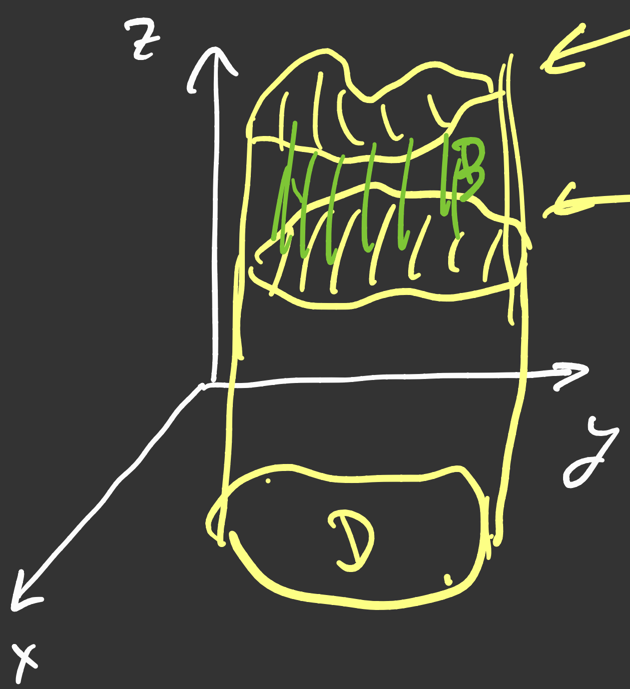
Volumes of simple bodies

← 3D domain, enclosed inside a surface

Consider a body which is analogous to a domain of type 1 in the plane



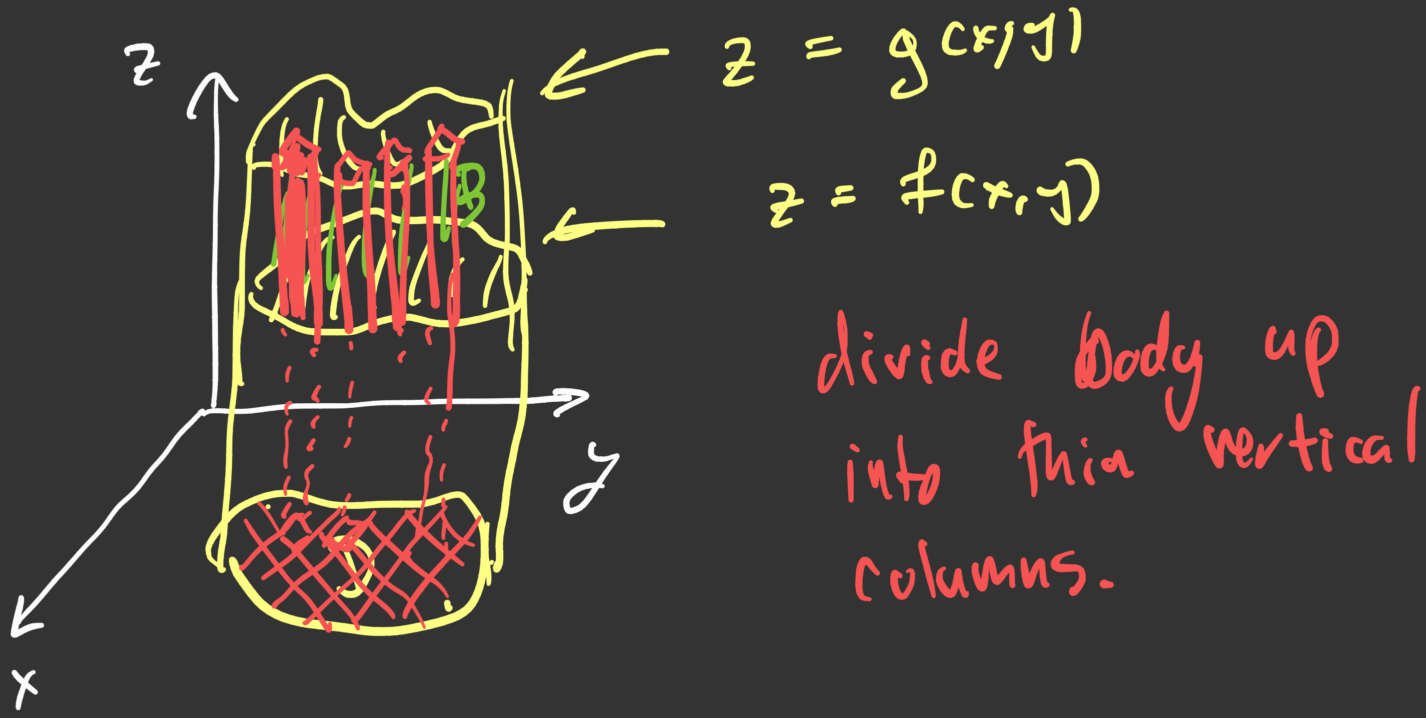
$z = g(x,y)$

$z = f(x,y)$

Body B in between the two surfaces

B: $(x,y) \in D$
$f(x,y) \le z \le g(x,y)$

Our goal is to compute the volume of this body.



