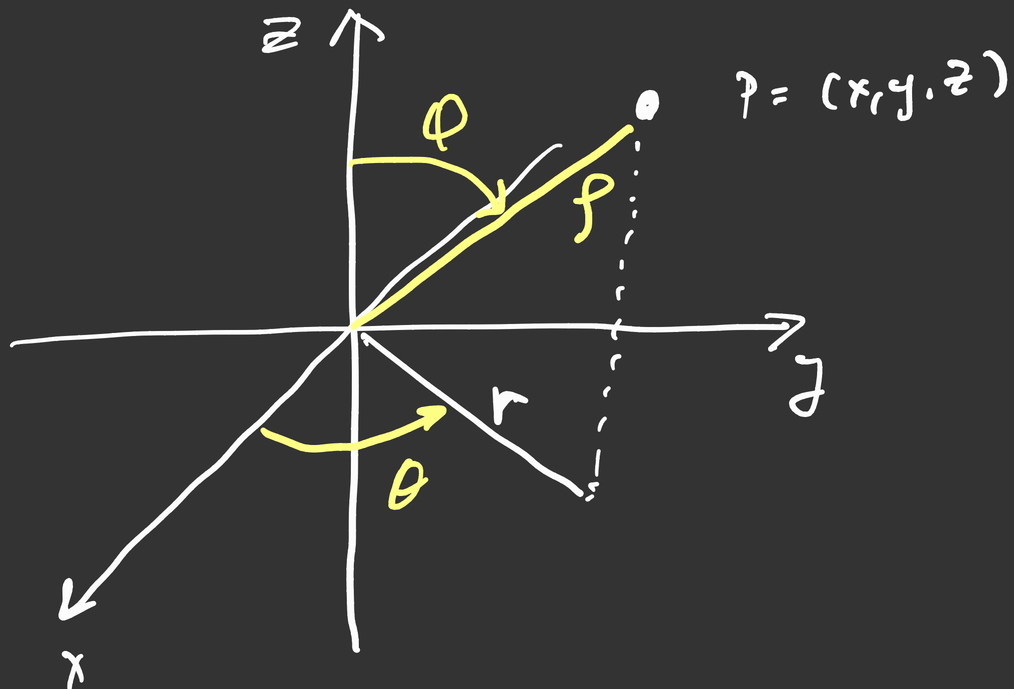
Spherical Coordinates



ρ = distance to the origin

φ = angle to the z-axis

θ = angle to x-axis

r = distance to z axis

$\varphi \in [0, \pi]$

$\varphi = 0$ is north pole, equator is $90°$, south pole is $180°$

