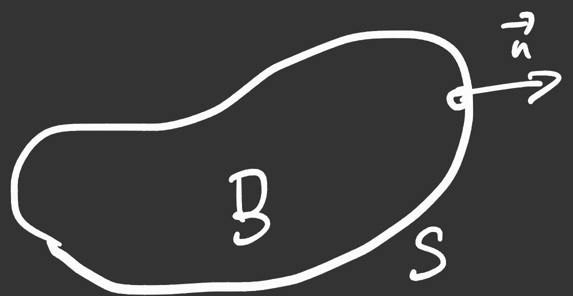
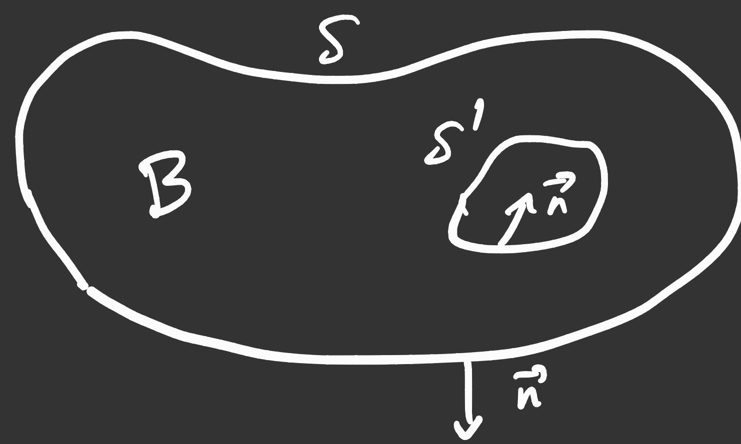
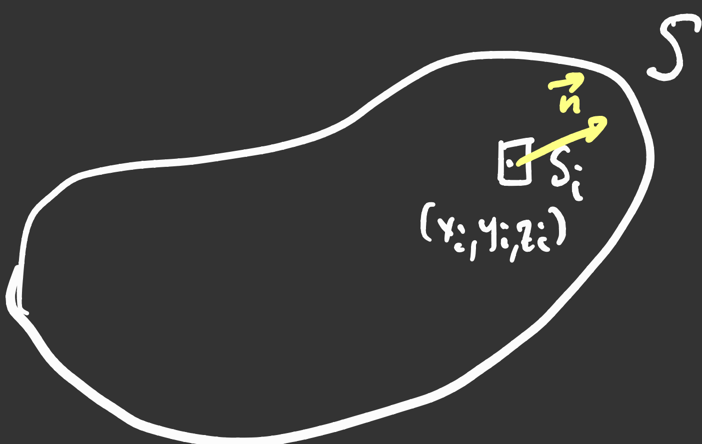
Divergence Theorem (Gauss-Ostrogradsky)

Consider a body B bounded by a surface S.





What is a flux? Given a closed surface



$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\, dS = \Phi$$

$$\sum_{i=1}^{n} \vec{F}(x_i, y_i, z_i)\cdot\vec{n}(x_i, y_i, z_i)\ \text{Area}(S_i)$$

$$\xrightarrow[\text{size}(S_i)\to 0]{n\to\infty} \iint_S \vec{F}\cdot\vec{n}\, dS \equiv \Phi$$

flux of $\vec{F}$ through S.

To compute it, we could divide the surface into pieces such that, within each piece, the surface is the graph of some function. Either $z=f(x,y)$, $x=g(y,z)$, $y=h(z,x)$.
In this case, we can write the integral as the corresponding integral over a 2d domain.
Then we compute the integral as we have been (if we can). Ideally, it reduces to some iterated integral.

Our goal here is to avoid this way of computing, and to do it in a <u>much simpler</u> way.

Theorem: Let S be a regular surface, which is the boundary of a 3d domain B. Let $\vec{F}(x,y,z)$ be a regular vector field in B. Then

$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_B \operatorname{div}\vec{F}\, dV$$

Example: B: $x^2+y^2+z^2 \le R^2$ (Ball of radius R)

S: $x^2+y^2+z^2 = R^2$ (sphere)

$$\vec{F}(x,y,z) = \left((\sin z + \cos y)^{e^2},\ (\sin x + \cos z)^{e^{x+e^2}},\ (\sin y + \cos x)^{xy^2} \right)$$

$\iint_S F\cdot\vec{n}\, dS$ looks complicated

but...

