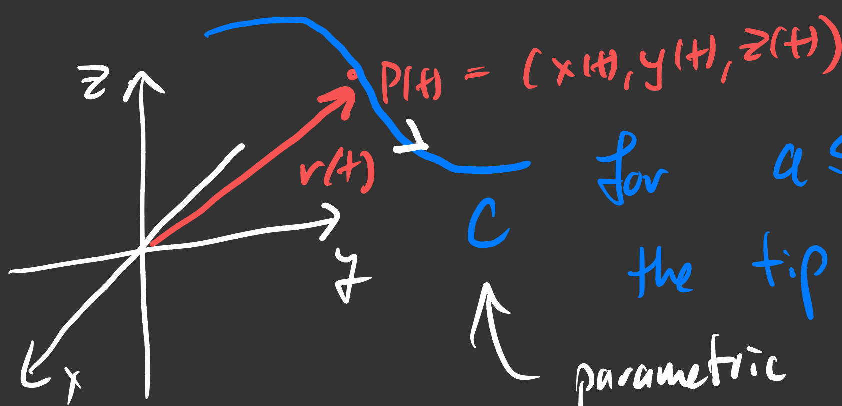
Vector functions: function whose values are vectors

$$\vec{r}(t) = (x(t), y(t), z(t))$$



for $a \le t \le b$, the tip sweeps out the curve C

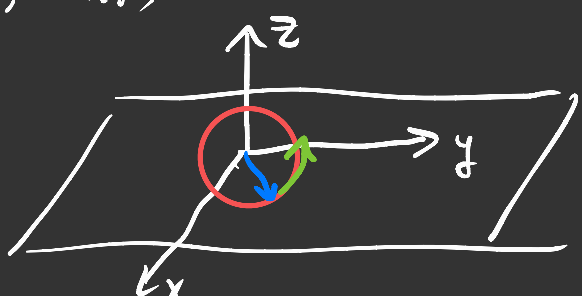
parametric curve oriented (directed) by the direction of motion of the tip.

Example: $r(t) = r_0 + t\vec{u}$. : motion along straight line with constant speed.

Example: $r(f) = (\cos t, \sin t, 0)$

- planar motion
- note that $(x(t))^2 + (y(t))^2 = (\cos t)^2 + (\sin t)^2 = 1$

thus this motion remains on unit circle



motion is counter clockwise observed from tip of $\vec{K}$.