$\theta \in [0, 2\pi)$

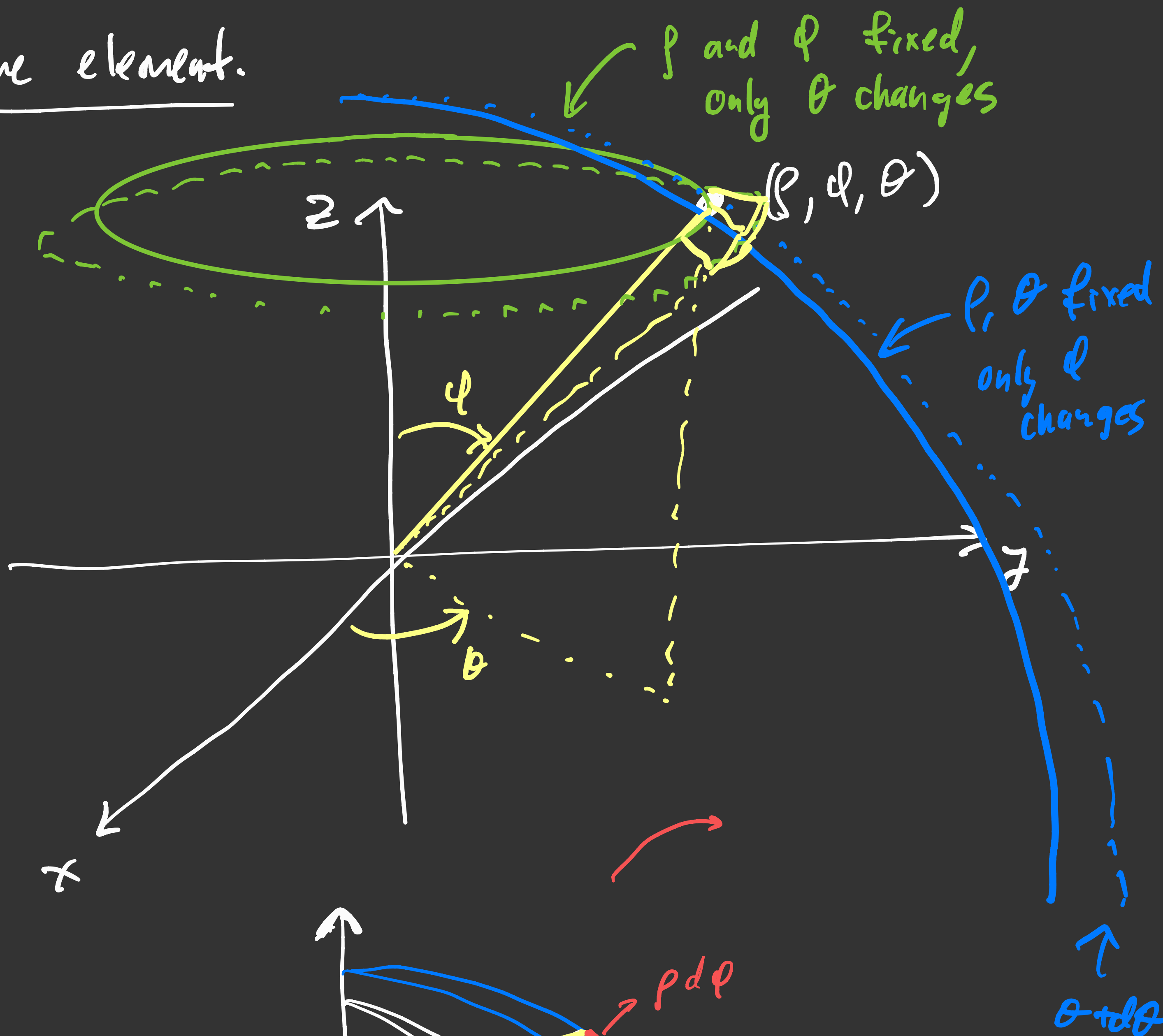
$r = \rho \sin\varphi$

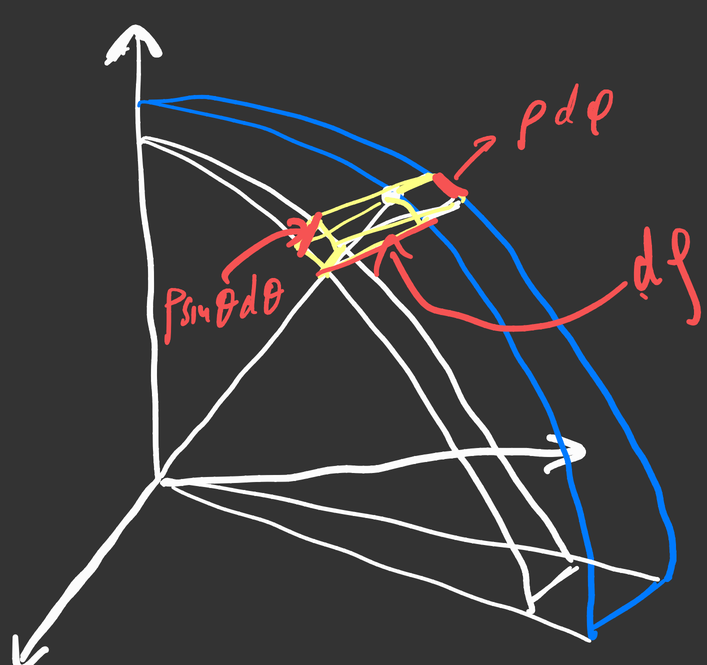
$x = r\cos\theta = \rho \sin\varphi \cos\theta$

$y = r\sin\theta = \rho\sin\varphi \sin\theta$

$z = \rho\cos\theta$

Volume element.