$\operatorname{div}\vec{F} = 0$! So $\iint_S \vec{F}\cdot\vec{n}\, dS = 0$.

This situation is typical in applications and very useful.

Proof of Divergence theorem:

$$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

$$= \vec{F}_1(\vec{r}) + \vec{F}_2(\vec{r}) + \vec{F}_3(\vec{r})$$

(when $\vec{F}_1(\vec{r}) = P(\vec{r})\vec{i}$, etc)

We shall prove the theorem for each $\vec{F}_i$ seperatly, the general case will follow by additivity. It is enough to prove it for one, say $\vec{F}_3$, as others follow similar argument.

Must show

$$\iint_S \vec{F}_3 \cdot \vec{n}\, dS = \iiint_B \operatorname{div} \vec{F}_3\, dV$$

We start proving this for simple domains B...

Suppose B is of type 1.

e.g. B: $(x,y) \in D$ and $g(x,y) \leq z \leq h(x,y)$

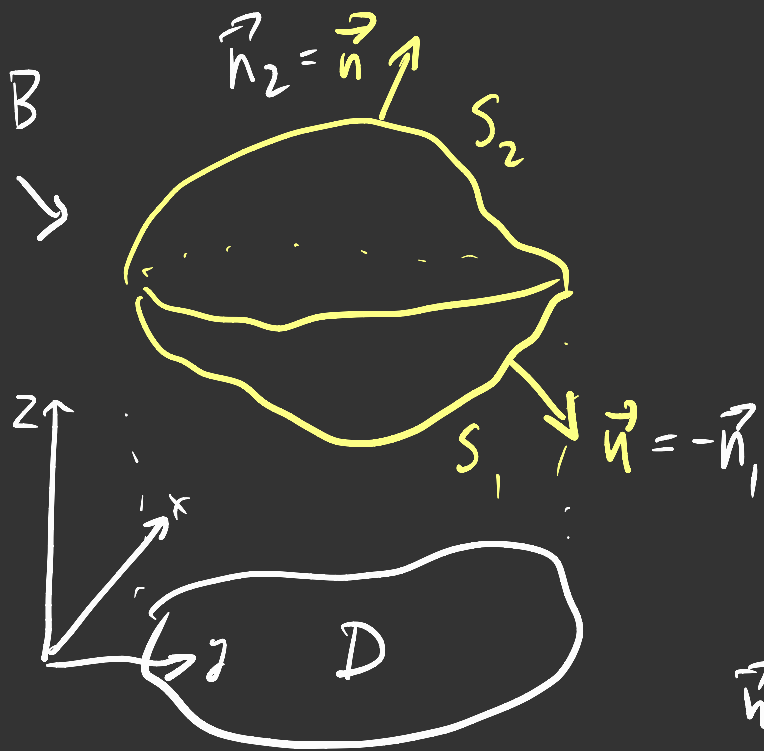
$$\operatorname{div} \vec{F}_3(\vec{r}) = \frac{\partial R}{\partial z} \qquad \left(\text{since } \vec{F}_3 = (0,0,R)\right)$$

Thus, the div integral is

$$\iiint_B \operatorname{div} \vec{F}_3 \, dV = \iint_D \int_{g(x,y)}^{h(x,y)} \frac{\partial R}{\partial z} \, dz \, dy \, dx$$

$$= \iint_D \Big(R(x,y,h(x,y)) - R(x,y,g(x,y)) \Big) \, dy \, dx$$

Now the flux...



$S_1 : \quad z = g(x,y)$

$S_2 : \quad z = h(x,y)$

$\vec{F}_3 = (0,0,R)$

$\vec{n}_1 = (-g_x, -g_y, 1)/\sqrt{1+g_x^2+g_y^2}$

$\vec{n}_2 = (-h_x, -h_y, 1)/\sqrt{1+h_x^2+h_y^2}$

$$\Phi = \iint_S \vec{F}_3 \cdot \vec{n}\, dS = \iint_{S_1} \vec{F}_3 \cdot \vec{n}_1\, dS + \iint_{S_2} \vec{F}_3 \cdot \vec{n}_2\, dS$$

$$\Phi_1 = -\iint_{S_1} \vec{F}_3 \cdot \vec{n}_1\, dS = -\iint_D (-g_x \cdot 0 - g_y \cdot 0, R)\, dA_{xy}$$

