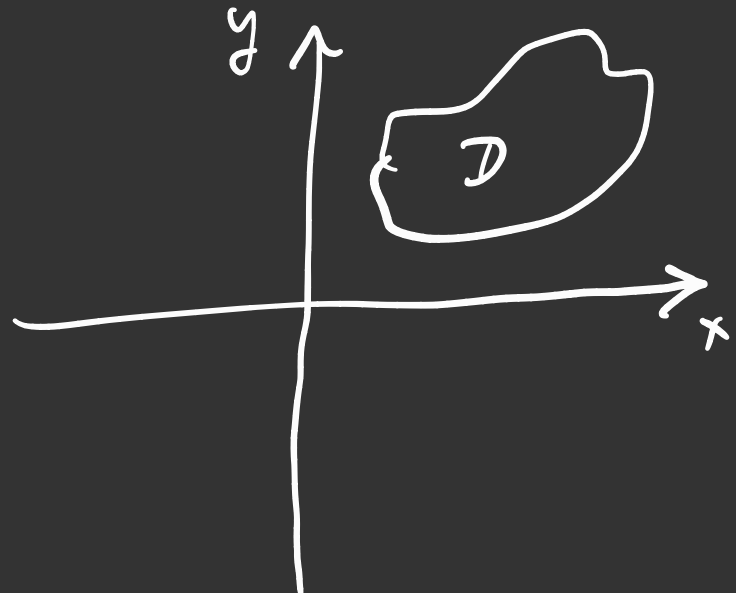
Change of variables in double integral

Suppose we have a domain $D \subseteq \mathbb{R}^2$



want to find

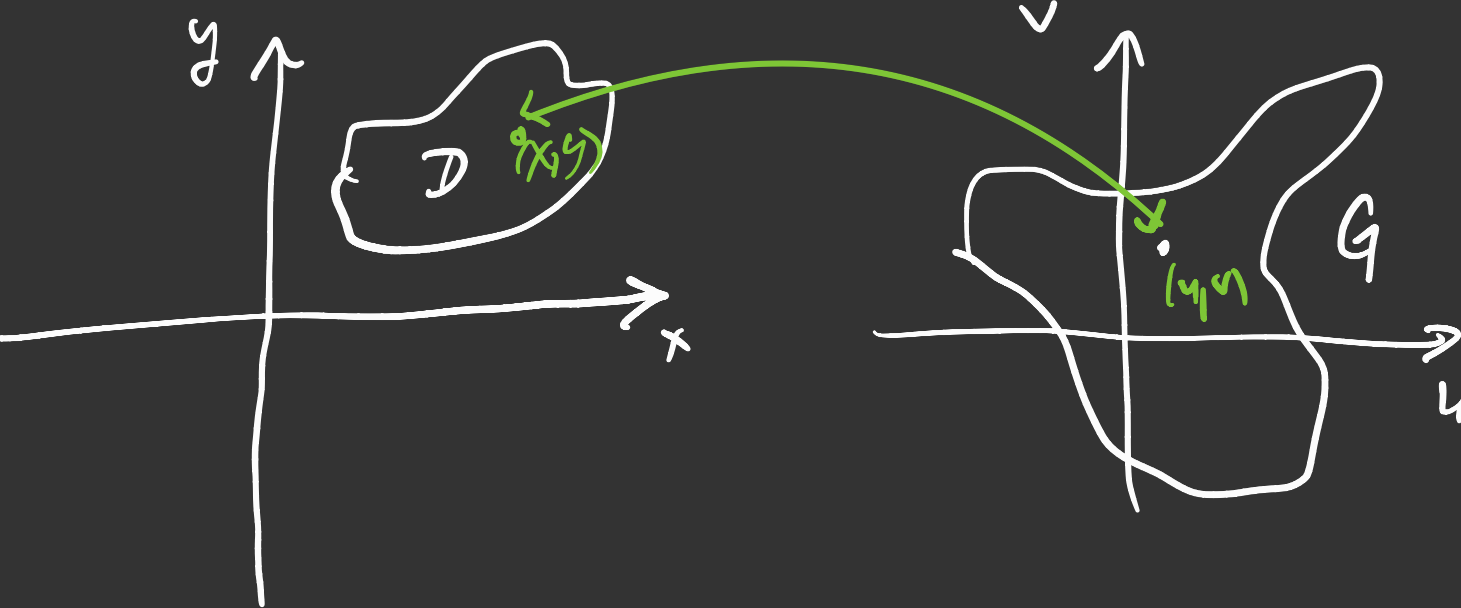
$$\iint_D f(x,y)\, dA_{xy}$$

Sometimes either domain of function is tricky, and we want to make a substitution to new variables (u,v) so that

