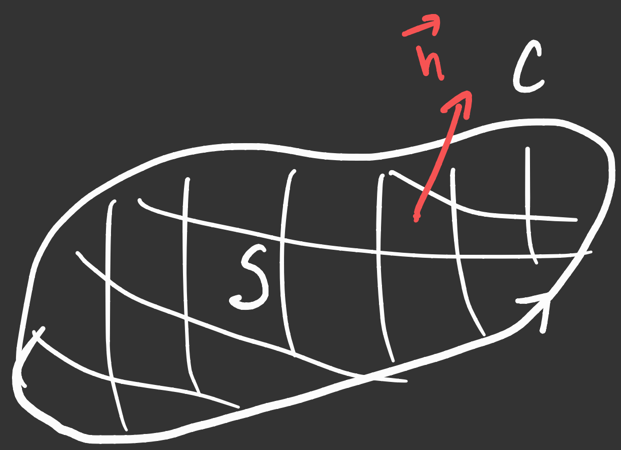
Stokes Formula



orientation is counter clockwise with respect to the normal (going in direction $\ominus \to \oplus$)

vector field $\vec{F}(\vec{r}) = (P, Q, R)$.

Stokes formula:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot \vec{n} \, dS$$

Left hand side is defined geometrically; independent of coordinates.

The same holds for the right-hand-side.

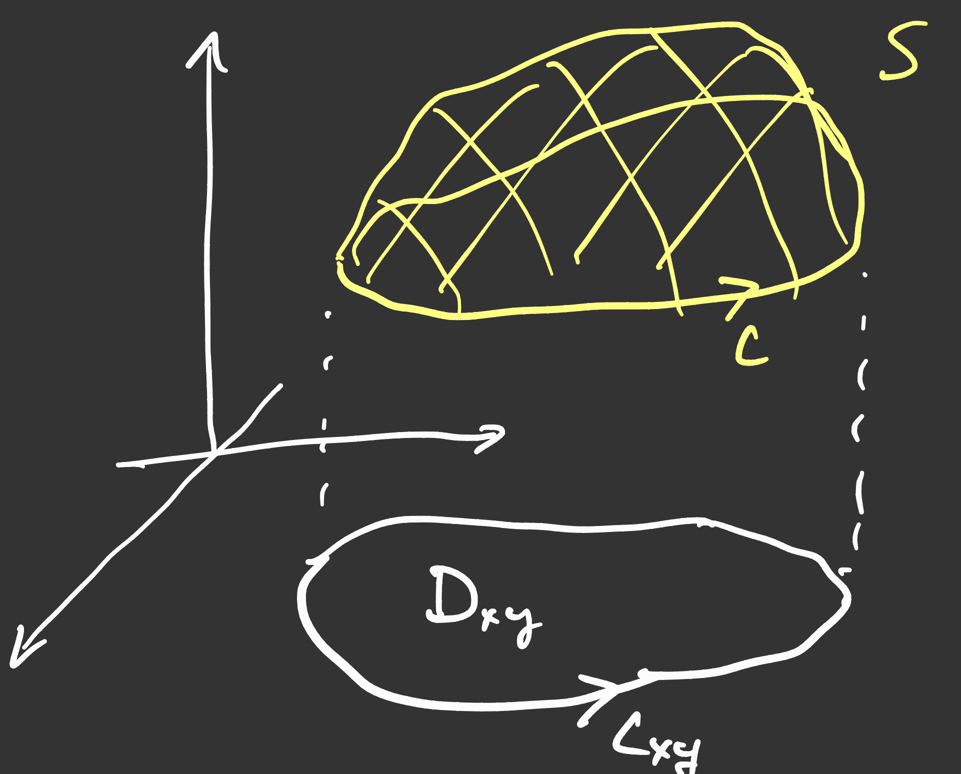
This formula is the <u>basis</u> for Maxwell's equations of electrodynamics. It is also the basis for some of hydrodynamics.

Why is this formula true?

