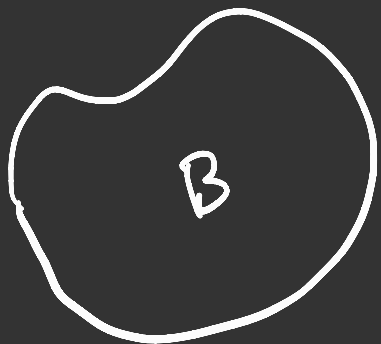
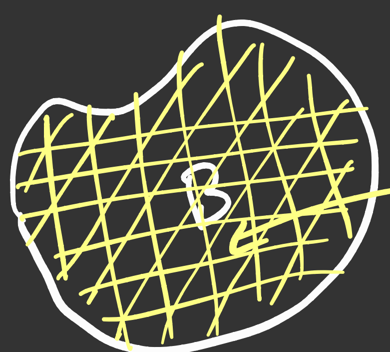
MAT 307 : Advanced Multivariable Calculus — Lecture 25

Triple integral : integral over domain in 3d space



$B \subset \mathbb{R}^3$

partition B into small pieces



B_i has center (x_i, y_i, z_i)

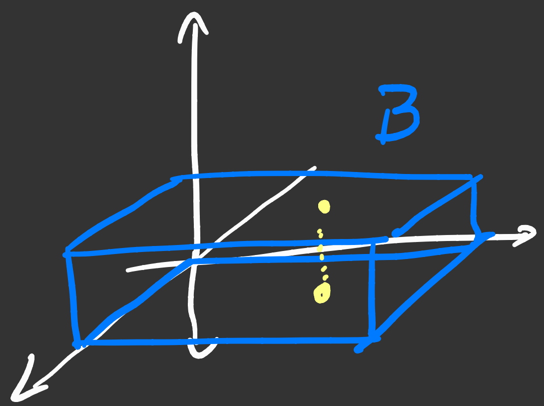
say $f(x,y,z)$ is a continuous function in B.

$$S_n := \sum_{i=1}^{n} f(x_i, y_i, z_i)\, V(B_i) \xrightarrow[\text{size}(B_i) \to 0]{n \to \infty} \iiint_B f(x,y,z)\, dV$$

for instance, if B_i is a small parallelepiped, volume is product of lengths.

Iterated integral

Ex: $B: \left\{ \begin{array}{l} a_1 \le x \le a_2 \\ b_1 \le y \le b_2 \\ c_1 \le z \le c_2 \end{array} \right\}$



$$\iiint_B f \, dV = \int_{a_1}^{a_2} \left(\int_{b_1}^{b_2} \left(\int_{c_1}^{c_2} f(x,y,z) \, dz \right) dy \right) dx$$

first fix x & y and integrate along z.

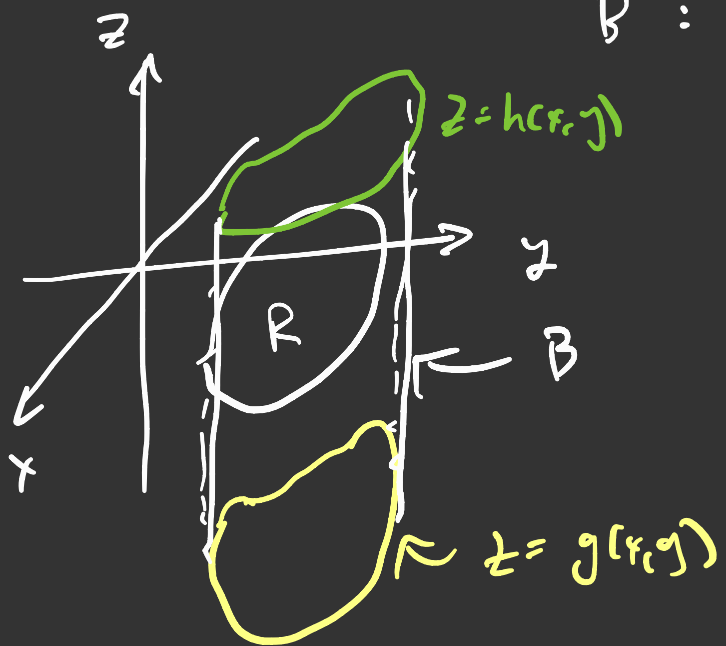
ex $B: \quad 0 \le x \le 1, \quad 0 \le y \le 1, \quad 0 \le z \le 1$