$$x = x(u,v) \quad \text{and} \quad y = y(u,v)$$

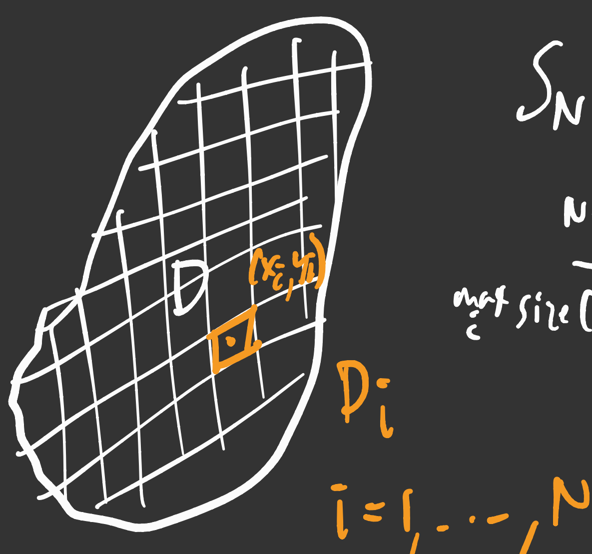
E.g. $x = r\cos\theta$, $y = r\sin\theta$, polar coordinates

$u = r$, $v = \theta$



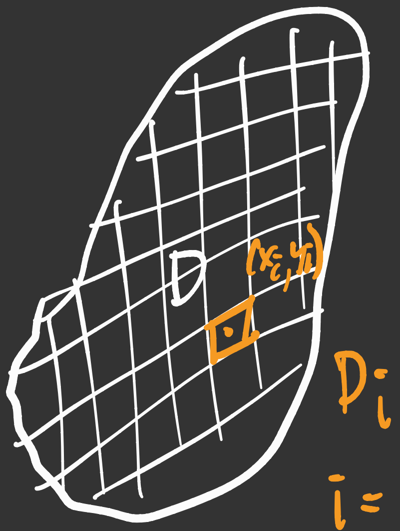
If for every $(x,y) \in D$ there is a point $(u,v) \in G$, then we can transform the integral in x,y to an integral in (u,v). How do we do it?

Let's return to defintion of integral.



$$S_N = \sum_{i=1}^{N} f(x_i, y_i)\, \text{Area}(D_i) \xrightarrow[\max_i \text{size}(D_i) \to 0]{N \to \infty} \iint_D f \, dA$$

Shape of D_i is unimportant; rectangles, parallelograms, triangles etc.



$$S_N = \sum_{i=1}^{N} f(x_i, y_i)\, \text{Area}(D_i)$$

$$\xrightarrow[\max_i \text{size}(D_i) \to 0]{N \to \infty} \iint_D f \, dA$$

$(x(u,v), y(u,v))$

maps partition in G to partition in D

$$(u_i, v_i) \to (x(u_i, v_i), y(u_i, v_i)) =: (x_i, y_i)$$

$$G_i \to D_i$$

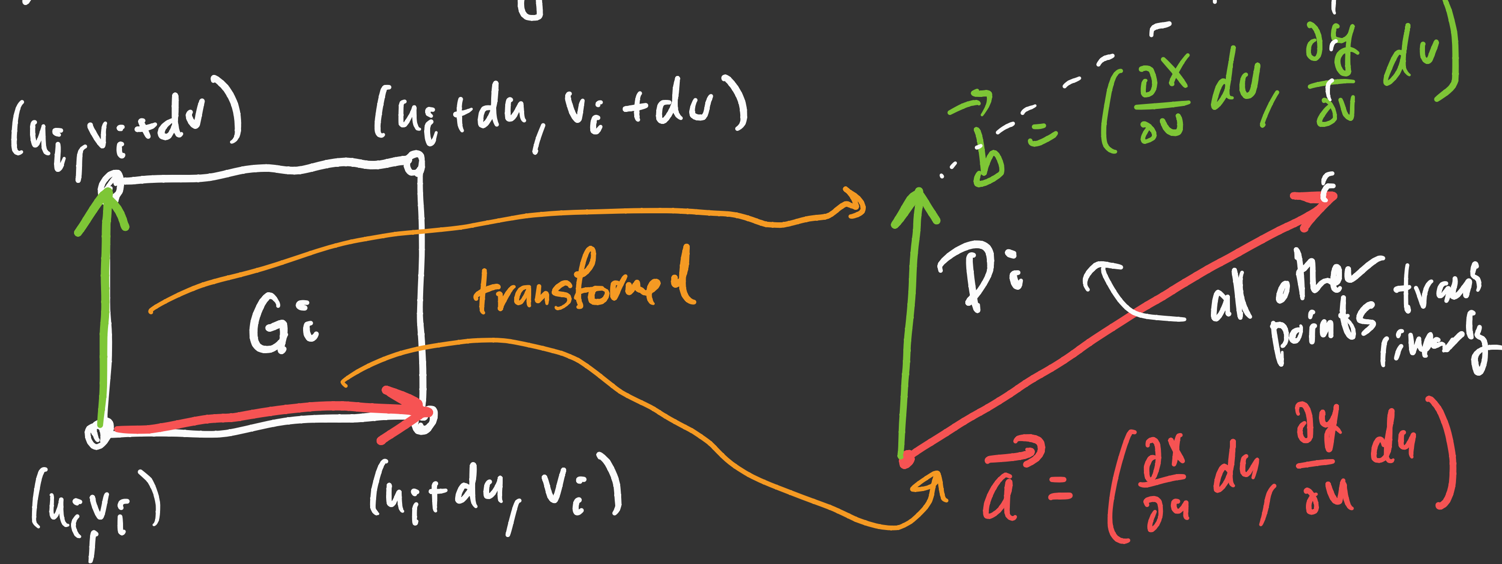
$$S_N = \sum_{i=1}^{N} f(x_i, y_i)\, \text{Area}(D_i)$$

