Green's Theorem

Consider some domain

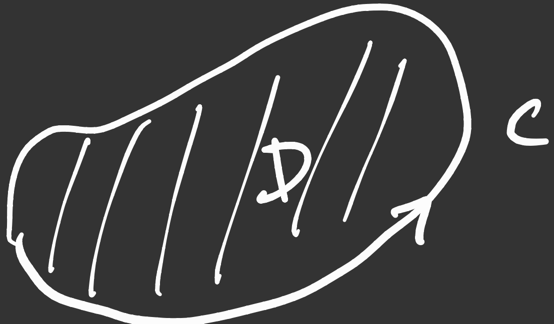
To define integration, we must give some direction on C.

Convention: if C is the boundary of a domain, the direction is chosen so that if you move in that direction, domain is always to your left.

counter-clockwise.

Rule also applies to domains with holes

Consider first simply connected domain:



Consider

$$\oint_C \left(P(x,y)\,dx + Q(x,y)\,dy\right) = \oint_C P\,dx + Q\,dy$$

Theorem:

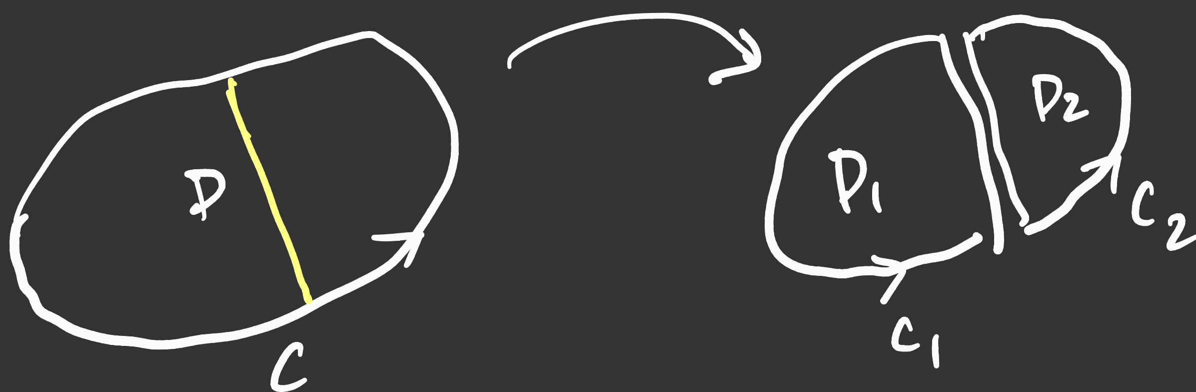
$$\oint_C P\,dx + Q\,dy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

- If $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, both sides are zero.
- If P, Q are zero on the boundary, both sides are zero even if $\frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y}$

Helps a lot to find integrals!

We will try to understand why this theorem is true, with an emphasis on seeing what the key idea is and how <u>you</u> could have discoved it.

<u>Proof:</u> divide domain in two parts



Will establish

① $$\int_C P\,dx + Q\,dy = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy$$

② $$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

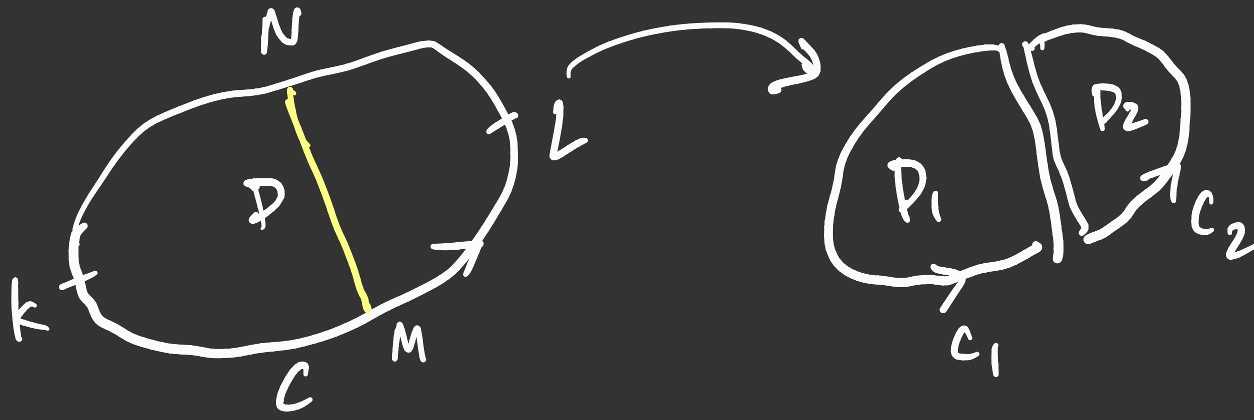
If we can prove ① and ②, and further that

