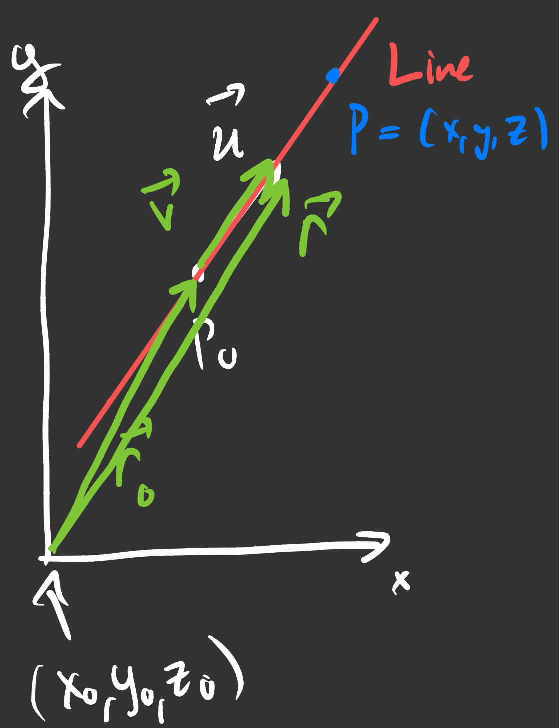
Lines and planes: problems to which we apply our vectorial machinery

Prob 1) Given a point $P_0 = (x_0, y_0, z_0)$ and vector $\vec{u} = (a, b, c)$. Find the equation of the line containing P_0 and parallel to $\vec{u}$.



Consider the vector $\vec{v} = \overrightarrow{P_0 P}$

From the picture, $\vec{v} = \vec{r} - \vec{r}_0$.

Then $\vec{v} \parallel \vec{u}$. This means that

$\vec{v} = t\vec{u}$ where $t \in \mathbb{R}$.

or $\vec{r} - \vec{r}_0 = t\vec{u}$

or $\vec{r} = \vec{r}_0 + t\vec{u}$

or $(x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$

$\vec{r} = (x, y, z)$

$\vec{r}_0 = (x_0, y_0, z_0)$

$$\boxed{\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}}$$

parametric equations of the line

t is parameter.

Symmetric equations.

We derive, if $a, b, c \neq 0$, then

$$t = \frac{x - x_0}{a}, \quad t = \frac{y - y_0}{b}, \quad t = \frac{z - z_0}{c}$$

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

If one has symmetric equations, one can always return to parametric eqns.

If $a = 0$, $b \neq 0$, $c \neq 0$

$$x = x_0 \qquad y = y_0 + tb \qquad z = z_0 + tc$$

or

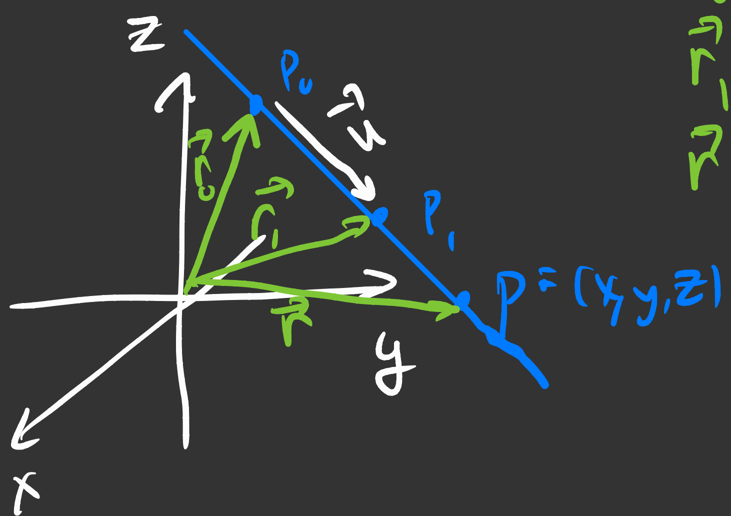
$$x = x_0 \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

If $a = b = 0$, $c \neq 0$, then

$$x = x_0, \quad y = y_0 \qquad z \in \mathbb{R} \text{ (arbitrary)}$$

vertical line through the point (x_0, y_0).