$$= \sum_{i=1}^{N} f(x(u_i, v_i), y(u_i, v_i))\, \text{Area}(D_i)$$

can be different from Area(G_i) since pieces can expand or contract.

We need to introduce a coefficient which captures how much Area(D_i) inflates or shrinks relative to Area$(G_i) = du\,dv$.

Consider more closely a small piece G_i



$$x(u_i + du, v_i) = x(u_i, v_i) + \partial_u x(u_i, v_i)\, du$$

$$y(u_i + du, v_i) = y(u_i, v_i) + \partial_u y(u_i, v_i)\, du$$

$$x(u_i, v_i + dv) = x(u_i, v_i) + \partial_v x(u_i, v_i)\, dv$$

$$y(u_i, v_i + dv) = y(u_i, v_i) + \partial_v y(u_i, v_i)\, dv$$

What is the area of this parallelogram?

$$\text{Area}(D_i) = \left|\begin{matrix} \frac{\partial x}{\partial u} du & \frac{\partial y}{\partial u} du \\ \frac{\partial x}{\partial v} dv & \frac{\partial y}{\partial v} dv \end{matrix}\right| = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) du\,dv$$

$$\text{Area}(D_i) = \begin{vmatrix} \frac{\partial x}{\partial u}du & \frac{\partial y}{\partial u}du \\ \frac{\partial x}{\partial v}dv & \frac{\partial y}{\partial v}dv \end{vmatrix} = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) du\,dv$$

In our picture $(\vec{a}, \vec{b})$ is a right pair

this can be negative if $(\vec{a}, \vec{b})$ is not a right pair

In general

$$\text{Area}(D_i) = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right|_{(x_i, y_i)} \text{Area}(G_i)$$

Jacobian: $\frac{\partial(x,y)}{\partial(u,v)}$

Thus

$$S_N = \sum_{i=1}^{N} f(x(u_i,v_i), y(u_i,v_i))\,\text{Area}(D_i)$$

$$= \sum_{i=1}^{N} f(x(u_i,v_i), y(u_i,v_i)) \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right|_{(x_i, y_i)} \text{Area}(G_i)$$

$$= \sum_{i=1}^{N} f(x(u_i,v_i), y(u_i,v_i)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|_{(x_i, y_i)} \text{Area}(G_i)$$

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$$\xrightarrow{N\to\infty} \iint_G f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \overbrace{dA_{u,v}}^{du\,dv}$$

Thus

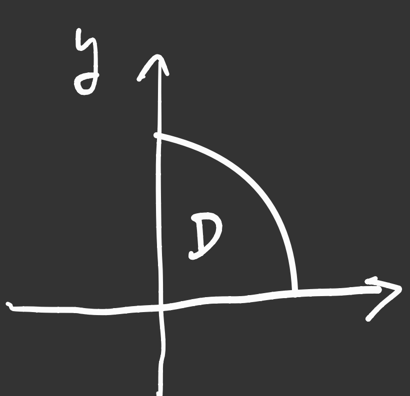
$$\iint_D f(x,y)\, dA_{x,y} = \iint_G f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA_{u,v}$$