$$\int_{C_i} P\,dx + Q\,dy = \iint_{D_i} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \qquad i=1,2$$

Then we prove the theorem for the domain.

To prove ②, we note that this is just a property of double integration called additivity. We saw it directly from the limit definition of the integral.

How about ①?



$$\int_{C_1} P\,dx + Q\,dy = \int_{NKM} \cdots + \int_{MN} \cdots$$

$$\int_{C_2} P\,dx + Q\,dy = \int_{NM} \cdots + \int_{MLN} \cdots$$

Taking the sum:

$$\int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy = \int_{NKM} \cdots + \int_{MN} \cdots + \int_{NM} \cdots + \int_{MLN} \cdots$$

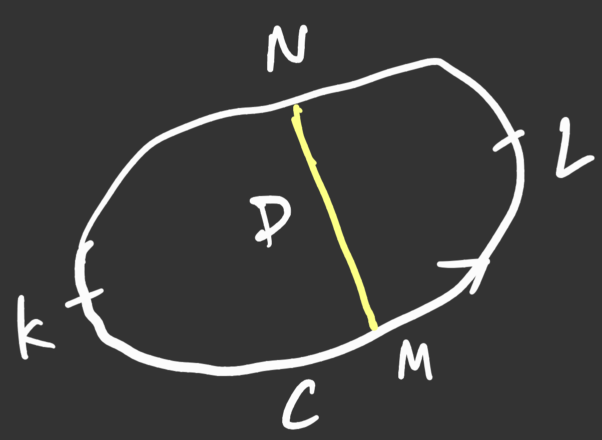
Now note that

$$\int_{MN} \cdots = - \int_{NM} \cdots$$

Since the orientation of the line integral has changed, so the sign is switched.

Thus, we have

$$\int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy = \int_{NKM} \cdots + \int_{MLN} \cdots$$



$$= \int_C P\,dx + Q\,dy$$

Since NKM and MLN make up C.

Thus we established ① and ②

① $$\int_C P\,dx + Q\,dy = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy$$