Prob 2) Given $P_0 = (x_0, y_0, z_0)$ and $P_1 = (x_1, y_1, z_1)$, find the equation of the line containing P_0 and P_1.



$$\vec{r_0} = \vec{OP_0}$$
$$\vec{r_1} = \vec{OP_1}$$
$$\vec{r} = \vec{OP}$$

$$\vec{u} = \vec{P_0P_1}$$

We know one vector which lyes on the line: $\vec{P_0P_1}$.

$$\vec{P_0P_1} = \vec{r_1} - \vec{r_0}$$

Now we return to previous problem:

$$\vec{r} = \vec{r_0} + t\vec{u}$$
$$= \vec{r_0} + t(\vec{r_1} - \vec{r_0}).$$

or

$$x = x_0 + t(x_1 - x_0)$$
$$y = y_0 + t(y_1 - y_0)$$
$$z = z_0 + t(z_1 - z_0)$$

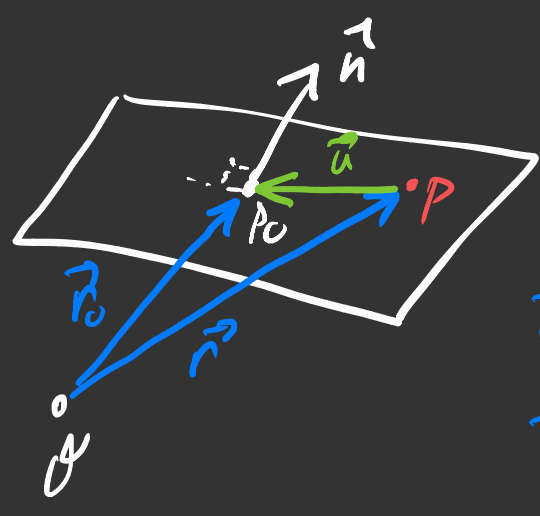
Example: $P_0 = (1,2,3)$ $P_1 = (3,2,1)$

parametric form: $x = 1 + 2t \quad y = 2 \quad z = 3 - 2t$

symmetric form: $\dfrac{x-1}{2} = \dfrac{z-3}{-2} \qquad y = 2$

equations for planes.
Intersection is our line.

Prob 3) Given $P_0 = (x_0, y_0, z_0)$ and $\vec{n} = (a,b,c)$ find a plane Π containing P_0 and normal (perpendicular) to the vector $\hat{n}$.



$P = (x,y,z)$
arbitrary point in plane

$\vec{r}_0 = (x_0, y_0, z_0)$

$\vec{r} = (x,y,z)$

$\vec{u} = \vec{r} - \vec{r}_0$

Note $\vec{u} \perp \vec{n}$. Then

$$\boxed{(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0 \quad \text{or} \quad \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0}$$

This is the equation since

$$\vec{n} \cdot \vec{r} = ax + by + cz$$

$$\vec{n} \cdot \vec{r_0} = ax_0 + by_0 + cz_0$$

equation for the plane:

$$\boxed{ax + by + cz = ax_0 + by_0 + cz_0}$$

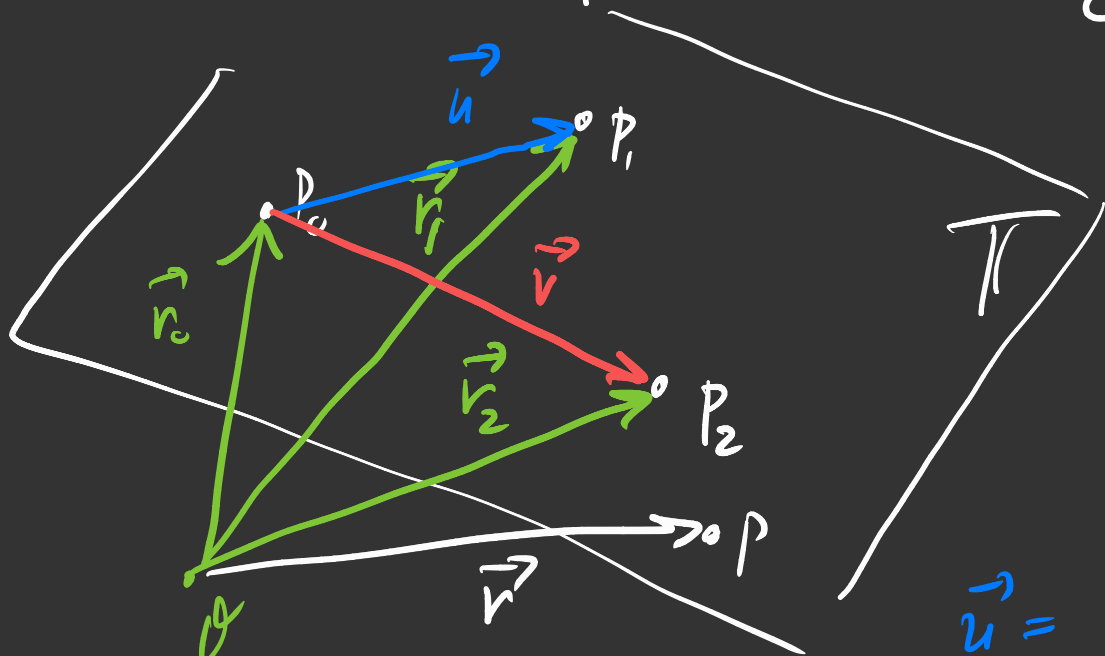
<u>Example:</u> $P_0 = (1, 2, -3)$ $\quad \vec{n} = (-3, 2, 2)$

$$-3x + 2y + 2z = -3 + 4 - 6 = -5$$

$$\Rightarrow \quad 3x - 2y - 2z = 5$$

Prob 4) Given 3 points $P_0 = (x_0, y_0, z_0)$, $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2)$ assume not colinear

Find the plane Π containing P_0, P_1, P_2.



$$\vec{r}_0 = \vec{OP_0} = (x_0, y_0, z_0)$$

...

$$\vec{u} = \vec{P_0P_1} = \vec{r}_1 - \vec{r}_0$$

$$\vec{v} = \vec{P_0P_2} = \vec{r}_2 - \vec{r}_0$$

The cross product of $\vec{u}, \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$. Since $\vec{u}, \vec{v}$ are in the plane Π, the cross product is orthogonal to Π.

$$\vec{n} = \vec{u} \times \vec{v} \quad \text{is orthogonal to } \Pi.$$

If $P = (x, y, z)$ is any point in the plane, the
$= \vec{r} - \vec{r}_0$

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

is equation for plane.

Example: $P_0 = (1,2,4)$, $P_1 = (-2,4,3)$, $P_2 = (0,-3,1)$

$$\underbrace{\vec{r}_1 - \vec{r}_0}_{\vec{u}} = (-3,2,-1) \qquad \underbrace{\vec{r}_2 - \vec{r}_0}_{\vec{v}} = (-1,-5,-3)$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 2 & -1 \\ -1 & -5 & -3 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -5 & -3 \end{vmatrix} \vec{i} - \begin{vmatrix} -3 & -1 \\ -1 & -3 \end{vmatrix} \vec{j} + \begin{vmatrix} -3 & 2 \\ -1 & -5 \end{vmatrix} \vec{k}$$