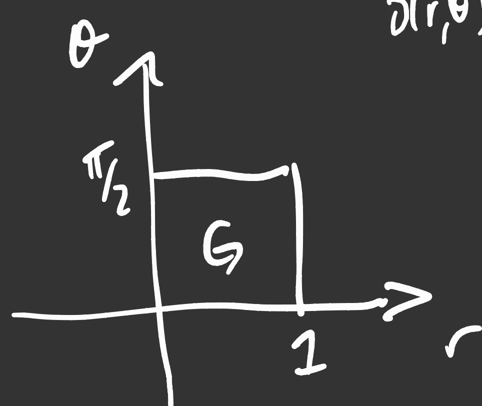
Example: polar coordinates

$x = r\cos\theta, \quad y = r\sin\theta$

$D = \{ r \in [0,1], \ \theta \in [0, \pi/2] \}$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r(\sin^2\theta + \cos^2\theta) = r$$

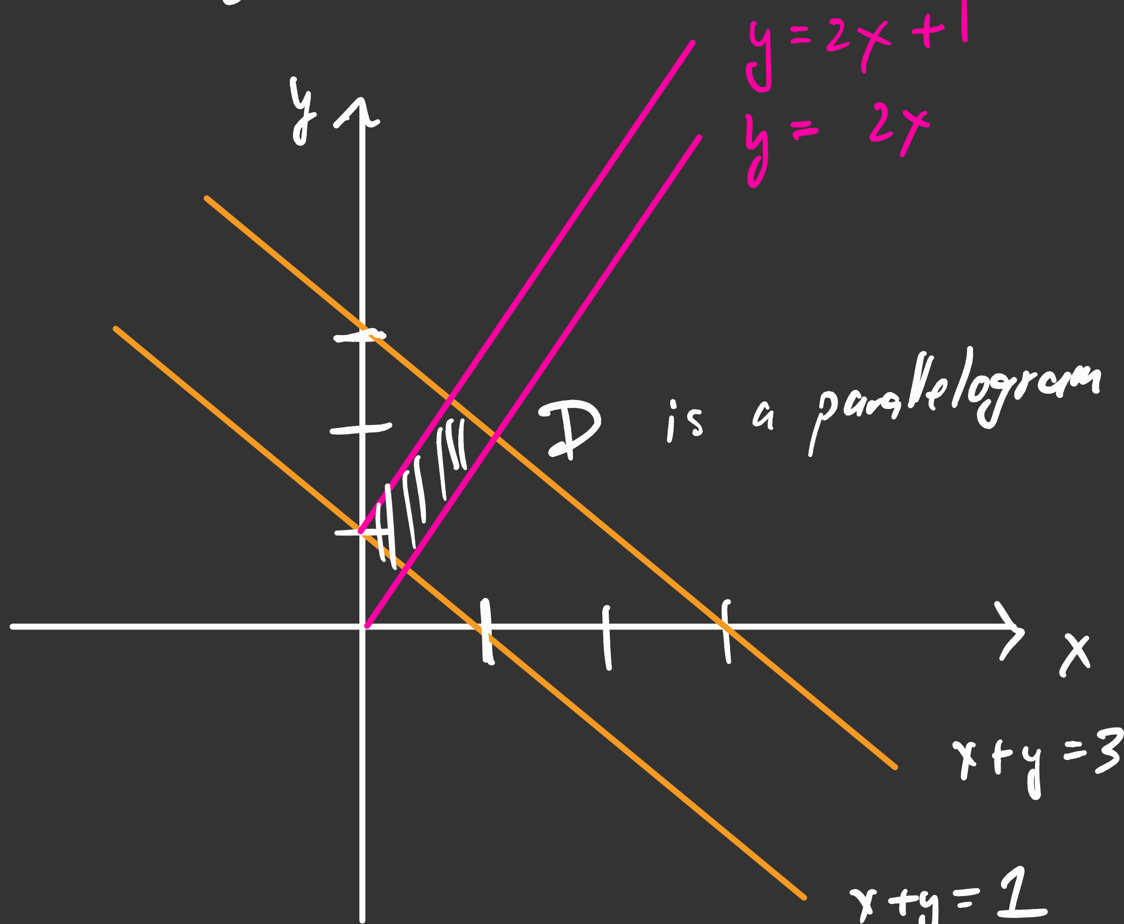




Thus $dA_{x,y} = r\, dA_{r,\theta} = r\,dr\,d\theta$.

Example:

$$D = \left\{ \begin{array}{l} 1 \le x+y \le 3 \\ 0 \le y-2x \le 1 \end{array} \right\}$$



Consider

$$\iint_D (x-y)\, dA_{xy}$$

Can do in, breaking D up into type one regions, but cumbersome...