We start from 2-d Stokes formula

Consider $\vec{F}(x,y,z) = (P(x,y), Q(x,y), 0)$

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & 0 \end{vmatrix} = (0, 0, \partial_x Q - \partial_y P)$$



$S: z = f(x,y)$
$(x,y) \in D$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C P(x,y)\,dx + Q(x,y)\,dy + 0\,dz$$

$$= \oint_C P(x,y)\,dx + Q(x,y)\,dy$$

since independent of z

$$= \oint_{C_{xy}} P(x,y)\,dx + Q(x,y)\,dy$$

On other hand, since $\hat{n} = (-f_x, -f_y, 1)$ for $z = f(x,y)$

$$\iint_S \operatorname{curl} F\cdot\hat{n}\,dS = \iint_{D_{xy}} \operatorname{curl} F(x,y,f(x,y))\cdot(-f_x, -f_y, 1)\,dA$$

$$= \iint_{D_{xy}} \left(Q_x(x,y) - P_y(x,y)\right) dA_{xy}$$

by Green's formula

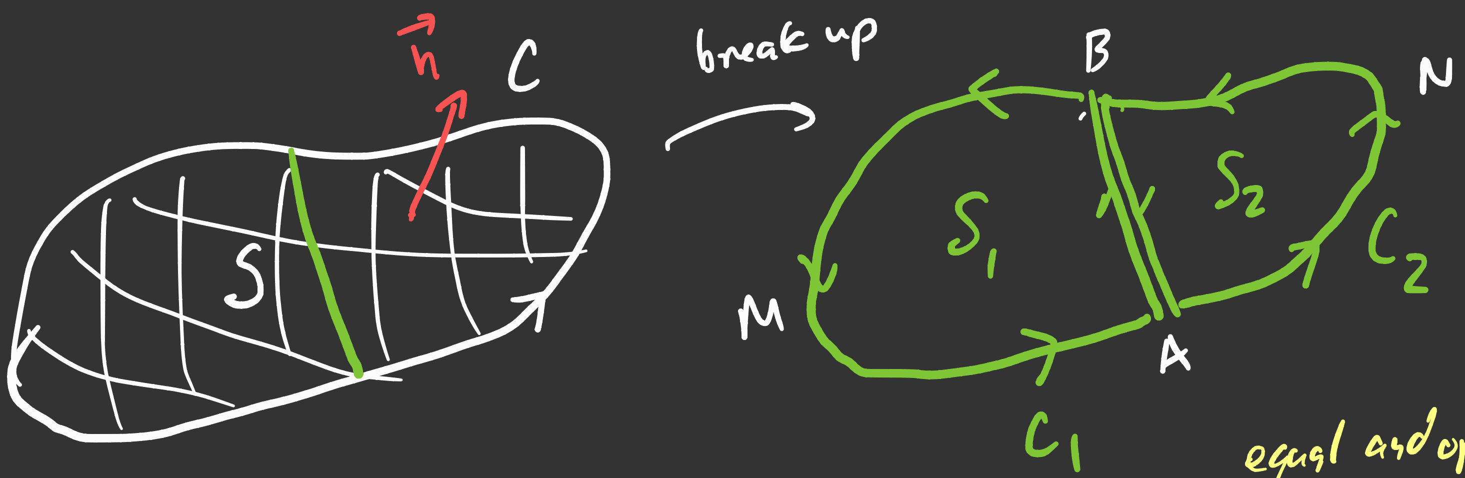
$$= \oint_{C_{xy}} P\,dx + Q\,dy$$

Thus, the Stokes formula for "2d fields" is proved

Applies also to:

$\vec{G}(\vec{r}) = (0, Q(y,z), H(y,z))$ $\quad \vec{H}(\vec{r}) = (P(x,z), 0, Q(x,z))$

Stokes formula: general case



equal and opposite

$$\oint_{C_1} \vec{F}\cdot d\vec{r} + \oint_{C_2} \vec{F}\cdot d\vec{r} = \int_{BMA} + \int_{AB} + \int_{ANB} + \int_{BA}$$

$$= \int_{BMA} + \int_{ANB}$$

$$= \oint_{C} \vec{F}\cdot d\vec{r}$$

Also:

$$\iint_S = \iint_{S_1} + \iint_{S_2} \qquad \text{by additivity.}$$

Our goal is to prove

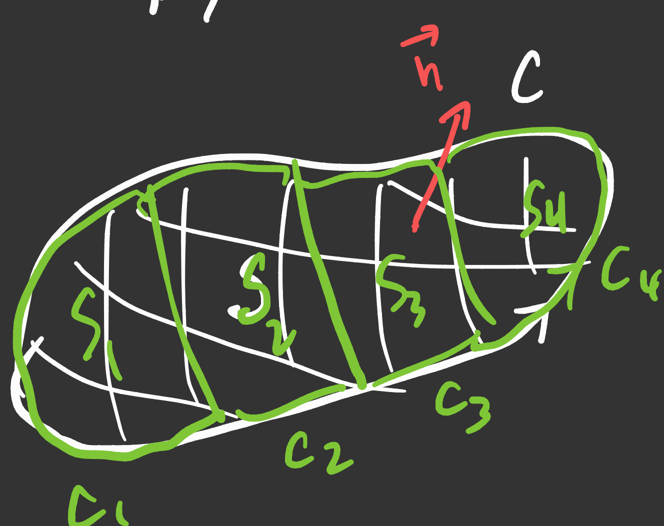
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot \vec{n} \, dS$$