$$= -\iint_D R(x,y,g(x,y))\, dA$$

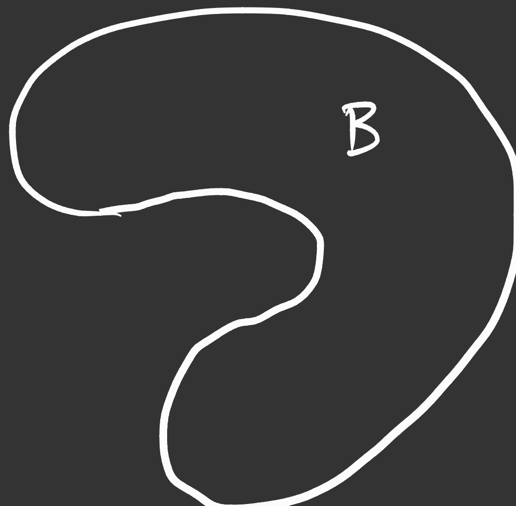
$$\Phi_2 = \iint_{S_2} \vec{F}_3 \cdot \vec{n}_2\, dS = \iint R(x,y,h(x,y))\, dA$$

agrees with div integral

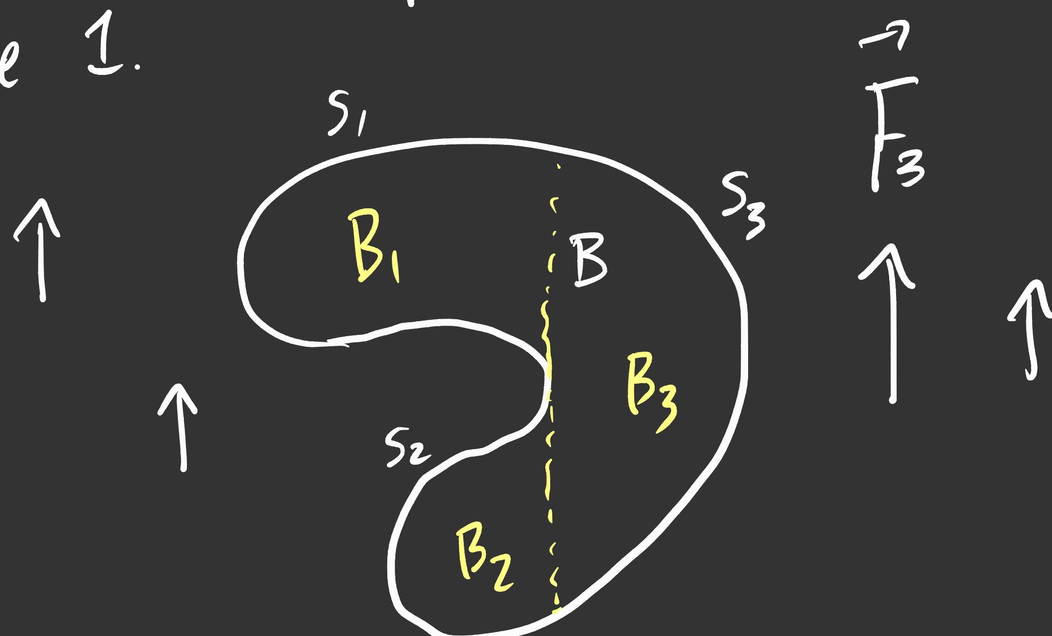
$$\Phi = \iint_D \left(R(x,y,h(x,y)) - R(x,y,g(x,y)) \right) dA \quad \checkmark$$

Thus, if B is a domain of type 1, the theorem is proved.

What to do in the more general case..



Here, we decompose into several domains of type 1.



Since $\vec{F}_3$ is a vertical vector field, it is tangent to the cut, so flux there is zero.