Let us change variables

$$u = x + y \qquad v = -2x + y$$

now we want to express (x, y) in terms of (u, v)

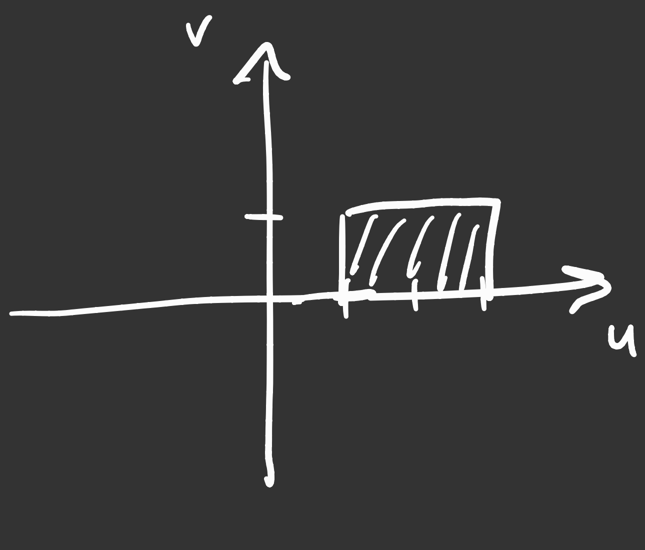
$$\begin{aligned} x + y &= u \\ -2x + y &= v \end{aligned} \implies 3x = u - v \implies x = \frac{1}{3}(u - v)$$

$$\implies y = u - x = \frac{2}{3}u + \frac{1}{3}v$$

$$(x, y) = \left(\frac{1}{3}u - \frac{1}{3}v, \ \frac{2}{3}u + \frac{1}{3}v\right)$$

$$\frac{\partial x}{\partial u} = \frac{1}{3} \qquad \frac{\partial y}{\partial u} = \frac{2}{3} \qquad \implies \qquad \frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{3} \cdot \frac{1}{3} - \left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)$$

$$\frac{\partial x}{\partial v} = -\frac{1}{3} \qquad \frac{\partial y}{\partial v} = \frac{1}{3} \qquad \qquad = \frac{4}{9}$$



$$G = \left\{ \begin{array}{l} 1 \le u \le 3 \\ 0 \le v \le 1 \end{array} \right\}$$

Now $x = \frac{1}{3}u - \frac{1}{3}v$

$y = \frac{2}{3}u + \frac{1}{3}v$

$$\iint_D (x-y)\, dA_{xy} = \iint_G \left(\frac{1}{3}u - \frac{1}{3}v - \frac{2}{3}u + \frac{1}{3}v\right) \cdot \frac{4}{9}\, du\, dv$$

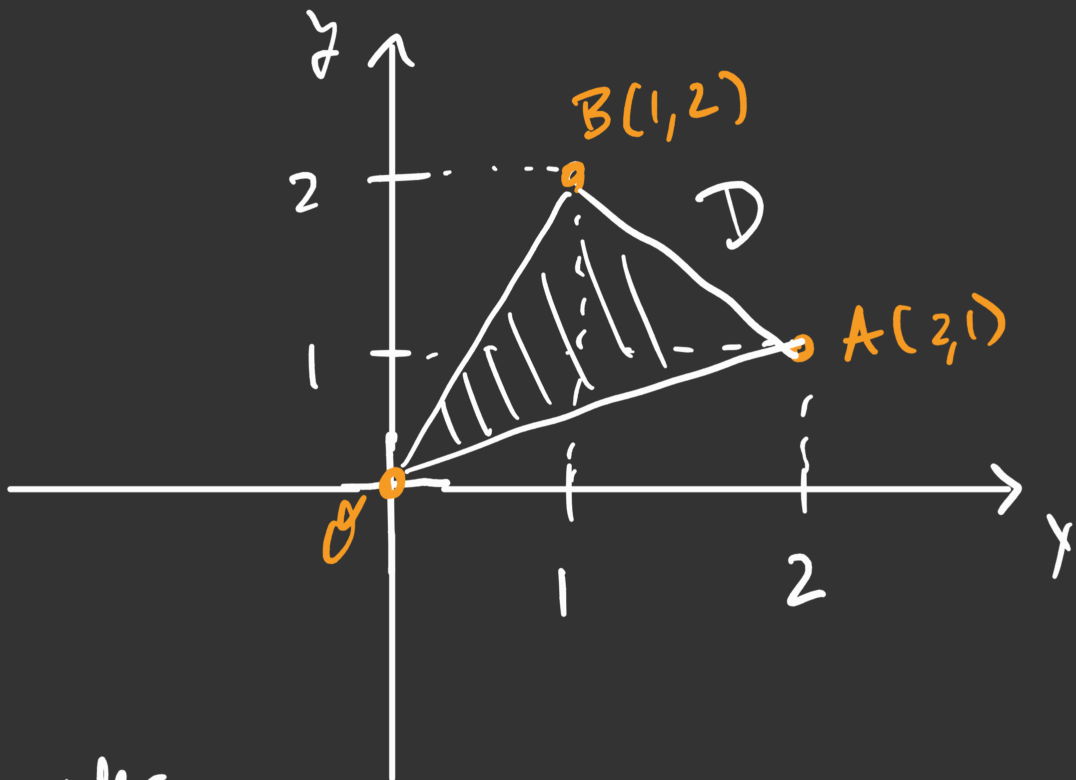
$$= -\int_0^1 \int_1^3 \frac{1}{3}u \cdot \frac{4}{9}\, du\, dv$$