$f(x,y,z) = xy - z^4$

$$\int_0^1 \int_0^1 \int_0^1 (xy - z^4) \, dz \, dy \, dx = \int_0^1 \int_0^1 \left(xy - \frac{1}{5} \right) dy \, dx$$

$$= \int_0^1 \left(\frac{x}{2} - \frac{1}{5} \right) dx = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$$

More general domain

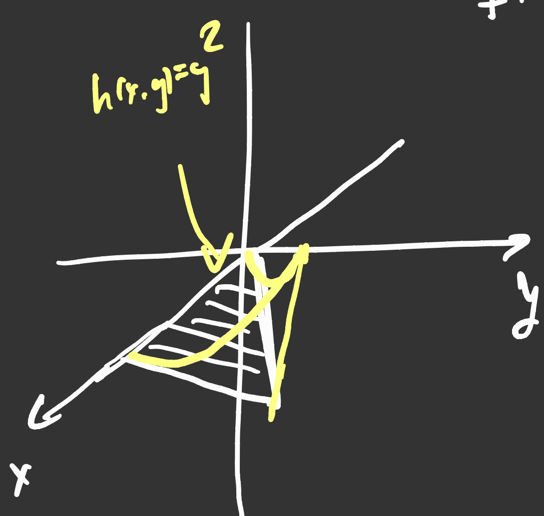


$B: \quad (x,y) \in R, \quad g(x,y) \le z \le h(x,y)$

$$\iiint_B f \, dV = \iint_R \int_{g(x,y)}^{h(x,y)} f(x,y,z) \, dz \, dA_{xy}$$

Ex: $B: \quad 0 \le x \le 1, \; 0 \le y \le x, \; 0 \le z \le y^2$

$f = x + y + z$



$$\iiint_B f \, dV = \int_0^1 \int_0^x \int_0^{y^2} (x+y+z) \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^x \left(xy^2 + y^3 + \frac{y^4}{2} \right) dy \, dx$$

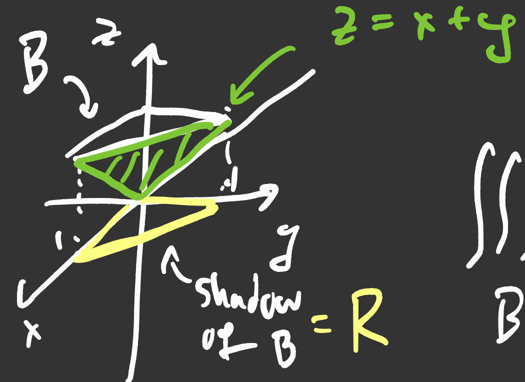
$$= \int_0^1 \left(\frac{x^4}{3} + \frac{x^4}{4} + \frac{x^5}{10} \right) = \frac{1}{15} + \frac{1}{20} + \frac{1}{60}$$

$$= \frac{2}{15}$$

Here we distinguished the z-direction, with our body extruded over a domain in xy place. But, we can change which axis is distinguished, having x or y distinguished, for example.

Sometimes changing the order of integration can make an integral doable.