② $$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Now, if

$$\int_{C_1} P\,dx + Q\,dy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

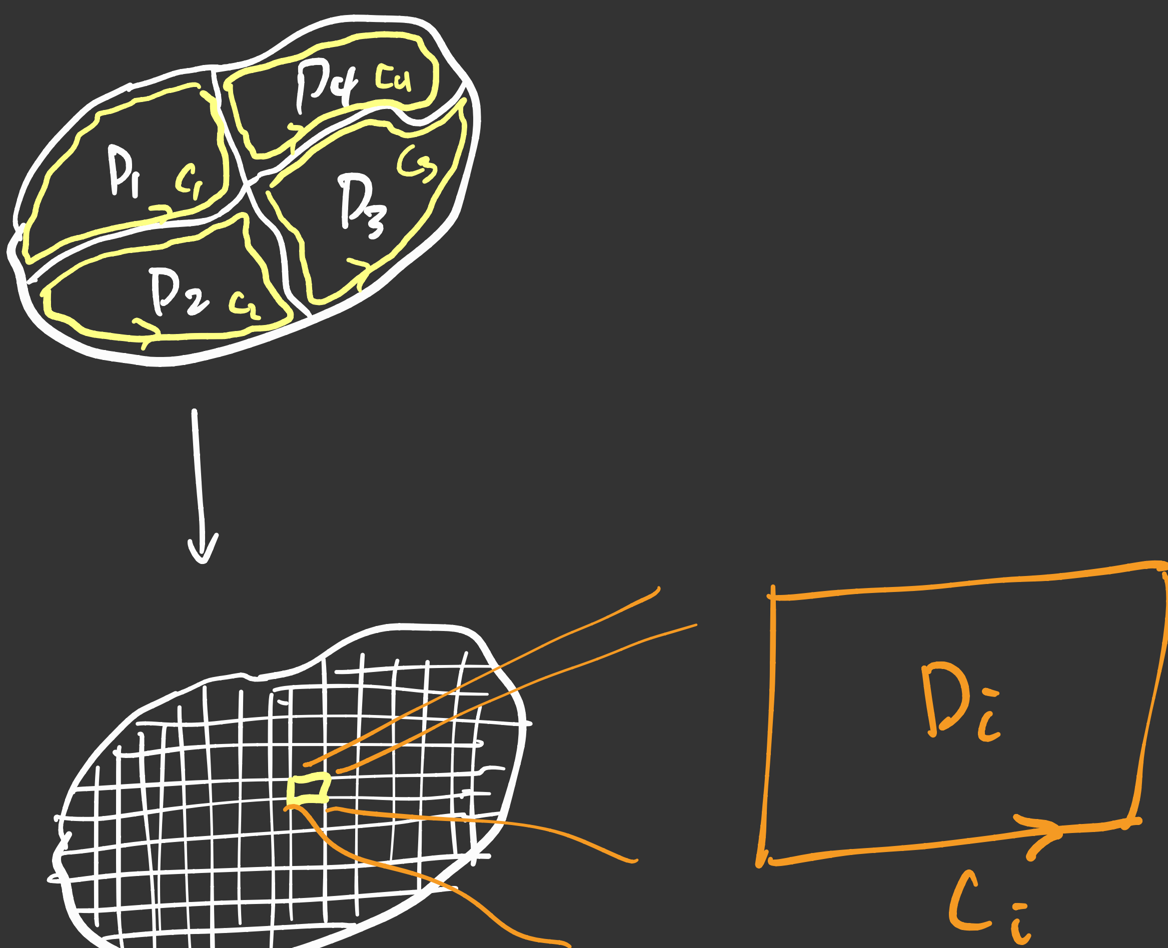
and

$$\int_{C_2} P\,dx + Q\,dy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA,$$

the

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

What did we achieve? We reduced our problem to two problems that are equally difficult. But now, we can repeat! We can continue dividing our domain:

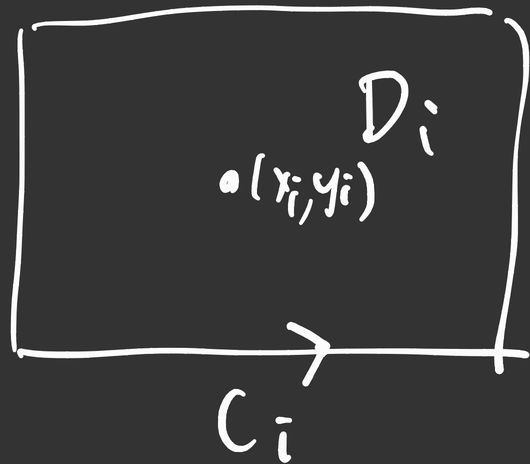


$D_1, D_2, \ldots, D_N$

$C_1, C_2, \ldots, C_N$

If we can prove the result for each piece, we can get the result by summing.

Let us consider one cell.



$P(x,y)$, $Q(x,y)$ differentiable ↗ assume twice

$$P(x,y) \approx P(x_i,y_i) + \frac{\partial P}{\partial x}(x_i,y_i)(x-x_i) + \frac{\partial P}{\partial y}(x_i,y_i)(y-y_i)$$

$$Q(x,y) \approx Q(x_i,y_i) + \frac{\partial Q}{\partial x}(x_i,y_i)(x-x_i) + \frac{\partial Q}{\partial y}(x_i,y_i)(y-y_i)$$

when size of cell goes to zero, the error goes to zero even faster

Rearranging:

$$P(x,y) \approx \underset{\text{constant}}{P_i} + ax + by$$

$$P_i = P(x_i,y_i) - P_x(x_i,y_i)x_i - P_y(x_i,y_i)y_i$$

$$a = \frac{\partial P}{\partial x}(x_i,y_i)$$

$$b = \frac{\partial P}{\partial y}(x_i,y_i)$$

$$Q(x,y) \approx q_i + ex + fy$$

$$q_i = Q(x_i,y_i) - \frac{\partial Q}{\partial x}(x_i,y_i)\,x_i - \frac{\partial Q}{\partial y}(x_i,y_i)\,y_i$$