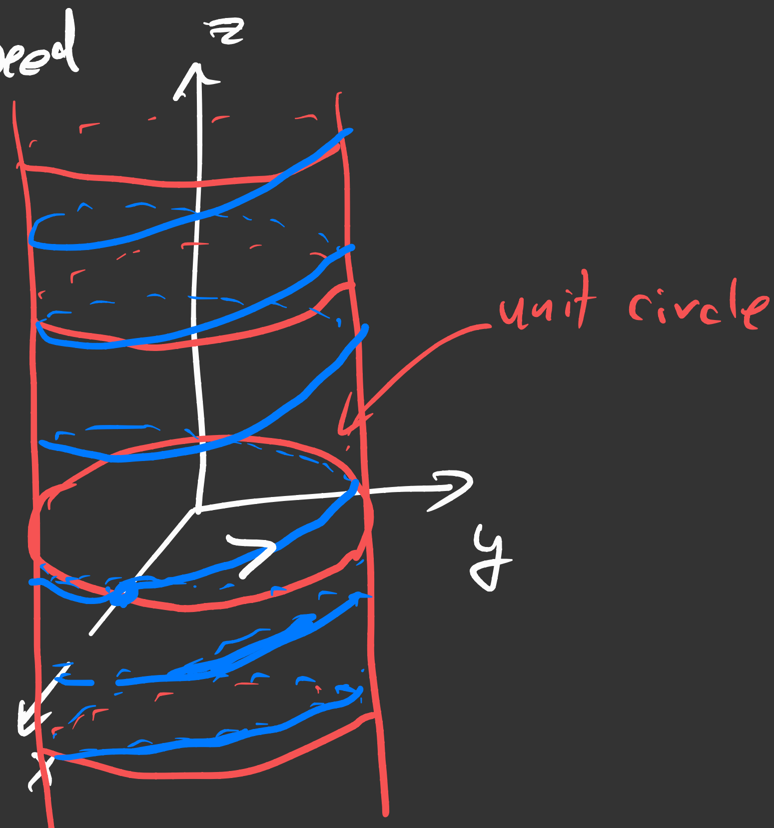
Example: $r(t) = (\cos(t), \sin(t), t)$

- $(x(t))^2 + (y(t))^2 = 1$

Moving up in z at const speed

$\boxed{\text{Helix}}$

ccw motion
with respect to $\vec{k}$.



Differentiation

$$\vec{r}(t) = (x(t), y(t), z(t))$$

$$\vec{r}\,'(t) = (x'(t), y'(t), z'(t))$$

$$\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

Properties of derivative

1) Linearity:

$$(r_1(t) + r_2(t))' = r_1'(t) + r_2'(t)$$

2) $k \in \mathbb{R}$, $(k r(t))' = k r'(t)$.

<u>Proof</u>: Limit definition.

3) Leibnitz Rules

scalar a) $(f(t) g(t))' = f'(t) g(t) + f(t) g'(t)$

scalar multiple b) $(k(t) \vec{r}(t))' = k'(t) \vec{r}(t) + k(t) \vec{r}\,'(t)$

dot product c) $(\vec{r}_1(t) \cdot \vec{r}_2(t)) = \vec{r}_1\,'(t) \cdot \vec{r}_2(t) + \vec{r}_1(t) \cdot \vec{r}_2\,'(t)$

cross product d) $(\vec{r}_1(t) \times \vec{r}_2(t))' = \vec{r}_1\,'(t) \times \vec{r}_2(t) + \vec{r}_1(t) \times \vec{r}_2\,'(t)$

all are proved in same way. Let's prove (d).

Write $$(\vec{r}_1(t) \times \vec{r}_2(t))' = \lim_{h \to 0} \frac{\vec{r}_1(t+h) \times \vec{r}_2(t+h) - \vec{r}_1(t) \times \vec{r}_2(t)}{h}$$

$$= \lim_{h \to 0} \frac{(\vec{r}_1(t+h) - \vec{r}_1(t)) \times \vec{r}_2(t+h) - \vec{r}_1(t) \times (\vec{r}_2(t) - \vec{r}_2(t+h))}{h}$$

$$= \lim_{h \to 0} \left[\left(\frac{\vec{r}_1(t+h) - \vec{r}_1(t)}{h} \right) \times \vec{r}_2(t+h) + \vec{r}_1(t) \times \left(\frac{\vec{r}_2(t+h) - \vec{r}_2(t)}{h} \right) \right]$$