$$= -\frac{4}{27} \left.\frac{u^2}{2}\right|_1^3 = -\frac{2}{27}(9-1)$$

$$= -\frac{16}{27}.$$

Easy!

Consider another example



Consider

$$I = \iint_D (x-y)\, dA_{xy}$$

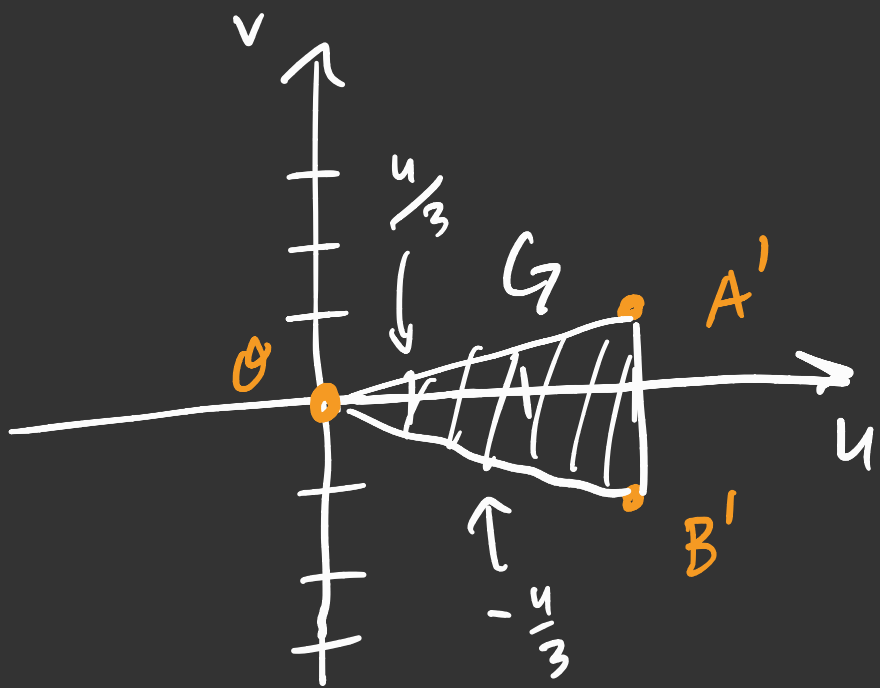
Note that A and B have same sum of coordinate

Consider then

$$u = x+y \qquad \text{and (for symmetry)} \qquad v = x-y$$

Then $\quad x = \frac{1}{2}(u+v) \qquad y = \frac{1}{2}(u-v).$

This triangle is transformed into



$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

Orientation reversed.

Then
$$I = \iint_G \left(\frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}u + \frac{1}{2}v\right) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$$

