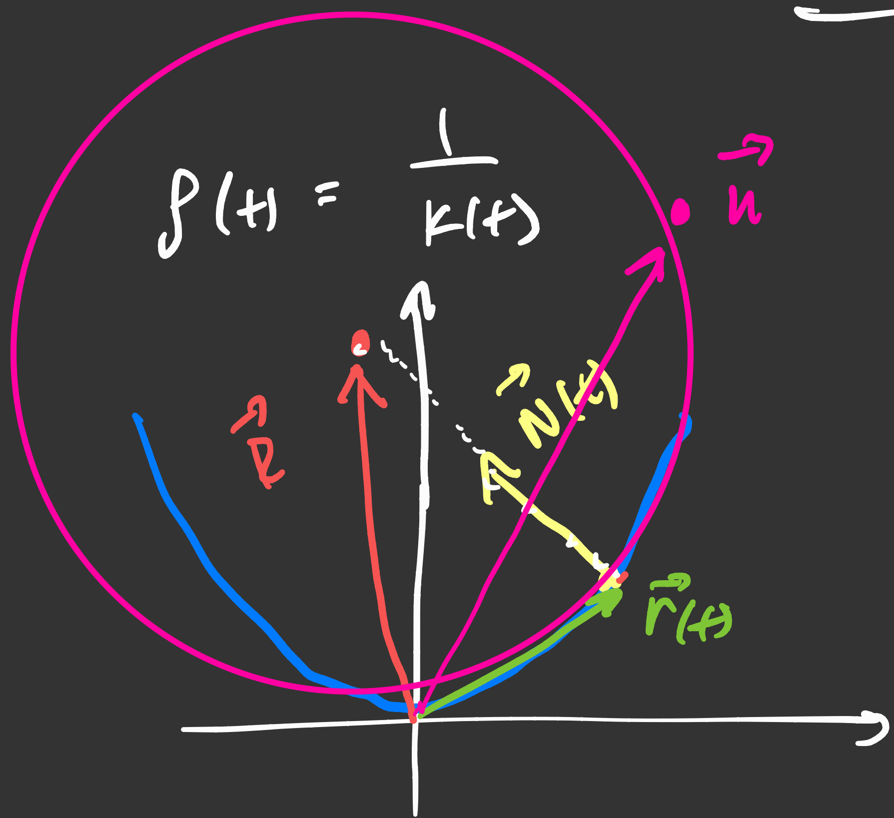
Example: $f(t) = t^2$

$$\kappa(t) = \frac{2}{(1 + 4t^2)^{3/2}}$$

$$\vec{N}(t) = \frac{(-2t, 1, 0)}{\sqrt{1 + 4t^2}}$$



Oscular circle



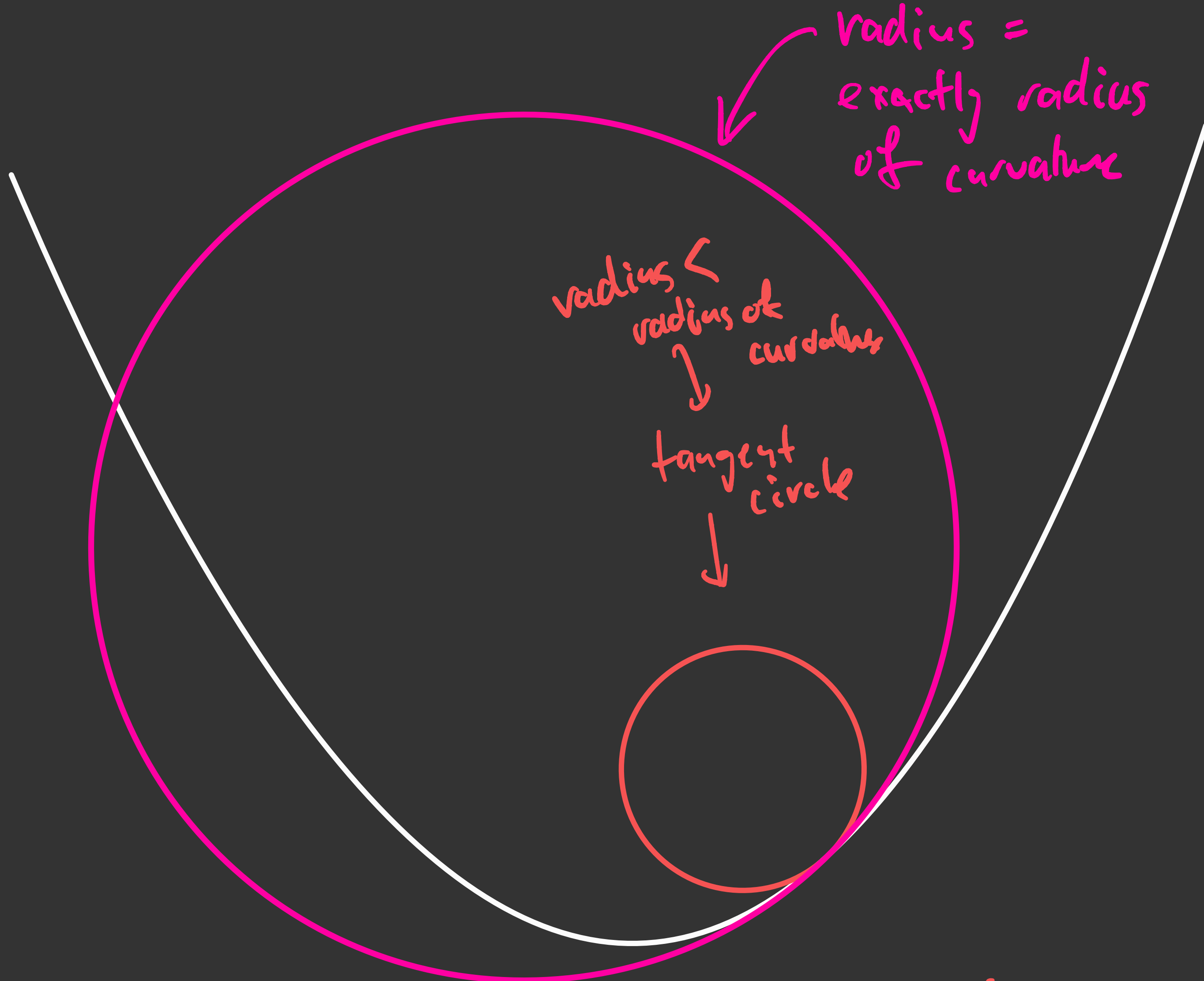
Oscular circle: circle whose center $\vec{R}(t)$ is on the line through $\vec{r}(t)$ in the direction of $\vec{N}(t)$, i.e.

$$\vec{R}(t) = \vec{r}(t) + \rho\,\vec{N}(t)$$

The equation

$$\|\vec{u} - \vec{R}(t)\| = \rho(t)$$

equation for osculating circle.



dist between parabola and circle, decreases like square of dist. entirely above parabola.

osculating circle: on one side, it is over parabola. On other, it is under.

$\boxed{\text{distance to parabola behaves as the cube of the distance}}$