$$e = \frac{\partial Q}{\partial x}(x_i,y_i)$$

$$f = \frac{\partial Q}{\partial y}(x_i,y_i)$$

Thus

$$\oint_{C_i} P\,dx + Q\,dy \approx \oint_{C_i} (P_i + ax + by)\,dx + \oint_{C_i} (q_i + ex + fy)\,dy$$

Now observe that

$$\oint_{C_i} P_i\,dx = 0 \quad \text{Since} \quad P_i = \frac{d}{dx}(P_i x)$$

$$\int_{C_i} q_i\,dy = 0 \quad \text{Since} \quad q_i = \frac{d}{dy}(q_i y)$$

e.g. $(P_i, q_i) = \nabla(P_i x, q_i y)$

and integrals of gradients over closed curves is zero

Thus

$$\oint_{C_i} P\,dx + Q\,dy \approx$$

$$\oint_{C_i} (ax + by)\,dx + \oint_{C_i} (ex + fy)\,dy$$

Now $\displaystyle\oint_{C_i} (ax\,dx + fy\,dy) = 0$

since $\quad (ax, fy) = \nabla\left(a\frac{x^2}{2} + f\frac{y^2}{2}\right).$

Thus, we found

$$\boxed{\oint_{C_i} P\,dx + Q\,dy \approx \oint_{C_i} (by\,dx + ex\,dy)}$$

$$\boxed{\oint_{C_i} P\,dx + Q\,dy \approx \oint_{C_i} (by\,dx + ex\,dy)}$$

Note

$$\int_{C_i} y\,dx = -\,\text{area}(D_i) \qquad \text{(we proved this before)}$$

$$\int_{C_i} x\,dy = \text{area}(D_i)$$

Thus we have

$$\int_{C_i} (P\,dx + Q\,dy) \approx (e-b)\,\text{area}(D_i)$$

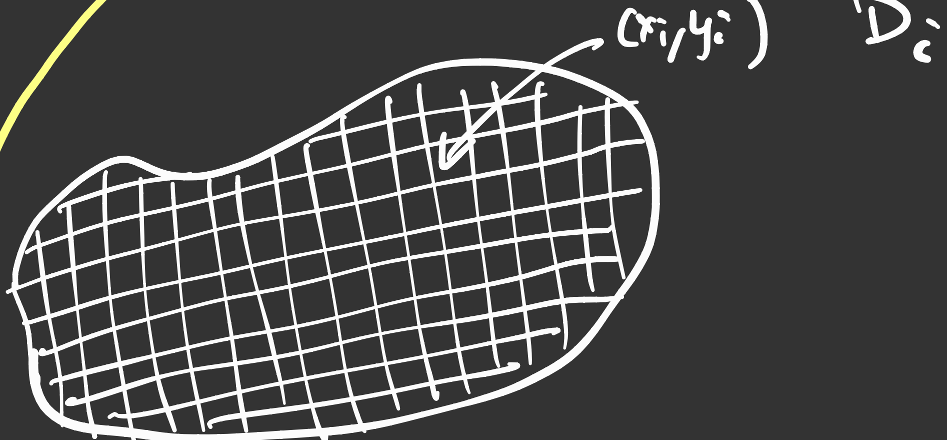
$$= \left(\frac{\partial Q}{\partial x}(x_i, y_i) - \frac{\partial P}{\partial y}(x_i, y_i)\right) \text{area}(D_i).$$

Thus

this requires careful estimates to justify as many errors are added up.

$$\int_C P\,dx + Q\,dy = \sum_{i=1}^{N} \oint_{C_i} P\,dx + Q\,dy$$

$$\approx \sum_{i=1}^{N} \left(\frac{\partial Q}{\partial x}(x_i,y_i) - \frac{\partial P}{\partial y}(x_i,y_i)\right) \text{area } D_i$$