3) <u>Chain Rule</u>

$\vec{r}(t)$ and $t = t(s)$

$$\left(\vec{r}(t(s))\right)' = t'(s)\, \vec{r}\,'(t(s))$$

Example: $r(t) = (\cos(t), \sin(t), t)$

$t(s) = s^2$

$$r(t(s)) = (\cos(s^2), \sin(s^2), s^2)$$

$$\frac{d}{ds} r(t(s)) = \frac{d}{ds}\left(\cos(s^2), \sin(s^2), s^2\right)$$

$$= \left(-\sin(s^2)\cdot(2s),\ \cos(s^2)\cdot(2s),\ (2s)\right)$$

$$= 2s\left(-\sin(s^2), \cos(s^2), 1\right)$$

$$= \frac{dt}{ds} \cdot \frac{d\vec{r}}{dt}(t(s))$$

✓

$\frac{dt}{ds} = 2s.$

4) Higher derivatives

$$\vec{r}(t) = (x(t), y(t), z(t))$$

$$\vec{r}\,'(t) = (x'(t), y'(t), z'(t))$$

$$\vec{r}\,''(t) = (x''(t), y''(t), z''(t))$$

$$\vdots$$

$$\vec{r}^{\,(n)}(t) = (x^{(n)}(t), y^{(n)}(t), z^{(n)}(t))$$

<u>Example</u>: $(\vec{r}_1(t) \cdot \vec{r}_2(t))' = \vec{r}_1\,'(t) \cdot \vec{r}_2(t) + \vec{r}_1(t) \cdot \vec{r}_2\,'(t)$

$$(\vec{r}_1(t) \cdot \vec{r}_2(t))'' = \vec{r}_1\,''(t) \cdot \vec{r}_2(t) + 2\,\vec{r}_1\,'(t) \cdot \vec{r}_2\,'(t) + \vec{r}_1(t) \cdot \vec{r}_2\,''(t)$$

Rate of change of volume of a parallelepiped.

Consider

$$\vec{a}(t),\ \vec{b}(t) \text{ and } \vec{c}(t).$$

$$V(\vec{a}(t), \vec{b}(t), \vec{c}(t)) = (\vec{a}(t) \times \vec{b}(t)) \cdot \vec{c}(t)$$

$$\frac{d}{dt} V(\vec{a}(t), \vec{b}(t), \vec{c}(t)) = \frac{d}{dt}\left((\vec{a}(t) \times \vec{b}(t)) \cdot \vec{c}(t) \right)$$

$$= (\vec{a}'(t) \times \vec{b}(t)) \cdot \vec{c}(t)$$

$$+ (\vec{a}(t) \times \vec{b}'(t)) \cdot \vec{c}(t)$$

$$+ (\vec{a}(t) \times \vec{b}(t)) \cdot \vec{c}'(t)$$

$$= V(\vec{a}'(t), \vec{b}(t), \vec{c}(t))$$

$$+ V(\vec{a}(t), \vec{b}'(t), \vec{c}(t))$$

$$+ V(\vec{a}(t), \vec{b}(t), \vec{c}'(t))$$

Integral of vector function

Primative:

$$\vec{r}(t) = (x(t), y(t), z(t))$$

Def: $\vec{R}(t)$ is called the primative if $\vec{R}'(t) = \vec{r}(t)$.

$$\vec{R}(t) = (\vec{u}(t), \vec{v}(t), \vec{w}(t))$$

$$\vec{R}(t) = (\vec{u}'(t), \vec{v}'(t), \vec{w}'(t))$$

Thus

$$\vec{u}'(t) = \vec{x}(t)$$
$$\vec{v}'(t) = \vec{y}(t)$$
$$\vec{w}'(t) = \vec{z}(t)$$

Then

$$\vec{u}(t) = \int \vec{x}(t)\,dt + C_1$$

$$\vec{v}(t) = \int \vec{y}(t)\,dt + C_2$$

$$\vec{w}(t) = \int \vec{w}(t)\,dt + C_2$$

or

$$\vec{R}(t) = \int \vec{r}(t)\,dt + \vec{C}.$$

Ex: $\vec{r}(t) = (\cos t, \sin t, t)$

$$\vec{R}(t) = \left(\int \cos t \, dt + C_1, \int \sin t \, dt + C_2, \int t \, dt + C_3 \right)$$

$$= \left(\sin t, -\cos t, \frac{t^2}{2} \right) + (C_1, C_2, C_3)$$

Definite integral:

$$\int_a^b \vec{r}(t) \, dt = \vec{R}(b) - \vec{R}(a)$$

$$= \left(\int_a^b x(t) dt, \int_a^b y(t) \, dt, \int_a^b z(t) dt \right)$$

Ex: $\displaystyle\int_0^{\pi/4} (\cos t, \sin t, t) \, dt$

$$= \left(\frac{1}{\sqrt{2}}, 1 - \frac{1}{\sqrt{2}}, \frac{1}{2} \left(\frac{\pi}{4} \right)^2 \right)$$