Thus, for B_1, B_2, B_3 are domains of type I and Theorem is proved

$$\iint_{S_i} \vec{F}_3 \cdot \vec{n}\, ds = \iint_{B_i} \operatorname{div} \vec{F}_3\, dV$$

since $\vec{F}_3$ is tangent to vertical boundaries

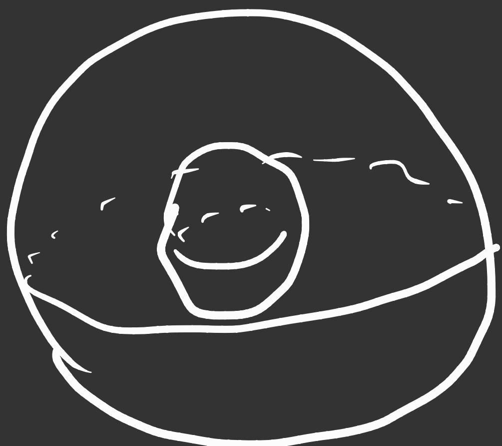
Adding up

$$\iint_{S} \vec{F}_3 \cdot \vec{n}\, ds = \iint_{S_1} \vec{F}_3 \cdot \vec{n}\, ds + \iint_{S_2} \vec{F}_3 \cdot \vec{n}\, ds + \iint_{S_3} \vec{F}_3 \cdot \vec{n}\, ds$$

$$\iiint_{B} \vec{F}_3 \cdot \vec{n}\, ds = \iiint_{B} \vec{F}_3 \cdot \vec{n}\, ds + \iiint_{B} \vec{F}_3 \cdot \vec{n}\, ds + \iiint_{B} \vec{F}_3 \cdot \vec{n}\, ds$$

Thus, in this case the theorem is proved.

You can consider more complicated domains.



Two consentric spheres.

Boundary is the union of outer sphere and inner sphere.

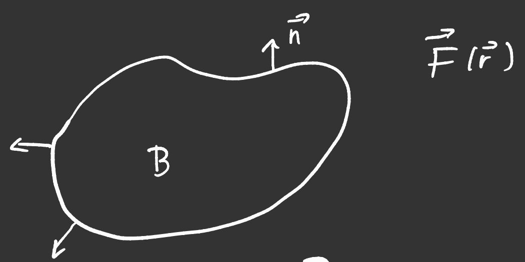
This domain is not of type I, but

We can cut it with a cylinder. Then B_1, B_2, B_3 are all type I domains.

Full mathematical proof applying to any regular domain is subtle, but this is main idea.

Some applications of Divergence theorem

①



Suppose $\operatorname{div}\vec{F} = 0$ in B, then $\Phi = \iint_S \vec{F}\cdot\vec{n}\, ds = 0$.

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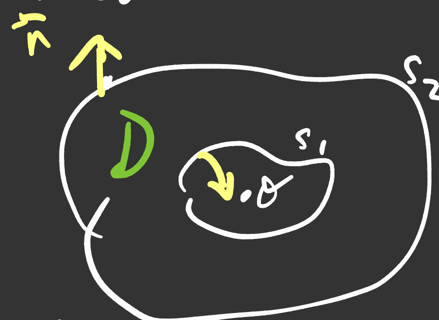
② $\vec{F}(\vec{r})$ is regular (continuous and differentiable) away from a point $\mathcal{O}$. Moreover $\operatorname{div}\vec{F} = 0$ away from $\mathcal{O}$. Now consider S_1 and S_2 surfaces cont

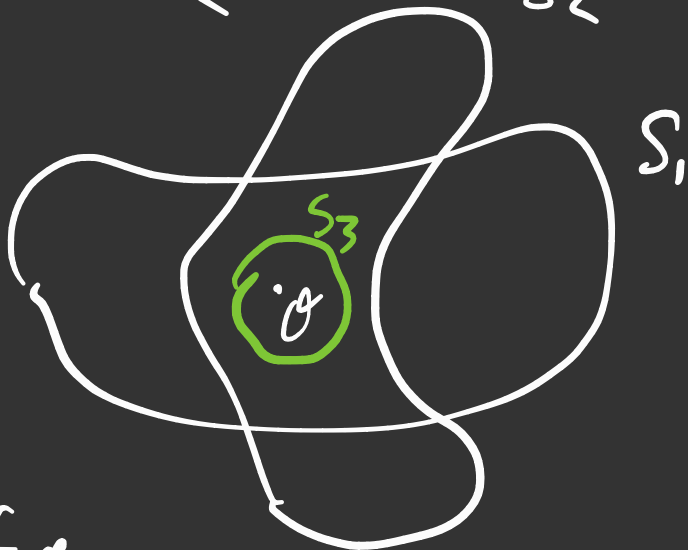


Then $\iint_{S_1} \vec{F}\cdot\vec{n}\, ds = \iint_{S_2} \vec{F}\cdot\vec{n}\, ds$

Pf: Apply div theorem to the domain D between S_1 and S_2. Normals look outside of domain. $0 = \iiint_D \operatorname{div}\vec{F}\, dv = \int_{S_2} \vec{F}\cdot\vec{n}\, ds - \int_{S_1} \vec{F}\cdot\vec{n}\, ds$.