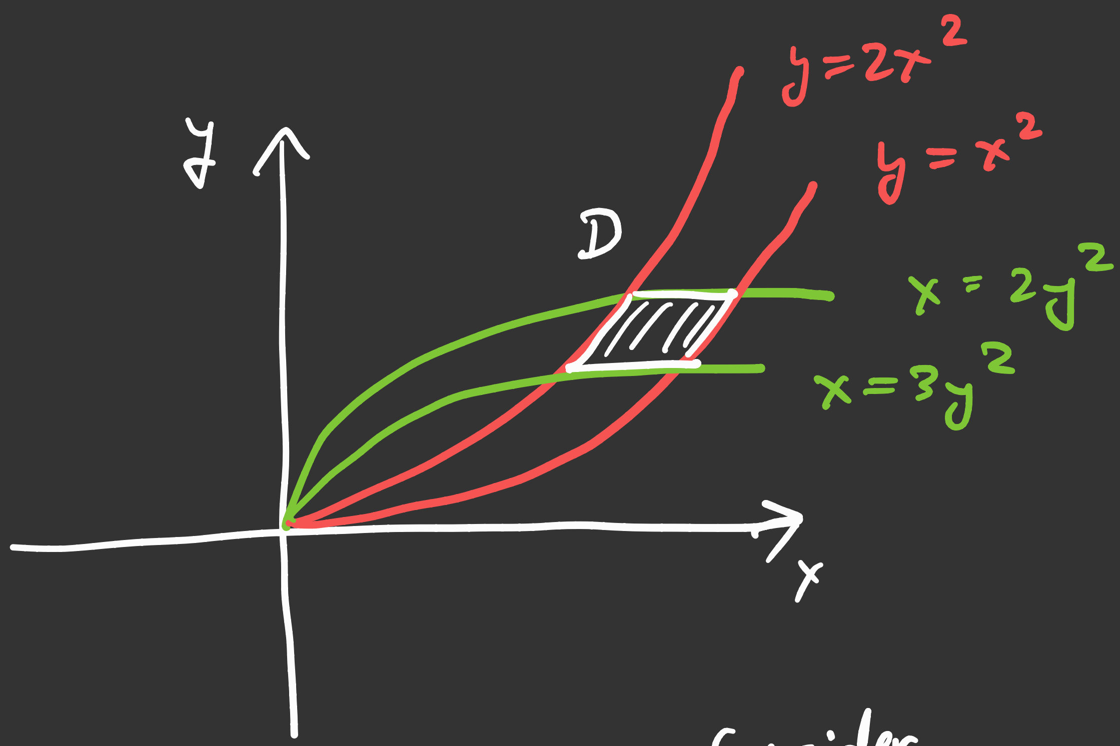
$$= \iint_G v \cdot \frac{1}{2}\, du\, dv$$

$$= \int_0^3 \underbrace{\int_{-u/3}^{u/3} \frac{v}{2}\, dv}_{=0}\, du$$

as integral of an odd function over a symmetric region.

$$= 0$$

Example



Consider

$$D = \left\{ \begin{array}{l} x^2 \le y \le 2x^2 \\ 2y^2 \le x \le 3y^2 \end{array} \right\} \qquad I = \iint_D xy \, dA_{xy}$$

We propose

$$u = \frac{y}{x^2} \quad \text{and} \quad v = \frac{x}{y^2}$$

In these variables, if $x^2 \le y \le 2x^2$, $2y^2 \le x \le 3y^2$

$$\Rightarrow \quad 1 \le u \le 2 \qquad 2 \le v \le 3$$

Now we have to find (x,y) as functions of (u,v)

$$\frac{y}{x^2} = u \quad \frac{x}{y^2} = v \quad \Longrightarrow \quad y^2 = u^2 x^4$$

$$\frac{x}{y^2} = v$$

$$v = \frac{x}{u^2 x^4} = \frac{1}{u^2 x^3} \quad \Longrightarrow \quad x^3 = \frac{1}{u^2 v}$$

$$y = u x^2$$

$$\boxed{x = u^{-2/3} v^{-1/3}} \qquad \boxed{y = u^{-\frac{1}{3}} v^{-2/3}}$$

$$\frac{\partial x}{\partial u} = -\frac{2}{3} u^{-5/3} v^{-1/3} \qquad \frac{\partial y}{\partial u} = -\frac{1}{3} u^{-4/3} v^{-2/3}$$

$$\frac{\partial x}{\partial u} = -\frac{1}{3} u^{-2/3} v^{-4/3} \qquad \frac{\partial y}{\partial v} = -\frac{2}{3} u^{-1/3} v^{-5/3}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{2}{3} u^{-5/3} v^{-1/3} & -\frac{1}{3} u^{-4/3} v^{-2/3} \\ -\frac{1}{3} u^{-2/3} v^{-4/3} & -\frac{2}{3} u^{-1/3} v^{\frac{5}{3}} \end{vmatrix}$$

$$= \frac{4}{9} u^{-2} v^{-2} - \frac{1}{9} u^{-2} v^{-2} = \frac{1}{3} u^{-2} v^{-2}.$$

Thus, as $xy = u^{-1}v^{-1}$, we have

$$I = \iint_D xy \, dA_{xy}$$

$$= \iint_G u^{-1}v^{-1} \cdot \frac{1}{3} u^{-2}v^{-2} \, dA_{uv}$$

$$= \int_1^2 \int_2^3 \frac{1}{3} u^{-3} v^{-3} \, du \, dv = \frac{1}{3} \int_1^2 v^{-3} dv \int_2^3 u^{-3} du$$