We know now that, if

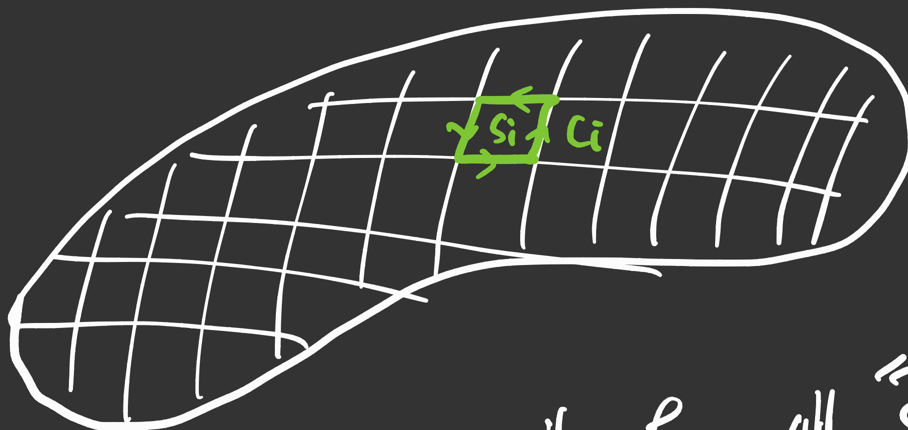
$$\oint_{C_1} \vec{F} \cdot d\vec{r} = \iint_{S_1} \operatorname{curl} \vec{F} \cdot \vec{n} \, dS$$

$$\oint_{C_2} \vec{F} \cdot d\vec{r} = \iint_{S_2} \operatorname{curl} \vec{F} \cdot \vec{n} \, dS$$

Then, we achieve our goal. Thus, as we did to "prove" Green's theorem, we reduced our problem to two subproblems. As before, we continue...

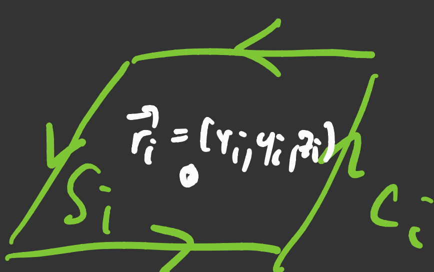


$$\oint_C \vec{F}\cdot d\vec{r} = \sum_{i=1}^{N} \oint_{C_i} \vec{F}\cdot d\vec{r} \qquad \iint_S \mathrm{curl}\,\vec{F}\cdot\vec{n}\,dS = \sum_{i=1}^{N} \iint_{S_i} \mathrm{curl}\,\vec{F}\cdot\vec{n}\,dS$$



If we can prove it for all "scales", then by the above (ignoring controlling errors) we have proved Stokes theorem.

Consider just one small piece



Our functions P, Q, R are assumed differentiable and are thus approximated by linear functions.

$$P(x,y,z) \approx P(\vec{r}_i) + P_x(\vec{r}_i)(x-x_i) + P_y(\vec{r}_i)(y-y_i) + P_z(\vec{r}_i)(z-z_i)$$

$$= P(\vec{r}_i) - P_x(\vec{r}_i)x_i - P_y(\vec{r}_i)y_i - P_z(\vec{r}_i)z_i$$

$$+ \quad P_x(\vec{r}_i)x + P_y(\vec{r}_i)y + P_z(\vec{r}_i)z$$

$$:= \bar{P}_i + P_x(\vec{r}_i)x + P_y(\vec{r}_i)y + P_z(\vec{r}_i)z$$

Similarly, in S_i,

$$Q(\vec{r}) \approx \bar{Q}_i + Q_x(\vec{r}_i)\, x + Q_y(\vec{r}_i)\, y + Q_z(\vec{r}_i)\, z$$

$$R(\vec{r}) \approx \bar{R}_i + R_x(\vec{r}_i)\, x + R_y(\vec{r}_i)\, y + R_z(\vec{r}_i)\, z$$