The volume of one of these columns is approx

(height of column)$\cdot$(Area of Base)

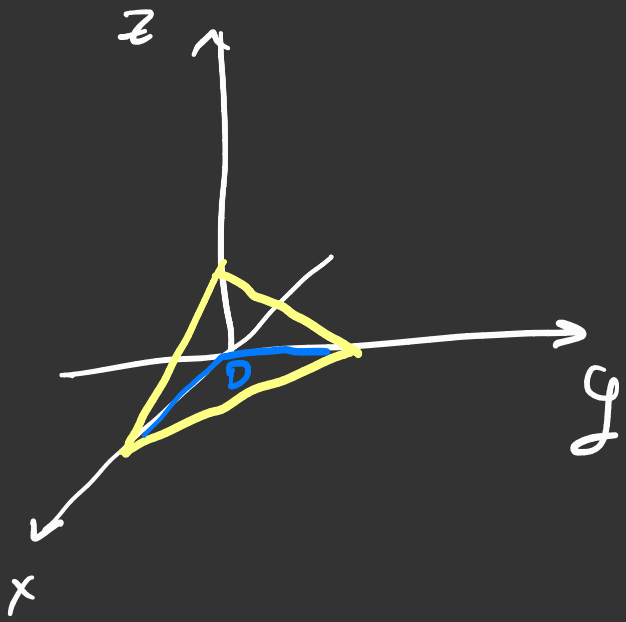
$$= (g(x,y) - f(x,y)) \; dA.$$

Summing up the little columns and taking the area of the base to zero, we have

$$V(B) = \iint_D (g(x,y) - f(x,y)) \, dA$$

If we are lucky, this reduces to iterated integral!