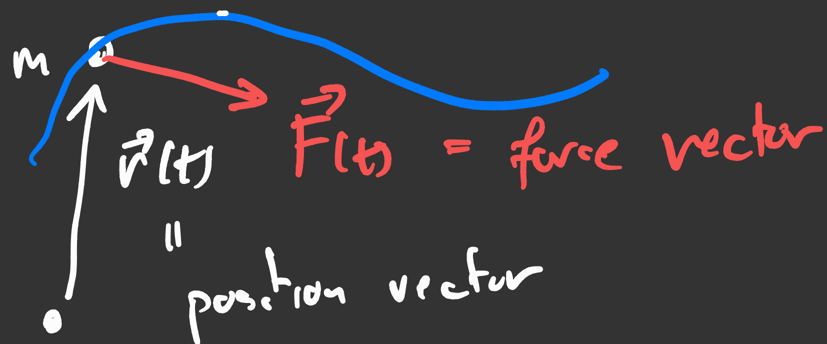
since

$$\int_0^{\pi/4} \cos t \, dt = \sin\left(\frac{\pi}{4}\right) - \sin(0) = \frac{1}{\sqrt{2}}$$

$$\int_0^{\pi/4} \sin t \, dt = -\cos\left(\frac{\pi}{4}\right) + \cos(0) = 1 - \frac{1}{\sqrt{2}}$$

Some dynamical problems.

① 2d Newton's Law : mass m



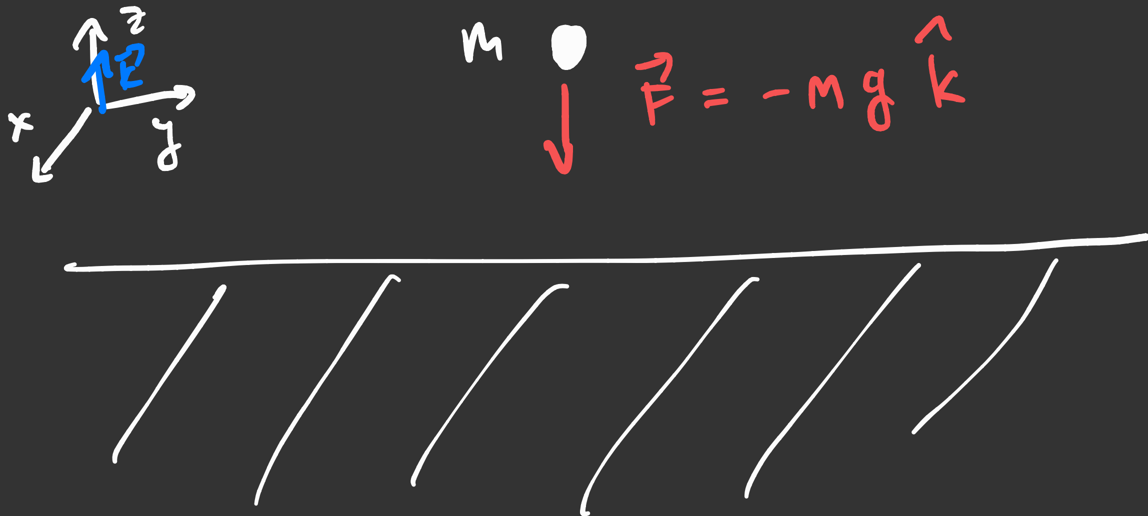
$$\boxed{\vec{F}(t) = m\,\vec{a}(t) = m\,\vec{r}\,''(t)}$$

acceleration.