Ex: B: $x \geq 0$, $y \geq 0$ $x+y \leq z$, $0 \leq z \leq 1$.



$$f(x,y,z) = \sin(z^3)$$

$$\iiint_B f \, dV = \iint_R \int_{x+y}^{1} f \, dz \, dA$$

$$= \int_0^1 \int_0^{1-x} \left(\int_{x+y}^{1} \sin(z^3) \, dz \right) dy \, dx.$$

cannot integrate in terms of simple functions

Instead, look from side:

$B = \ 0 \leq z \leq 1, \ 0 \leq x \leq z, \ 0 \leq y \leq z - x$

$$\iiint_B f \, dV = \int_0^1 \int_0^z \int_0^{z-x} \sin(z^3) \, dy \, dx \, dz$$

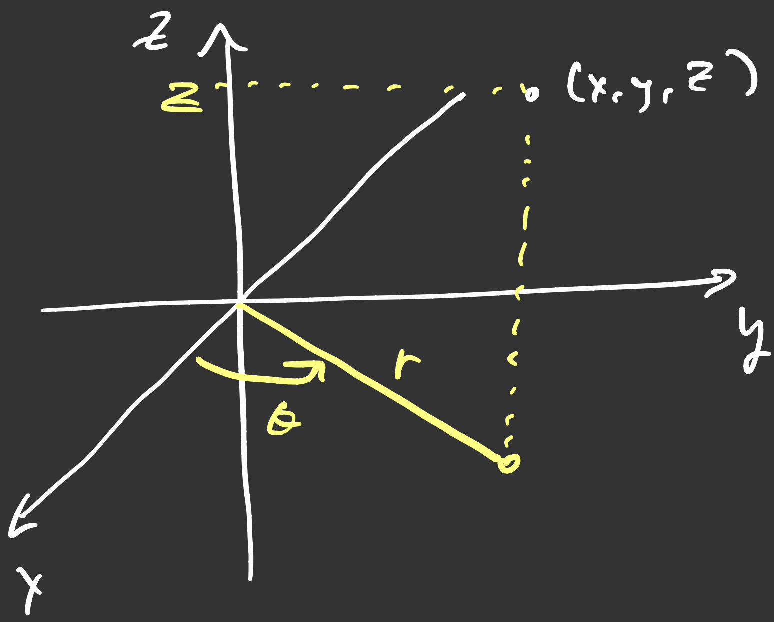
$$= \int_0^1 \int_0^z \sin(z^3)(z-x) \, dx \, dz = \int_0^1 \sin(z^3) \frac{z^2}{2} \, dz =$$

$$= \frac{1}{6} \int_0^1 \sin(t) \, dt$$

$$= \frac{1}{6} (-\cos t)\Big|_0^1$$

$$= \frac{1}{6} (1 - \cos(1))$$

Cylindrical Coordinates



three parameters,

r: distance to z axis

θ: angle to x axis

z: height

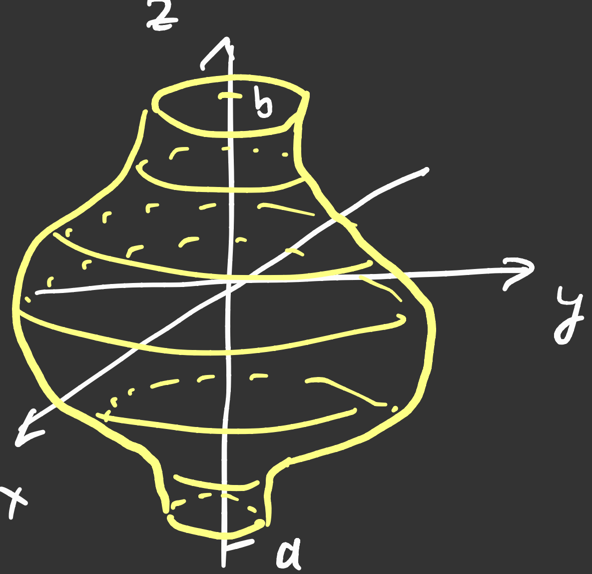
Characterise our point (x, y, z)