Ex: Volume of tetrahedron

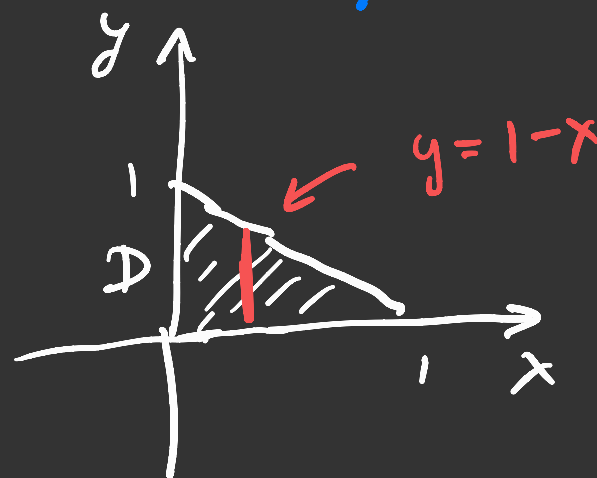


B: $x \geq 0,\ y \geq 0,\ z \geq 0$

$x + y + z \leq 1$

D: $x \geq 0,\ y \geq 0$

$x + y \leq 1$



$h(x,y) = 1 - x - y$

$g(x,y) = 0$

$$V(B) = \iint_D (1-x-y)\, dA$$

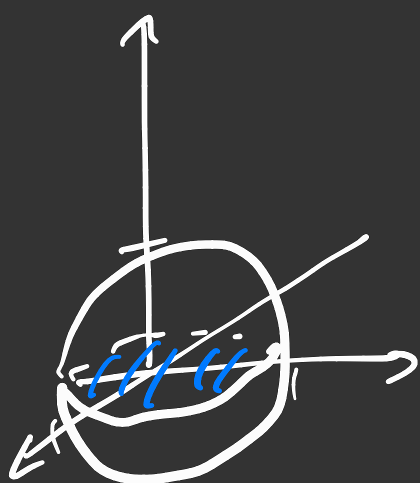
$$= \int_0^1 \left(\int_0^{1-x} (1-x-y)\, dy \right) dx$$

$$= \int_0^1 \left[(1-x)^2 - \frac{1}{2}(1-x)^2 \right] dx = \frac{1}{2} \int_0^1 (1-x)^2\, dx$$

$$= \frac{1}{2} \int_0^1 x^2\, dx = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}.$$

Volume of unit Ball

$$B = x^2+y^2+z^2 \le 1$$



Base (projection on x-y plane)

$$D = x^2+y^2 \le 1$$

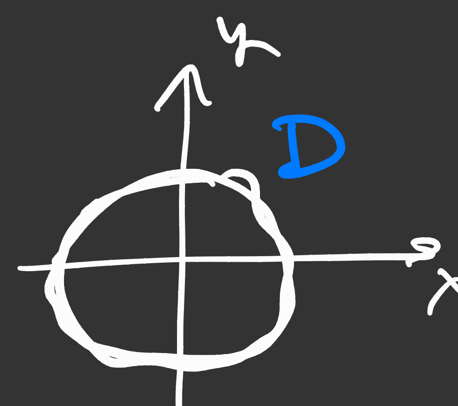
$=$ unit disk

$$z^2+x^2+y^2 = 1$$

$$\Rightarrow \quad z = \pm\sqrt{1-x^2-y^2}$$

B: $x^2+y^2 \le 1$

$$\underbrace{-\sqrt{1-x^2-y^2}}_{f(x,y)} \le z \le \underbrace{\sqrt{1-x^2-y^2}}_{g(x,y)}$$



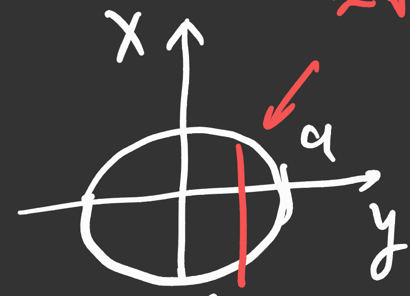
$$V(B) = \iint_D 2\sqrt{1-x^2-y^2}\, dA$$

$$= \int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1-x^2-y^2}\, dy$$

$$I = \int_{-a}^{a} 2\sqrt{a^2 - y^2}\, dy$$

$a = \sqrt{1-x^2}$

length of segment $2\sqrt{a^2-y^2}$



circle of radius a.

$$= \text{Area of disk of radius } a$$

$$= \pi a^2 = \pi(1-x^2)$$

Thus

$$V(B) = \pi \int_{-1}^{1} (1-x^2)\, dx = \pi\left(2 - \frac{2}{3}\right) = \frac{4\pi}{3}.$$

Volume of unit ball (radius 1) is $\frac{4\pi}{3}$.

Archemedes did this 2.5 millenia ago. He essentially did the same computation without machinary of calculus.

Lamina : what is it?

D

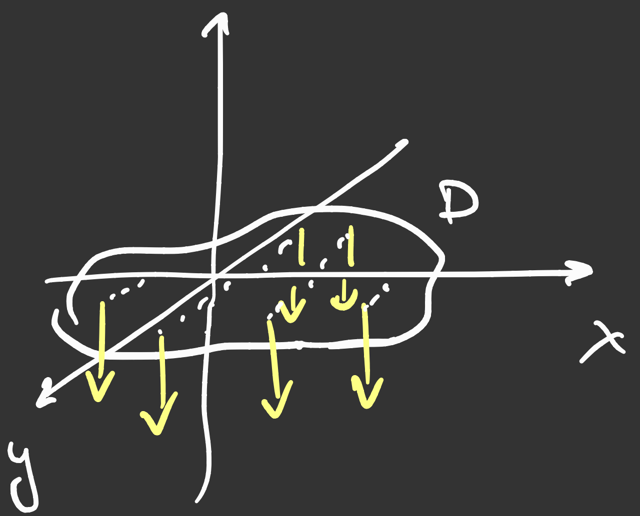
plane, solid plate D with variable density $\rho(x,y)$ (thickness of material) $\left[\frac{kg}{m^2}\right]$

example of lamina is a door.