Not fully general...

Could have



namely S_2 doesn't contain S_1.

Introduce S_3 . . .

By before,

$$\iint_{S_3} \vec{F} \cdot \vec{n} \, dS = \iint_{S_1} \vec{F} \cdot \vec{n} \, dS$$

but also

$$\iint_{S_3} \vec{F} \cdot \vec{n} \, dS = \iint_{S_2} \vec{F} \cdot \vec{n} \, dS.$$

Thus

$$\iint_{S_1} \vec{F} \cdot \vec{n} \, dS = \iint_{S_2} \vec{F} \cdot \vec{n} \, dS$$

One prominent vector field in all Mathematics and physics...

$$\vec{F}(x,y,z) = \frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}}$$

or

$$\vec{F}(\vec{r}) = \frac{\vec{r}}{\|\vec{r}\|^3}$$

This field enters in

a) Newton's gravitational law

b) Electro-static field (Coulomb)

c) arises also in fluid dynamics.

$\vec{F}$ is regular for all points except $\vec{r}=0$.

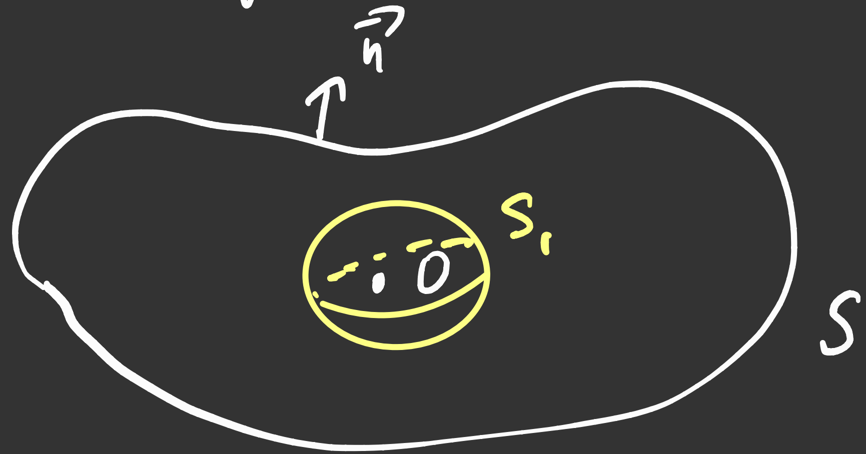
Moreover $\|\vec{F}\| = \frac{\|\vec{r}\|}{\|\vec{r}\|^3} = \frac{1}{\|\vec{r}\|^2}$.

But for $\vec{r} \neq 0$,

$$\operatorname{div} \vec{F}(\vec{r}) = 0$$

Check this!

Thus, if you take any surface S around O



$$\Phi = \iint_S \vec{F}\cdot\vec{n}\,dS$$

$$= 4\pi$$

for any surface S.

$S_1: \; x^2+y^2+z^2=1$

on S_1, $\vec{F}(\vec{r}) = (x,y,z)$

also $\vec{n}(x,y,z) = (x,y,z)$

Thus $\vec{F}(\vec{r})\cdot\vec{n} = x^2+y^2+z^2 = 1$

Thus $\displaystyle\iint_{S_1} \vec{F}\cdot\vec{n}\,dS = \text{Area}(S_1)$

$$= 4\pi$$

For instance, a shifted sphere, or scaled

Indeed, directly $\|\vec{F}\| = \dfrac{1}{\|\vec{r}\|^2}$

$$\iint_{S_R} \vec{F}\cdot\vec{n}\,dS = \frac{4\pi}{R^2}\,\text{Area}(S_R)$$

$$= \frac{4\pi}{R^2}R^2 = 4\pi$$

Flow of incompressible fluid

incompressible means every element of fluid keeps its volume over time.

Water is close to incompressible.

$\vec{v}(\vec{r})$ = velocity of incompressible fluid

$\operatorname{div} \vec{v}(\vec{r}) = 0$ implies fluid is incompressible.