$$dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

Volume of a Ball

$B: \quad 0 \le \theta \le 2\pi, \quad 0 \le \varphi \le \pi, \quad 0 \le \rho \le R$

$$V(B) = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= 2\pi \int_0^R \rho^2 \, d\rho \int_0^{\pi} \sin\varphi \, d\varphi$$

$$= \frac{2\pi}{3} R^3 \cdot (-\cos\varphi)\Big|_0^{\pi}$$

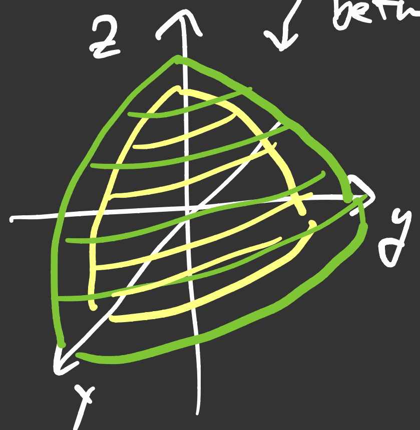
$$= \frac{4\pi}{3} R^3$$

✓

Ex: $B: \quad 0 \le \varphi \le \pi/2 \qquad 0 \le \theta \le \pi/2$

$R_1 \le \rho \le R_2$

between two spheres: $\frac{1}{8}$ of spherical shell.



density $\delta = \text{const}$

$$\text{Mass} = \iiint_B \delta \, dV = \delta \int_0^{\pi/2} \int_0^{\pi/2} \int_{R_1}^{R_2} \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \delta \frac{\pi}{2} \int_0^{\pi/2} \sin\varphi \, d\varphi \int_{R_1}^{R_2} \rho \, d\rho$$

$$= \delta \frac{\pi}{2} \cdot \frac{1}{3} \left(R_2^3 - R_1^3 \right) \int_0^{\pi/2} \sin\varphi \, d\varphi$$

$$= \frac{\delta \pi}{6} \left(R_2^3 - R_1^3 \right)$$

moment:

$z = \rho\cos\varphi$

$$M_{xy} = \iiint_B z\,\delta\,dV$$

$$= \delta \int_0^{\pi/2}\int_0^{\pi/2}\int_{R_1}^{R_2} \rho^3 \sin\varphi\cos\varphi \, d\rho\, d\varphi\, d\theta$$

$$= \frac{\delta\pi}{2} \int_0^{\pi/2} \sin\varphi\cos\varphi\, d\varphi \int_{R_1}^{R_2} \rho^3\, d\rho$$

$$= \frac{\delta\pi}{8}\left(R_2^4 - R_1^4\right) \int_0^{\pi/2} \sin\varphi\cos\varphi\, d\varphi$$

$u = \cos\varphi$

$du = \sin\varphi\, d\varphi$

$$= \frac{\delta\pi}{8}\left(R_2^4 - R_1^4\right) \int_0^1 u\, du$$

$$= \frac{\delta\pi}{16}\left(R_2^4 - R_1^4\right)$$

Since domain is symmetric,

$$M_{xy} = M_{yz} = M_{zx}$$

Center of mass: $\bar{z} = \frac{M_{xy}}{M} = \frac{3}{8}\,\frac{R_2^4 - R_1^4}{R_2^3 - R_1^3}.$

Extremes: first shell is the ball:

If $R_1 = 0$, then

$$\boxed{\bar{z} = \frac{3}{8} R_2} = \bar{x} = \bar{y}$$

other extreme: $R_1 = R_2 - \varepsilon$, $\varepsilon \ll R_2$

$$\bar{z} = \frac{3}{8} \frac{R_2^4 - (R_2 - \varepsilon)^4}{R_2^4 - (R_2 - \varepsilon)^3}$$

But, $f(x - \varepsilon) \approx f(x) + f'(x)(-\varepsilon)$

Thus

$$\bar{z} \approx \frac{3}{8} \frac{R_2^4 - R_2^4 - 4R_2^3(-\varepsilon)}{R_2^3 - R_2^3 - 3R_2^2(-\varepsilon)}$$

$$= \frac{3}{8} \cdot \frac{4R_2^3(-\varepsilon)}{3R_2^2(-\varepsilon)} = \boxed{\frac{1}{2} R_2.}$$

For the thin shell, center of mass moves up 1/8 of ball.
relative to solid

Ex: (with important mechanical implications)

$B: \quad 0 \le \varphi \le \varphi_0, \quad 0 \le \theta \le 2\pi, \quad 0 \le \rho \le R$

drumstick domain

Let δ be density

$$M = \iiint_B \delta \, dV = \delta \int_0^{2\pi} \int_0^{\varphi_0} \int_0^R \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= 2\pi\delta \int_0^{\varphi_0} \int_0^R \rho^2 \sin\varphi \, d\rho \, d\varphi$$

$$= \frac{2}{3}\pi\delta R^3 \cdot (-\cos)\Big|_0^{\varphi_0}$$

$$= \frac{2\pi}{3}\delta R^3 (1 - \cos\varphi_0).$$

Now we want to find center of mass.