Vectorial formulation of Newton's Law

Examples: Motion in gravitational field

Assume the earth is horizontal and flat, and the force of gravity acts vertically:



$$\vec{a} = \frac{\vec{F}}{m} = -g\hat{k} = (0, 0, -g)$$

$$\vec{a} = \frac{d}{dt}\vec{v}(t) \Rightarrow \vec{v}(t) = \int \vec{a}(t)\, dt + \vec{v}_0$$

$$= -\int g\vec{k}\, dt + \vec{v}_0$$

$$= -gt\vec{k} + \vec{v}_0$$

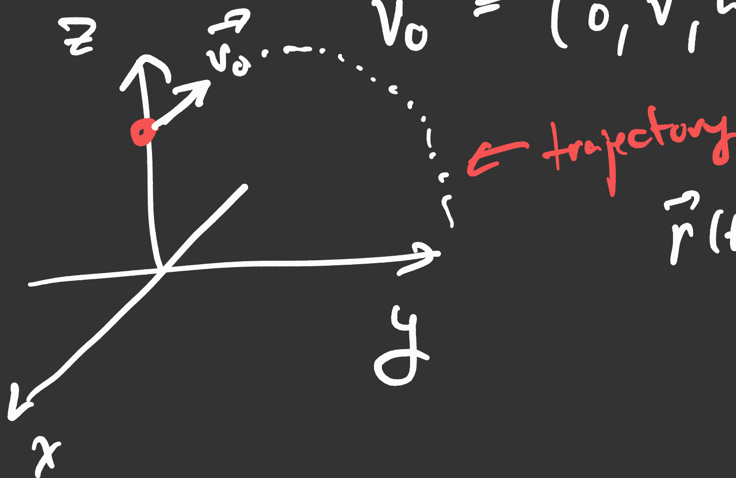
$$\vec{v}(t) = \frac{d}{dt}\vec{r}(t) \qquad \vec{r}(t) = \int \vec{v}(t)\, dt + \vec{C}$$

$$= -g\frac{t^2}{2}\vec{k} + t\vec{v}_0 + \vec{r}_0$$

$$\boxed{\vec{r}(t) = \vec{r}_0 + t\vec{v}_0 - g\frac{t^2}{2}\vec{k}}$$

Example: $\vec{r}_0 = (0, 0, H)$ height $= H$

$\vec{v}_0 = (0, v, w)$ ← motion in y-z plane



$$\vec{r}(t) = \left(0, 0, -g\frac{t^2}{2}\right) + (0, tv, tw) + (0, 0, h)$$

$$= \left(0, tv, h + tw - g\frac{t^2}{2}\right.$$

Let's find equation for the trajectory in the y-z plane, e.g. $z = z(y)$

Note $y = tv \Rightarrow t = \frac{y}{v}$

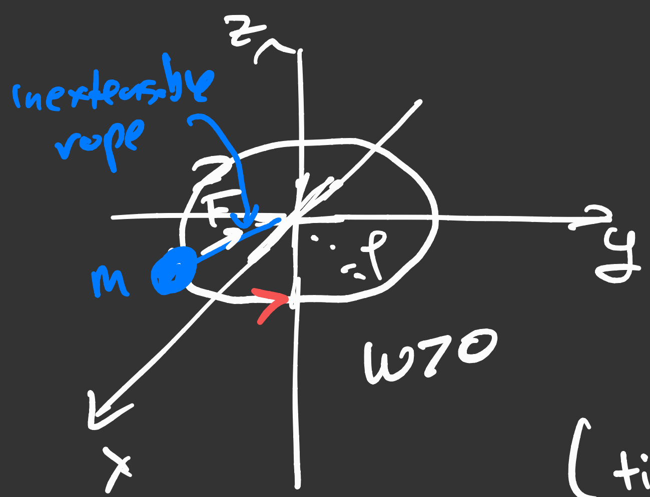
$$z = H + tw - g\frac{t^2}{2}$$

$$= H + \frac{w}{v}y - \frac{g}{2v^2}y^2$$

upside-down parabola.