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

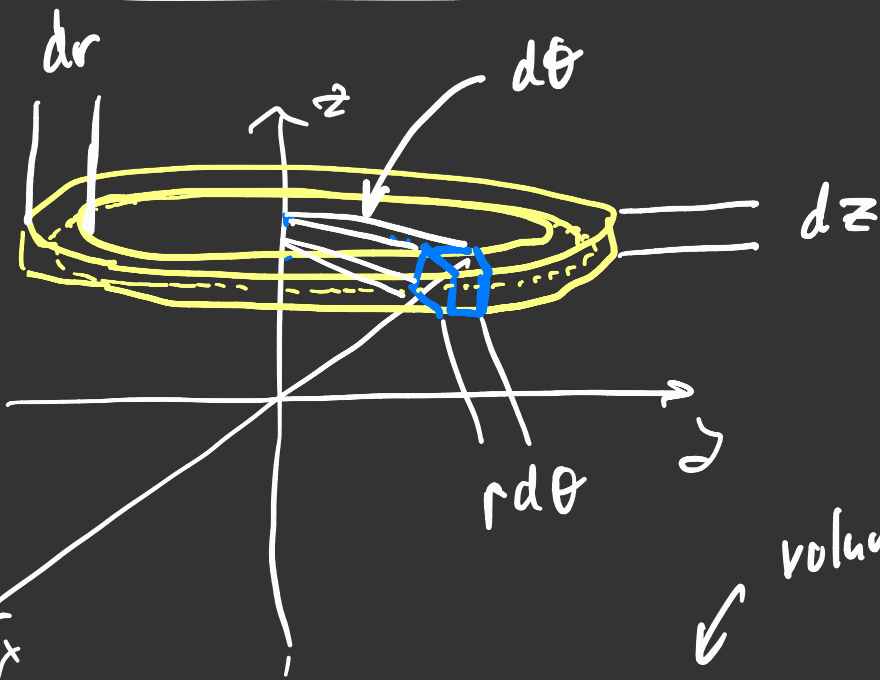
$$r = \sqrt{x^2+y^2} \qquad \cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r} \qquad z = z$$

Body of revolution

$a \le z \le b$, $0 \le r \le g(z)$ $0 \le \theta \le 2\pi$



Volume element in cylindrical coordinates



volume of small piece

$dV = r\,dr\,dz\,d\theta$

Take small piece

$z_0 \le z \le z_0 + dz$

$r_0 \le r \le r_0 + dr$

$\theta_0 \le \theta \le \theta_0 + d\theta$

$B: \quad a \le z \le b \quad 0 \le \theta \le 2\pi \quad 0 \le r \le g(z)$

$$\iiint_B f\,dV = \int_0^{2\pi}\int_a^b \int_0^{g(z)} f(r,z,\theta)\, r\, dr\, dz\, d\theta$$

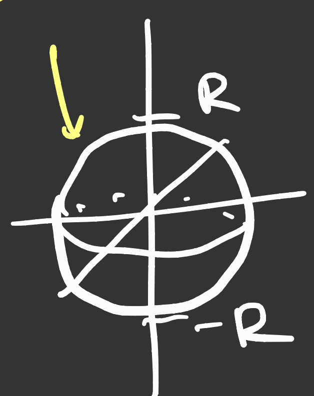
ex: Volume of B:

$$\iiint_B 1\,dV = \int_0^{2\pi}\int_a^b\int_0^{g(z)} r\,dr\,dz\,d\theta$$

$$= 2\pi \int_a^b \frac{g(z)^2}{2}\,dz = \pi \int_a^b g(z)^2\,dz.$$

For example, if B is a ball of radius R

$z^2 + r^2 = R^2$



$B: \quad -R \le z \le R$

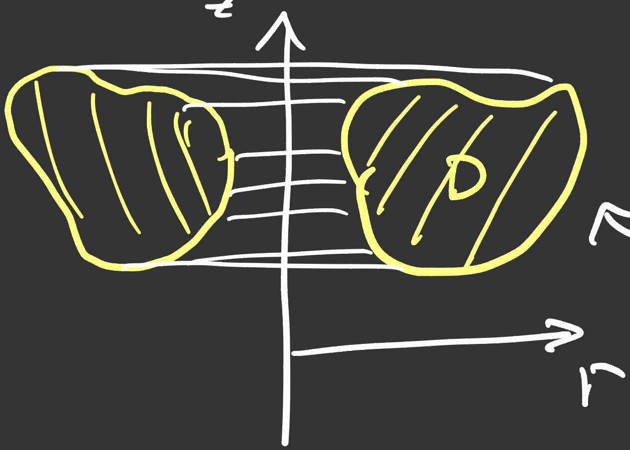
$\quad\quad 0 \le r \le \sqrt{R^2 - z^2}$

Thus $g(z) = \sqrt{R^2 - z^2}$

$$\mathrm{Vol}(B) = \pi \int_{-R}^{R} (R^2 - z^2)\,dz = \pi\left(2R^3 - 2\frac{R^3}{3}\right)$$

$$= \frac{4\pi}{3}R^3.$$

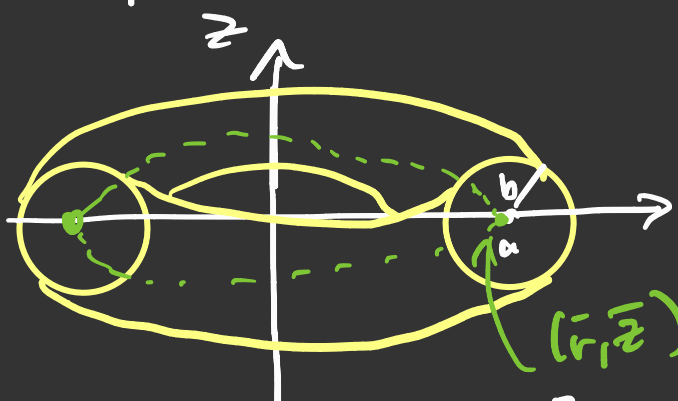
General body of revolution.



$B: \ (r,z) \in D \quad \theta \in [0, 2\pi).$

← sweep around z axis

Example: Volume inside torus (doughnut)



$D: (r-a)^2 + z^2 \le b^2$

$B: (r,z) \in D, \quad \theta \in [0, 2\pi)$

$(\bar{r}, \bar{z})$ center of mass

$$V(B) = \iiint_B dv = \int_0^{2\pi} \iint_D r\,dr\,dz\,d\theta = 2\pi \iint_D r\,dr\,dz$$

moment w.r.t. z

$$= 2\pi M_z(D) = 2\pi \bar{r} A(D)$$

← Pappus's first theorem (Gulden Theorem)

$=$ (length of circle generated by rotating center of mass around z-axis)

$\times$ (area of cross-section in z-r plane).

$$\bar{r} = \frac{M_z(D)}{M(D)}$$

$M(D) = \text{area}(D)$

In words, Pappus's first theorem says;

The volume V of a solid of revolution is equal to the area S of the figure whose rotation generates the solid, multiplied by the circumference $2\pi\bar{r}$ of the circle described in the process of rotation by the center of gravity of the figure.

assuming the figure is a homogeneous plate

Pappus of Alexandria (end of 3rd century AD) was the last of the great Greek mathematicians

Sometimes this theorem is attributed to Paul Guldin (1577-1643), a Swiss Monk and amateur mathematician, who wrote proofs of (weaker forms) of Pappus' statements. Stronger proofs given by Cavalieri and Kepler.