$$= (-6-5)\vec{i} - (9-1)\vec{j} + (15+2)\vec{k}$$

$$= (-11,-8,17).$$

Then

$$(x,y,z) \cdot (-11,-8,17) = (1,2,4) \cdot (-11,-8,17)$$

$$-11x - 8y + 17z = -11 - 16 + 68$$

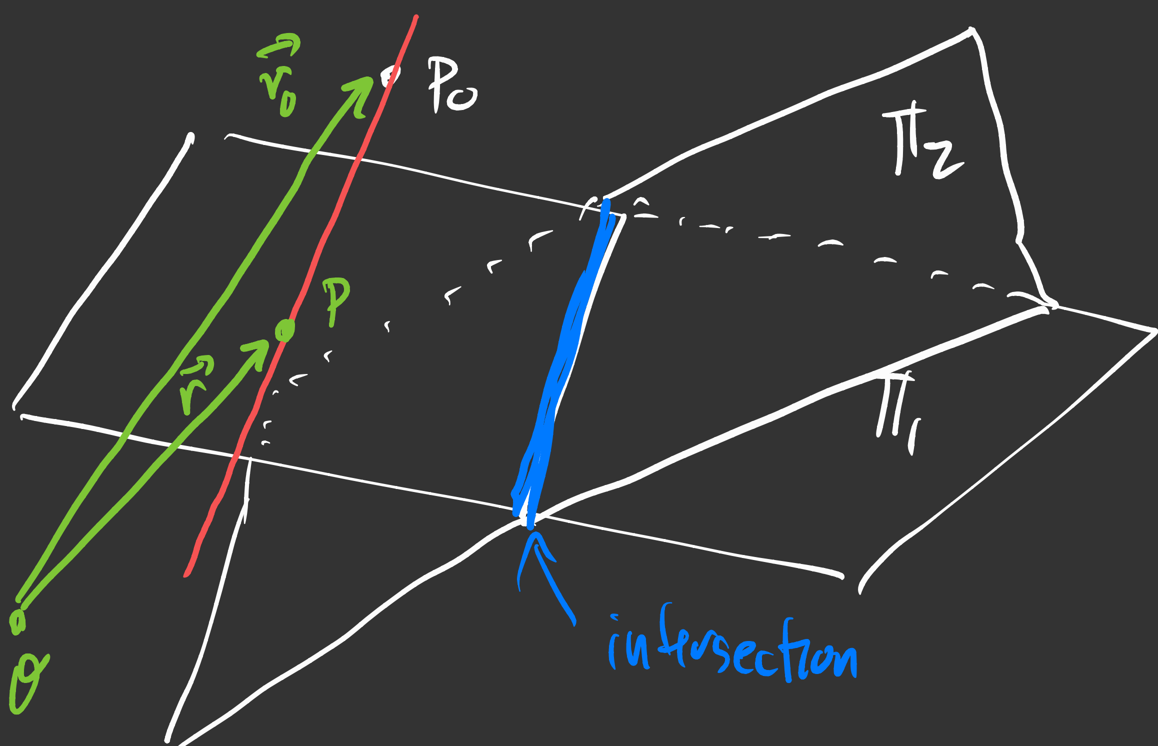
$$= -27 + 68$$

$$= 41$$

Prob 5) Given a point $P_0 = (x_0, y_0, z_0)$ and two planes $\Pi_1 : a_1 x + b_1 y + c_1 z = d_1$

$\Pi_2 : a_2 x + b_2 y + c_2 z = d_2$

Find a line containing P_0 and parallel to both planes.



To solve this problem, we must find a vector which is parallel to both planes. What we have are two vectors which are perpendicular

$\vec{u} = (a_1, b_1, c_1)$ perp to Π_1

$\vec{v} = (a_2, b_2, c_2)$ Perp to Π_2.

$$\vec{w} = \vec{u} \times \vec{v}$$

is perpendicular to both $\vec{u}$ and $\vec{v}$.

then $\vec{w}$ is parallel to Π_1, since $\vec{w} \perp \vec{u}$

and $\vec{w}$ is parallel to Π_2, since $\vec{w} \perp \vec{v}$.

Thus, the equation for the line is

$$\vec{r} = \vec{r}_0 + tw$$

$$(x, y, z) = (x_0, y_0, z_0) + t\vec{w}$$

Example $\quad \Pi_1: \; 2x - y - z = 1 \qquad P_0 = (1, 2, 3)$

$\qquad\qquad \Pi_2: \; x + 3y - 4z = 1$

$\vec{u} = (2, -1, -1) \qquad \vec{v} = (1, 3, -4)$

$$\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & -1 \\ 1 & 3 & -4 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 3 & -4 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & -1 \\ 1 & -4 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} \vec{k}$$

$$= (4 + 3)\vec{i} - (-8 + 1)\vec{j} + (6 + 1)\vec{k}$$

$$= (7, 7, 7)$$

$$(x, y, z) = (1 + 7t, 2 + 7t, 3 + 7t).$$