2) Circular motion

$\rho > 0$ ← radius
w = frequency



$$\vec{r}(t) = (\rho\cos wt, \rho\sin wt, 0)$$

period $T = \frac{2\pi}{w}$

(time it takes to make one turn)

Rope pulls body to the center, so force $\vec{F}$ is directed in. Let's find $\vec{F}$:

$$\vec{F} = m\vec{a} = m\vec{r}''(t)$$

while

$$\vec{r}'(t) = w(-\rho\sin(wt), \rho\cos(wt), 0)$$

$$\vec{r}''(t) = -w^2(\rho\sin(wt), \rho\cos(wt), 0)$$

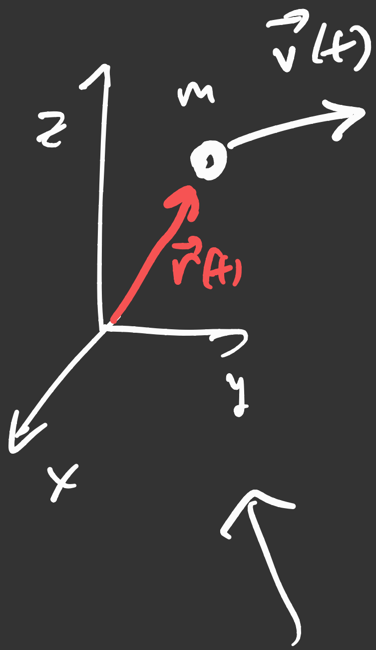
$$= -w^2\vec{r}(t).$$

Thus

$$\boxed{\vec{F}(t) = -mw^2\vec{r}(t)}$$

centrifugal force

3) Angular momentum and torque.



$$\boxed{\begin{array}{l}\text{Angular Momentum}\\ \vec{M} = m\vec{r}\times\vec{v}\end{array}}$$

perpendicular to $\vec{r}, \vec{v}$ and directed so that $(\vec{r}, \vec{v}, \vec{M})$ form a right triple

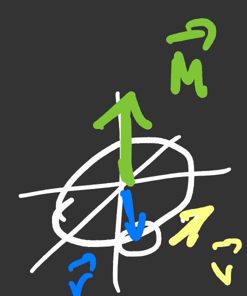
In this case, $\vec{M}$ points inside the page.

In previous example

$$\vec{r} = (\rho\cos(\omega t), \rho\sin(\omega t), 0)$$

$$\vec{v} = (-\rho\sin(\omega t), \rho\cos(\omega t), 0)$$

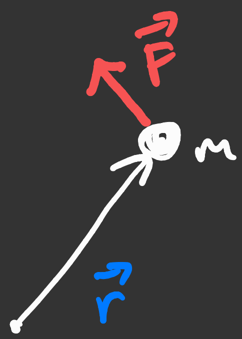
$$\vec{M} = m\vec{r}\times\vec{v} = m\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \rho\cos\omega t & \rho\sin\omega t & 0\\ -\rho\omega\sin\omega t & \rho\omega\cos\omega t & 0\end{vmatrix}$$



$$= m\begin{vmatrix}\rho\cos(\omega t) & \rho\sin(\omega t)\\ -\rho\omega\sin(\omega t) & \rho\omega\cos(\omega t)\end{vmatrix}\vec{k}$$

$$= m\rho^2\omega\left(\cos^2(\omega t) + \sin^2(\omega t)\right)\vec{k} = m\rho^2\omega\vec{k}$$

Torque



Torque

$$\vec{\tau} = \vec{r} \times \vec{F}$$

If a force is acting on a body, it changes it's angular momentum. How does it change? Analogue of 2nd Newton's Law.

$$\boxed{\begin{array}{l}\text{Theorem: If the mass is moving along a trajectory } \vec{r}(t) \text{ under the influence} \\ \text{of force } \vec{F}, \text{ then} \\ \qquad \dfrac{d}{dt}\vec{M}(t) = \vec{\tau}(t) \\ \text{where } \vec{M} = m\,\vec{r} \times \vec{v} \text{ is the angular momentum.}\end{array}}$$

(Mass) $\times$ velocity = linear momentum.