In the case of our doughnut,

length of circle $= 2\pi a$

$A(D) = \pi b^2$

Thus

$$\boxed{V(B) = 2\pi^2 a b^2.}$$

Note, we may compute directly

$$\iint_D r\,dr\,dz = \iint_D (r-a)\,dr\,dz + a\iint_D dr\,dz$$

$$\overset{\rho = r-a}{=} \iint_{\{\rho^2+z^2\le b^2\}} \rho\,d\rho\,dz + a(\pi b^2)$$

$$= \int_0^{2\pi}\int_0^b r\cos\theta\,dr\,d\theta + \pi a b^2$$

$$= \pi a b^2$$

Thus, again

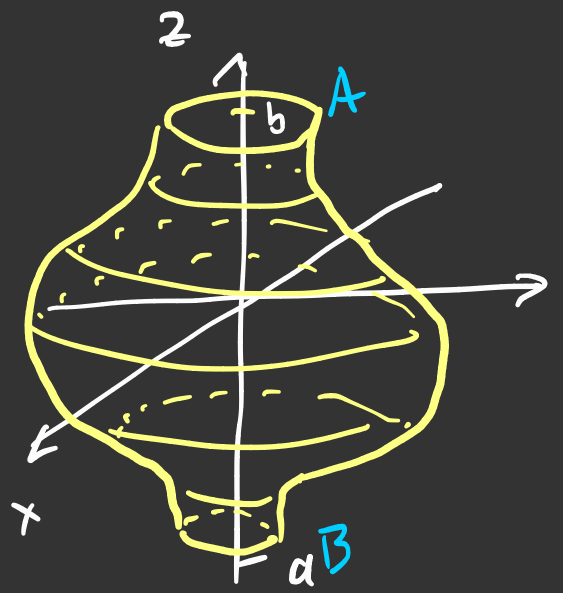
$$V(B) = 2\pi\iint_D r\,dr\,dz = 2\pi^2 a b^2.$$

Recall now our discussion of Surface area:

$0 \le \theta \le 2\pi$

$a \le z \le b$,

$0 \le r \le g(z)$



Think of the surface as being swept out by a heavy string formed in the shape of $\overset{\frown}{AB}$

Let $\bar{r}$ be the string's center of mass

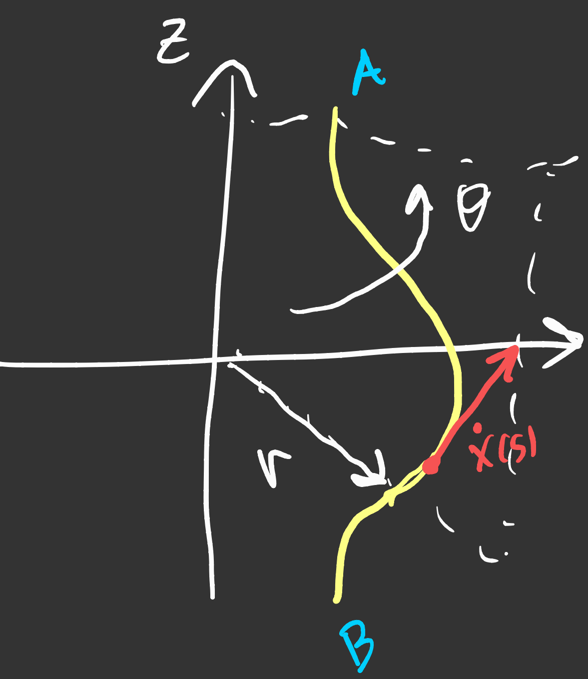
$$\bar{r} = \frac{1}{L}\int_{s_1}^{s_2} g(x(s))\,ds \qquad x(s_1)=a,\ x(s_2)=b$$

ds = arclength measure

$$= \frac{1}{L}\int_a^b g(z)\sqrt{1+(g'(z))^2}\,dz$$

where L is the curve's length

$$L = \int_{s_0}^{s_1} ds = \int_a^b \sqrt{1+(g'(z))^2}\,dz$$



$$S = 2\pi\int_a^b g(z)\sqrt{1+(g'(z))^2}\,dz = 2\pi L\bar{r}$$