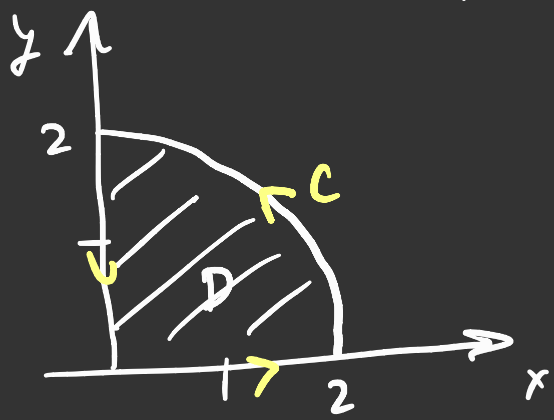
$N \to \infty$ and size$(D_i) \to 0$

$$\longrightarrow \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Applications of Greens Theorem

Mostly these are for computation of integrals.

Ex



D: $x \geq 0$, $y \geq 0$

$x^2 + y^2 \leq 4$

Find

$$I = \int_C (x^2 y - 2y^3)\,dx + (-x^3 + 3y^3)\,dy$$

By Greens formula,

$$I = \iint_D (-3x^2 - x^2 + 6y^2)\,dA$$

$$= \iint_D (-4x^2 + 6y^2)\,dA$$

It is natural to evaluate the integral using polar coordinates

$$D: \quad 0 \le \theta \le \pi/2 \qquad 0 \le r \le 2$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$I = \int_0^{\pi/2}\int_0^{2} \left(-4r^2\cos^2\theta + 6r^2\sin^2\theta\right) r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{2} r^3\left(-4\cos^2\theta + 6\sin^2\theta\right) dr\,d\theta$$

$$= \frac{2^4}{4}\int_0^{\pi/2}\left(-4\cos^2\theta + 6\sin^2\theta\right) d\theta$$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

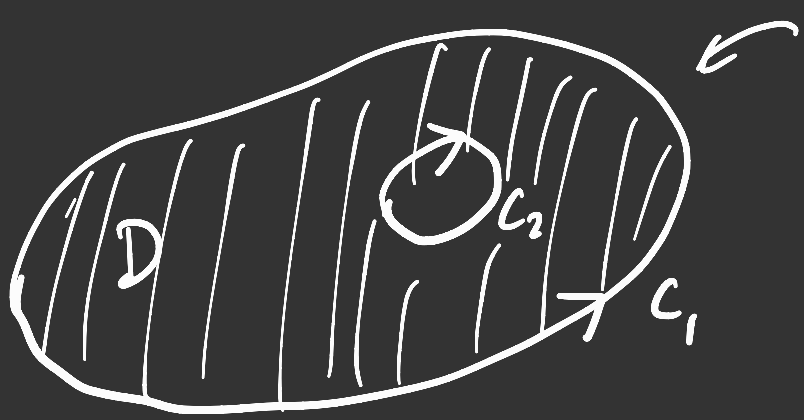
$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$= 4\int_0^{\pi/2}\left(-2 - 2\cos(2\theta) + 3 - 3\cos(2\theta)\right) d\theta$$

$$= 4\left[\frac{\pi}{2} - 5\int_0^{\pi/2}\cos 2\theta\,d\theta\right] = 4\left[\frac{\pi}{2} - \frac{5}{2}\overset{0}{\cancel{\sin 2\theta}}\Big|_0^{\pi/2}\right]$$

$$= 2\pi$$

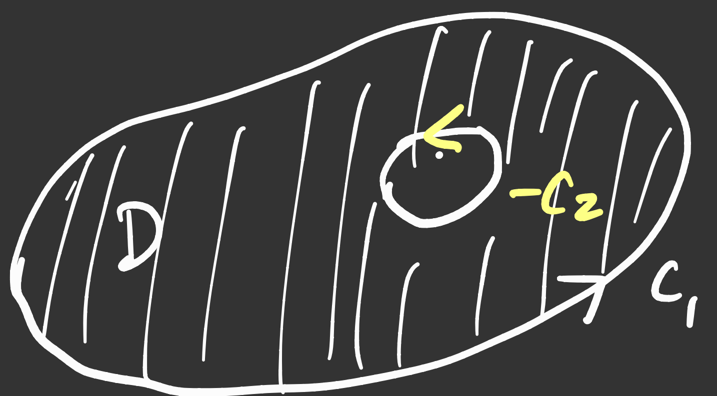
Generalization of Greens formula



convention for orientation.