Since it is a body of revolution, center of mass must be on z axis, so

$$\bar{x} = \bar{y} = 0.$$

$$\bar{z} = \frac{M_{xy}}{M}$$

$z = \rho \cos\varphi$

$$M_{xy} = \iiint_B \delta\, z \, dV$$

$$= \delta \int_0^{2\pi} \int_0^{\varphi_0} \int_0^R \rho^3 \cos\varphi \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \frac{2\pi\delta}{4} R^4 \int_0^{\varphi_0} \cos\varphi \sin\varphi \, d\varphi$$

$$= \frac{\pi\delta}{2} R^4 \int_0^{\sin\varphi_0} u \, du$$

$$= \frac{\pi\delta}{4} R^4 \sin^2\varphi_0$$

Thus $$\bar{z} = \frac{\pi\delta R^3 \sin^2\varphi_0 / 4}{\frac{2}{3}\pi\delta R^3 (1-\cos\varphi_0)} = \frac{3}{8} \frac{R \sin^2\varphi_0}{1-\cos\varphi_0}$$

$$\bar{x} = \bar{y} = 0$$

If $\varphi_0 = \pi$ (whole ball) then $\bar{z} = 0$ (center of mass is center of ball)

Moment of Inertia

$$I_z = \iiint_B \delta\,(x^2+y^2)\,dV$$

$E = \frac{1}{2} I_z \omega^2$ where ω = angular velocity in rad/sec.

plays role of mass if you consider rotation instead of linear motion.

$$I_z = \int_0^{2\pi}\int_0^{\varphi_0}\int_0^{R} \delta\,\rho^2 \sin^2\varphi \cdot \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= 2\pi\delta \int_0^{\varphi_0}\int_0^{R} \rho^4 \sin^3\varphi \, d\rho\, d\varphi$$

$$= \frac{2\pi\delta}{5} R^5 \int_0^{\varphi_0} \sin^3\varphi\, d\varphi$$

$$= \int_0^{\varphi_0} (1-\cos^2\varphi)\sin\varphi\, d\varphi$$

$c = \cos\varphi \qquad dc = -\sin\varphi\, d\varphi$

$$= -\int_1^{\cos\varphi_0} (1-c^2)\,dc$$

$$= \int_{\cos\varphi_0}^{1} (1-c^2)\,dc$$

$$= 1 - \cos\varphi_0 - \frac{1}{3} + \frac{1}{3}\cos^3\varphi_0$$

$$= \frac{2}{3} - \cos\varphi_0 + \frac{1}{3}\cos^3\varphi_0$$

$$= \frac{2\pi\delta}{15} R^5 \left(2 - 3\cos\varphi_0 + \cos^3\varphi_0\right)$$

What is interesting here?

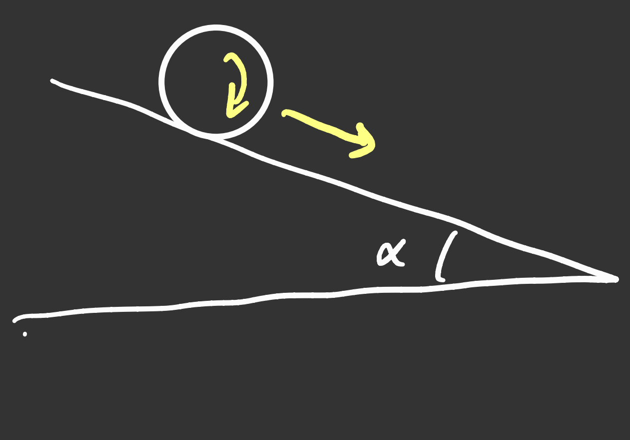
If $\varphi_0 = \pi$ (all the ball), then