$$= \ldots = \frac{5}{576}$$

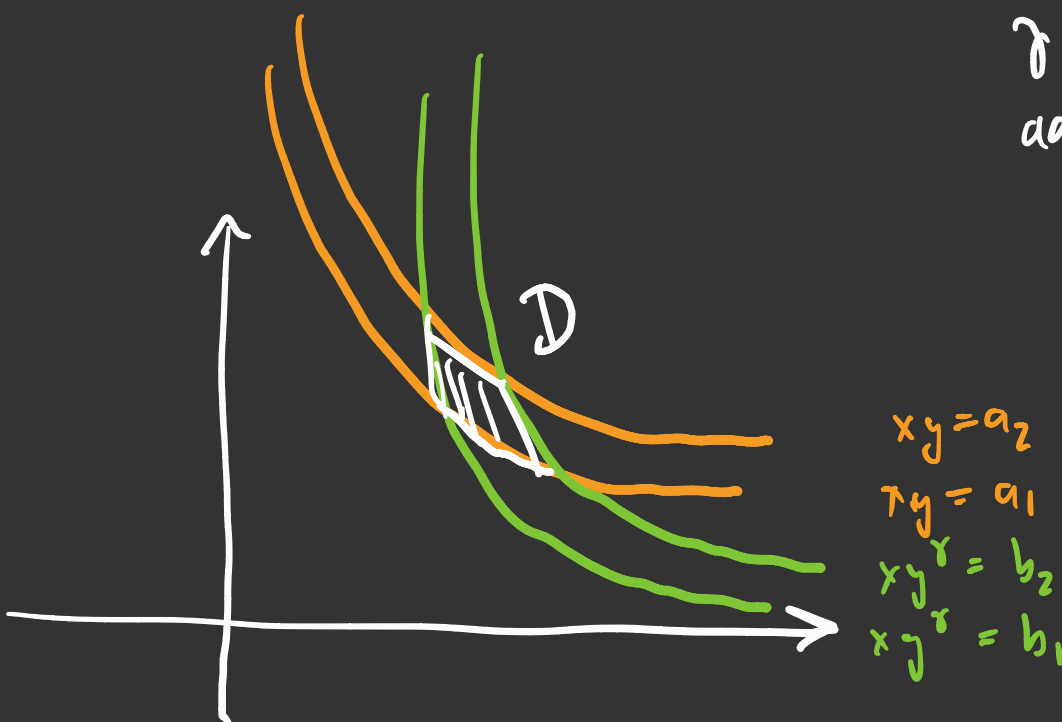
Example (with great historical significance)

It relates to the famous Carnot cycle in thermodynamics. This is an ideal cycle of a heat engine.

Problem: Find the area of the domain

$$D = \{ a_1 \le xy \le a_2, \quad b_1 \le xy^\gamma \le b_2 \}.$$

$\gamma > 1$
adiabatic exponent



Introduce $u = xy \quad v = xy^\gamma$

$$G = \{ a_1 \le u \le a_2, \quad b_1 \le v \le b_2 \}$$

Express (x,y) in terms of (u,v)

$$xy = u \quad \Rightarrow \quad y^{\gamma-1} = u^{-1}v$$
$$xy^{\gamma} = v \quad \Rightarrow \quad y = u^{-\frac{1}{\gamma-1}} v^{\frac{1}{\gamma-1}}$$

Let $\delta = \frac{1}{\gamma-1}$. then $\boxed{y = u^{-\delta} v^{\delta}}$

$\delta > 0$

Now, $xy = u$, so $\boxed{x = u^{1+\delta} v^{-\delta}}$

$$\frac{\partial x}{\partial u} = (1+\delta) u^{\delta} v^{-\delta} \qquad \frac{\partial y}{\partial u} = -\delta u^{-(1+\delta)} v^{\delta}$$

$$\frac{\partial x}{\partial v} = -\delta u^{1+\delta} v^{-(1+\delta)} \qquad \frac{\partial y}{\partial v} = \delta u^{-\delta} v^{\delta-1}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} (1+\delta) u^{\delta} v^{-\delta} & -\delta u^{-(1+\delta)} v^{\delta} \\ -\delta u^{1+\delta} v^{-(1+\delta)} & \delta u^{-\delta} v^{\delta-1} \end{vmatrix}$$

$$= (1+\delta)\delta v^{-1} - \delta^2 v^{-1} = \delta v^{-1}$$

$$\text{Area}(D) = \iint_D dA_{x,y}$$

$$= \iint_G \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

$$= \delta \int_{a_1}^{a_2}\int_{b_1}^{b_2} v^{-1}\, dv\, du$$

$$= \delta (a_2 - a_1) \ln\left(\frac{b_2}{b_1}\right).$$

$$\boxed{I = \frac{a_2 - a_1}{\gamma - 1} \ln\left(\frac{b_2}{b_1}\right)}$$