$\vec{M}$ is angular momentum

Proof: $\vec{M} = \vec{r} \times (m\vec{v})$

$$\frac{d}{dt}\vec{M}(t) = m\vec{r}\,' \times \vec{v} + m\vec{r} \times \vec{v}\,' \qquad \text{(Leibnitz)}$$

$$= \cancelto{0}{m\vec{v}\times\vec{v}} + m\vec{r} \times \vec{r}\,''$$

$$= \vec{r} \times (m\vec{r}\,'')$$

$$= \vec{r} \times \vec{F}.$$

$$= \vec{\tau}.$$

(since $\vec{a}\times\vec{a} = 0$ for all $\vec{a}$)

Pf: change places $\vec{u}\times\vec{w} = -\vec{w}\times\vec{u}$

$\blacksquare$

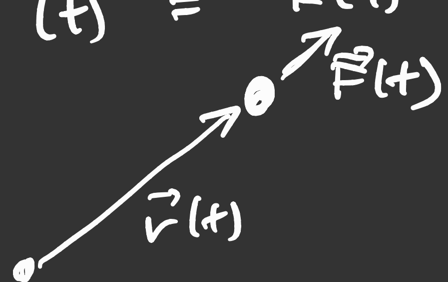
Remarkable particular case:

If $\vec{\tau}(t) = 0$, then $\vec{M}(t) = \text{const}$!

$\left(\frac{d}{dt}\vec{M}(t) = \vec{0} \quad \text{so} \quad \vec{M}(t) = \vec{M}_0 \right)$