$$I_z = \frac{2}{15} \pi \delta R^5 (2 + 3 - 1)$$

$$= \frac{8}{15} \pi \delta R^5$$

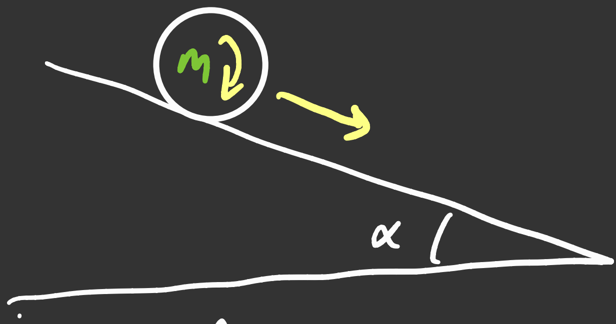
With this result, we can correct a mistake in the textbooks of Mechanics.

Some such texts make the following claim:
It is hard to measure acceleration of ball in free fall
In order to find the law of free fall,
we can perform the following experiment instead:

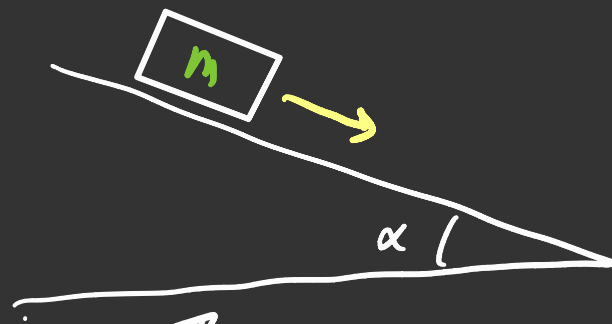


If α is small,
we can measure its displacement easier and exp. will say it moves with constant acceleration.

In fact, the rolling ball moves with different acceleration!



freely rolls

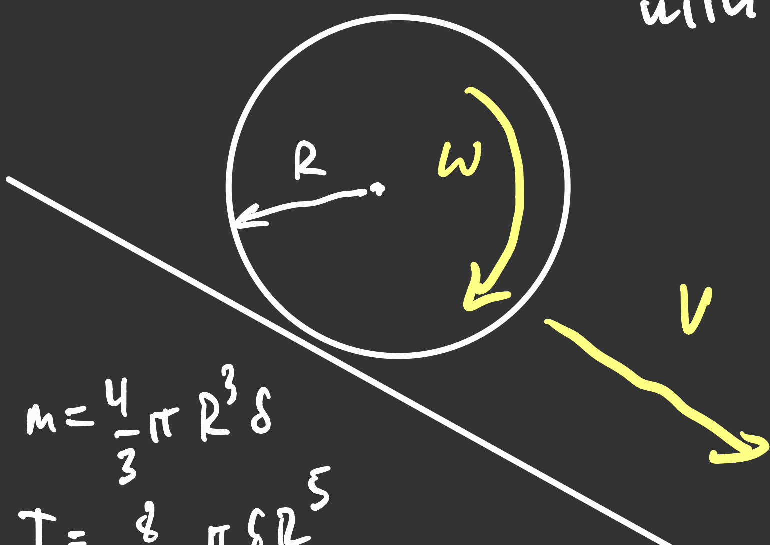


consider instead a frictionless block of equal mass m sliding down

Do they move down with same acceleration?

<u>NO</u>.

When ball moves, moves down at speed v with angular velocity



$$\omega = \frac{v}{R}$$

$$m = \frac{4}{3}\pi R^3 \delta$$

$$I = \frac{8}{15}\pi \delta R^5$$

Kinetic energy

$$= \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2$$

translation + rotation

$$E = \frac{4}{3}\pi R^3 \delta v^2 + \frac{8}{15}\pi \delta R^5 \left(\frac{v^2}{R^2}\right) = \left(\frac{4}{3} + \frac{8}{15}\right)\pi R^3 \delta \frac{v^2}{2}$$

$$E = \underbrace{\left(\frac{4}{3} + \frac{8}{15}\right)}_{M_{eff}} \pi R^3 \delta \frac{v^2}{2}$$

rolling ball

$$E = \frac{4}{3}\pi R^3 \delta \frac{v^2}{2}$$

sliding block

M_{eff} is effective mass that arises since some of the energy is expressed in "rolling" degrees of freedom

M_{eff} is 40% larger than m.

If you put a ball (or brick) that slides down, its acceleration is 40% larger than the rolling ball.

That is, rolling ball will move down slower than the sliding ball.