Now for the integral:

$$\oint_{C_i} P\,dx + Q\,dy + R\,dz$$

$$= \oint_{C_i} (\bar{P}_i + P_x x + P_y y + P_z z)\,dx + (\bar{Q}_i + Q_x x + Q_y y + Q_z z)\,dy + (\bar{R}_i + R_x x + R_y y + Q_z z)\,dz$$

Now, many terms are zero:

$$\oint_{C_i} (\bar{P}_i\,dx + \bar{Q}_i\,dy + \bar{R}_i\,dz)$$

$$= \oint_{C_i} \frac{\partial}{\partial x}(\bar{P}_i x)\,dx + \frac{\partial}{\partial y}(\bar{Q}_i y)\,dy + \frac{\partial}{\partial z}(\bar{R}_i z)\,dz$$

potential field

$$= \oint_{C_i} \nabla(\bar{P}_i x + \bar{Q}_i y + \bar{R}_i z)\cdot d\vec{r} = 0$$

Similarly

$$\oint_{C_i} P_x x\,dx + Q_y y\,dy + R_z z\,dz$$

Potential field

$$= \oint_{C_i} \nabla\left(\frac{P_x}{2} x^2 + \frac{Q_y}{2} y^2 + \frac{R_z}{2} z^2 \right) \cdot d\vec{r} = 0$$

What remains is

$$\oint_{C_i} (P_y y + P_z z)\,dx + (Q_x x + Q_z z)\,dy + (R_x x + R_y y)\,dz$$

Rearranging

$$= \oint_{C_i} (P_y y\,dx + Q_x x\,dy)$$

$$+ \oint_{C_i} (P_z z\,dx + R_x x\,dz)$$

$$+ \oint_{C_i} (Q_z z\,dy + R_y y\,dz)$$

Consider

Simple vector field!

$$\oint_{C_i} P_y\, y\, dx + Q_x\, x\, dy = \int_{C_i} (P_y y, Q_x x, 0) \cdot d\vec{r}$$

By the 2d Stokes formula, this is equal to

$$= \iint_{S_i} \mathrm{curl}\, (P_y y, Q_x x, 0) \cdot \vec{n}\, dS$$

For the same reason

$$\oint_{C_i} (P_z z, 0, R_x x) \cdot d\vec{r} = \iint_{S_i} \mathrm{curl}\, (P_z z, 0, R_x x) \cdot \vec{n}\, dS$$

$$\oint_{C_i} (0, Q_z z, R_y y) \cdot d\vec{r} = \iint_{S_i} \mathrm{curl}\, (0, Q_z z, R_y y) \cdot \vec{n}\, dS$$

Thus, combining, for the little piece  we proved:

$$\oint_{C_i} \vec{F} \cdot d\vec{r} = \iint_{S_i} \mathrm{curl}\, \vec{F} \cdot \vec{n}\, dS$$

$$\oint_{C_i} F \cdot d\vec{s} = \iint_{S_i} \operatorname{curl}(P_y y, Q_x x, 0) \cdot \vec{n}\, dS$$

$$+ \iint_{S_i} \operatorname{curl}(P_z z, 0, R_x x) \cdot \vec{n}\, dS$$

$$+ \iint_{S_i} \operatorname{curl}(0, Q_z z, R_y y) \cdot \vec{n}\, dS$$

$$= \iint_{S_i} \operatorname{curl}\left(P_y y + P_z z,\ Q_x x + Q_z z,\ R_x x + R_y y\right) \cdot \hat{n}\, dS$$

Now note that the curl does not differentiate in the variable corresponding to the component. Thus

$$= \iint_{S_i} \operatorname{curl}\left(P_x x + P_y y + P_z z,\ Q_x x + Q_y y + Q_z z,\ R_x x + R_y y + R_z z\right) \cdot \hat{n}\, dS$$

also the curl of a constant vector, $(\bar{P}, \bar{Q}, \bar{R})$ is zero, so

$$\approx \iint_{S_i} \operatorname{curl} F \cdot \vec{n}\, dS$$

Logical scheme of "proof".

① derive theorem for simple case of a two-dimensional vector field by an application of Green's theorem.

② Reduce our problem to consideration to small pieces, just one "scale".

③ Within a scale, we approximate of functions P, Q, R by linear functions. We then see some terms disappear. What remains are three terms which represent 2d vector fields and use step ① to conclude.