What's an example of this? <u>Central force</u>

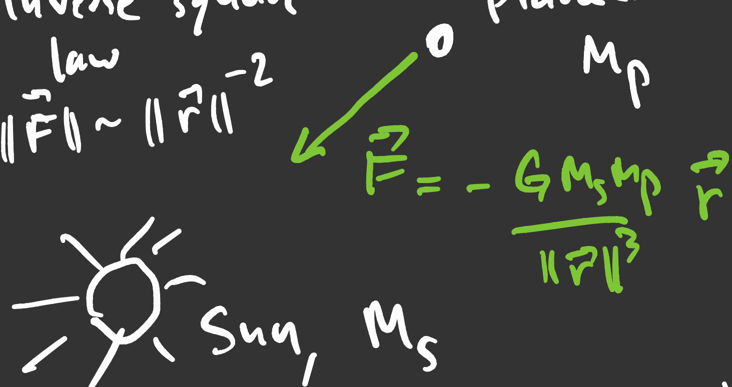
$\vec{F}(t) = k(t)\,\vec{r}(t)$



$\tau = \vec{r}\times\vec{F}$
$= k\,\vec{r}\times\vec{r} = 0.$

<u>Example:</u>

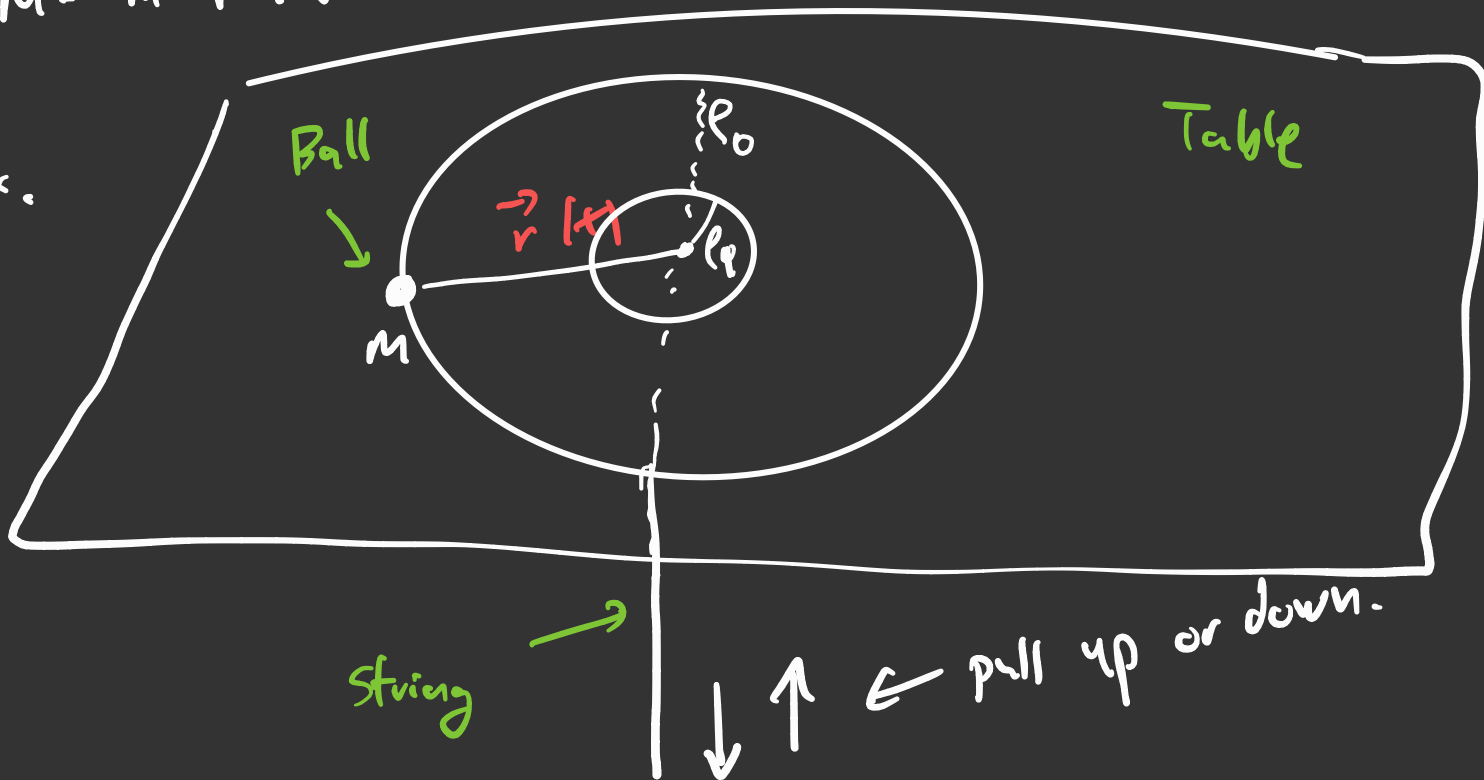
Inverse square law
$\|\vec{F}\| \sim \|\vec{r}\|^{-2}$



Thus, if $\vec{F}$ is a central force, then

$\vec{M} = m\,\vec{r} \times \vec{v}$ is constant.

Ex.



Thus we can prescribe $\|\vec{r}(t)\|$ arbitrarily.

$$\vec{M} = m\,\vec{r}(t) \times \vec{v}(t) = \text{constant}$$

$$\vec{M} = m\rho^2\omega\,\vec{k} \quad \text{angular momentum.}$$

$$\vec{M}_0 = m\rho_0^2\omega_0\,\vec{k} \qquad \vec{M}_1 = m\rho_1\omega_1^2\,\vec{k}$$

$$\Rightarrow \quad \frac{\omega_1}{\omega_0} = \frac{\rho_0^2}{\rho_1^2} \iff \boxed{\omega_1 = \frac{\rho_0^2}{\rho_1^2}\omega_0}$$

If you reduce length by half, $\rho_1 = \frac{1}{2}\rho_0$

then $\omega_1 = 4\omega_0$ and $\|v_1\| = 2\|v_0\|$.