But Greens formula uses c.c.w curves

So replace by "$-C_2$"



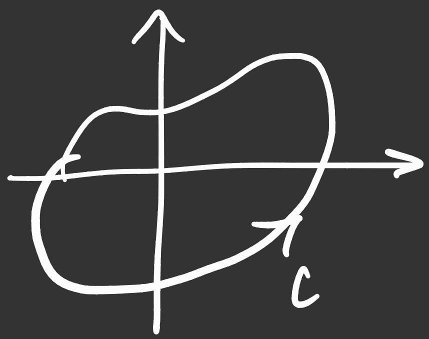
Thm:

$$\oint_{C_1} P\,dx + Q\,dy - \int_{C_2} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Thus, if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then the integrals over C_1 and C_2 are the same.

This can be useful...

Ex: $P(x,y) = \dfrac{-y}{x^2+y^2}$ $\quad Q(x,y) = \dfrac{x}{x^2+y^2}$



denominator goes to zero faster than numerator as $x,y \to 0$.

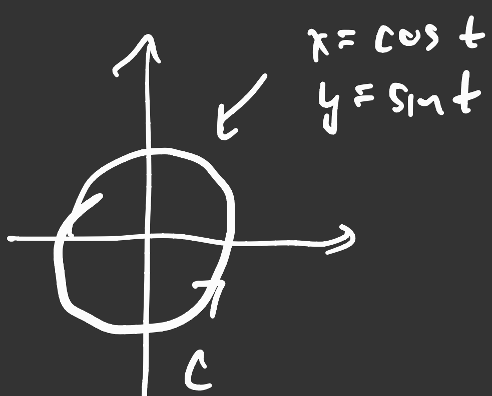
Functions not bounded at zero!

Makes finding integral over curve containing origin tricky. But

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = \frac{x^2+y^2-x\,2x}{(x^2+y^2)^2} - \frac{(x^2+y^2-2y^2)}{(x^2+y^2)^2}$$

$$= 0 \,!$$

Except at one point, $(0,0)$, where makes no sense. Indeed:

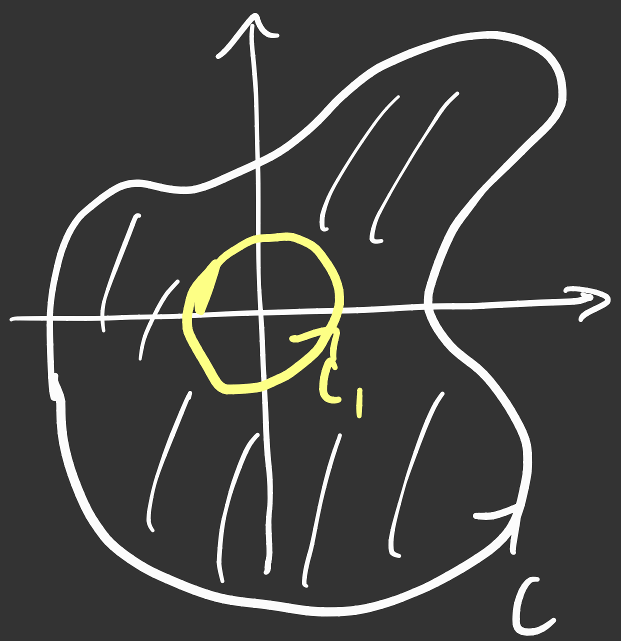


$$\int_C P\,dx + Q\,dy = \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt$$

$$= 2\pi$$

Thus Green's theorem does not apply.

But now consider, for any other contour C,



$$\oint_C \frac{-y\,dx + x\,dy}{x^2+y^2}$$

Remark: we computed for unit circle, but it could be for any size.

$$\left(\oint_C - \oint_{C_1}\right) P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= 0 \; !$$

since (P,Q) is nice in D.

Thus, for any such contour

$$\oint_C P\,dx + Q\,dy = 2\pi.$$

This is the basis for the theory of complex functions!