Consider

$$\vec{u}(\vec{r}) = \frac{Q}{4\pi} \frac{\vec{r}}{\|\vec{r}\|^3}$$

Q is a constant flow rate

point source of water.

$$\Phi = \iint_S \vec{u} \cdot \vec{n} \, ds = Q$$

flux of fluid through surface S, for any surface

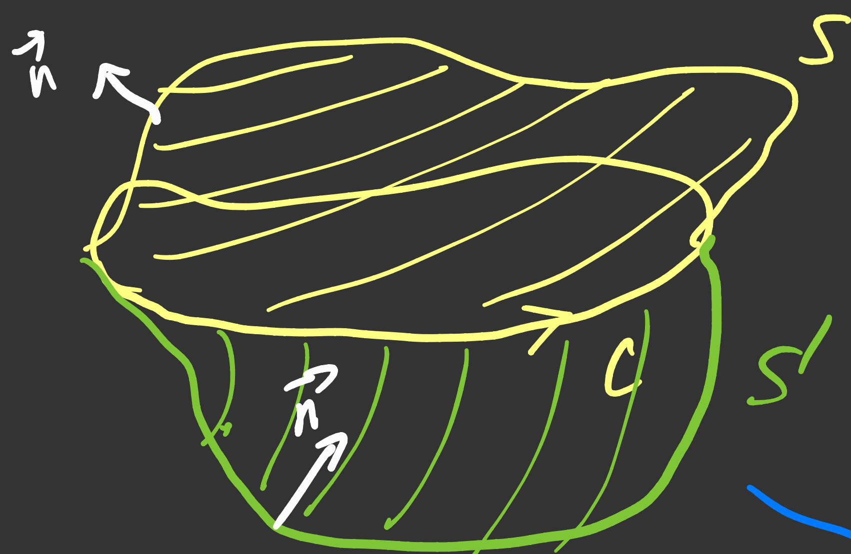
Amount of water through surface is always equal to what is put in at zero

Consider now the situation

$\vec{F}(\vec{r})$ with $\operatorname{div} \vec{F} > 0$.

Find $\iint_S \vec{F} \cdot \vec{n}\, dS$. This can be hard!

But consider also another surface.



e.g.

$z = h(x,y)$

$z = g(x,y)$

normal switches orientation

Claim: $\displaystyle\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_{S'} \vec{F} \cdot \vec{n}\, dS.$

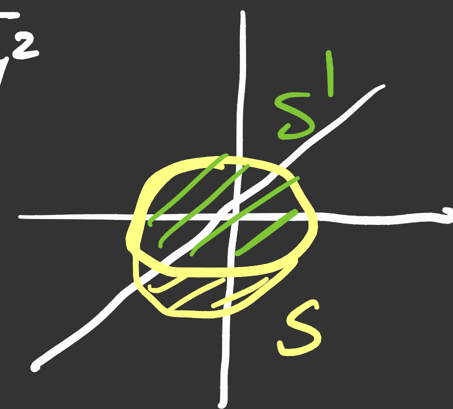
Indeed, consider B: $g(x,y) \leq z \leq h(x,y)$

$$0 = \iiint_B \operatorname{div} \vec{F}\, dV = \iint_{\text{boundary of } B} \vec{F} \cdot \vec{n}\, dS = \iint_S \vec{F} \cdot \vec{n}\, dS - \iint_{S'} \vec{F} \cdot \vec{n}\, dS$$

This can aid in computing integrals.

Consider

$$S: \quad z = \frac{e^{\sqrt{x^2+y^2}} + e^{-\sqrt{x^2+y^2}}}{2}$$



$$0 \le x^2+y^2 \le 1$$

$$\vec{F} = (x, y, -2z) \qquad \operatorname{div} \vec{F} = 0$$

what is contour on the boundary C

$$z = \frac{e+e^{-1}}{2} \qquad x^2+y^2 = 1 \quad .$$

Let us replace S by S':

$$z = \frac{e+e^{1}}{2} \qquad x^2+y^2=1$$

Now $\vec{n} = (0,0,1)$. Thus $\vec{F}\cdot\vec{n} = -2z = e+e^{-1}$.

Thus

$$\iint_S \vec{F}\cdot\vec{n}\, ds = \iint_{S'} \vec{F}\cdot\vec{n}\, ds = (e+e^{-1})\, \text{Area(disk)}$$

$$= (e+e^{-1})\, \pi$$