Mass of lamina

$$M = \iint_D \rho(x,y)\, dA$$

Moments w.r.t. x-axis



Imagine x-axis is a solid rod. Balance D on it. Because of gravity there is a force acting on it.

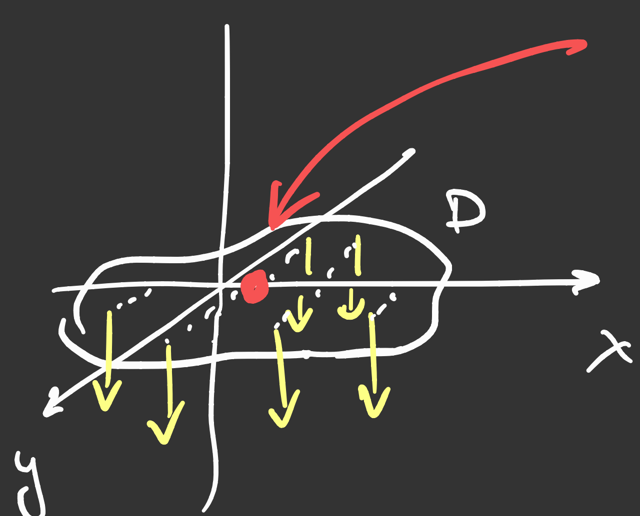
$$M_x = \iint_D y\,\rho(x,y)\,dA$$

moment is product of the force and distance to axis of rotation

Similarly moment w.r.t. y-axis

$$M_y = \iint_D x\,\rho(x,y)\,dA$$

Center of mass: $(\bar{x}, \bar{y})$



If you put it on the x-axis then the total moment is zero

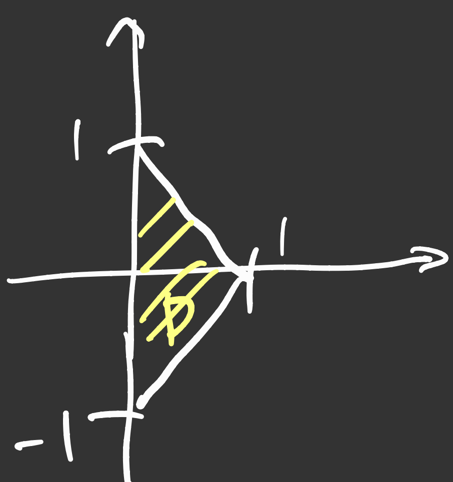
$$\bar{x} = \frac{M_x}{M} \qquad \bar{y} = \frac{M_y}{M}$$

you can balance the lamina with a support at center of mass

at equilibrium

Class of problems: given a domain D and density ρ, find all these things.

<u>Example</u>: $\rho(x,y) = x^2$



$$M = \iint_D x^2\, dA = \int_0^1 dx \left(\int_{-1+x}^{1-x} x^2\, dy \right)$$

$$= \int_0^1 dx \left(x^2 (2-2x) \right)$$

$$= \int_0^1 (2x^2 - 2x^3)\, dx$$

$$= \frac{2}{3} - \frac{2}{4} = \frac{1}{6}$$

Now we find moments

$$M_x = \iint_D y\, \rho(x,y)\, dA$$

y is odd function, integral is symmetric

$$= \int_0^1 dx \left(\int_{-1+x}^{1-x} y\, x^2\, dy \right) = 0$$

$$M_y = \iint_D x\, \rho(x,y)\, dA$$

$$= \int_0^1 dx \left(\int_{-1+x}^{1-x} x^3\, dy \right)$$

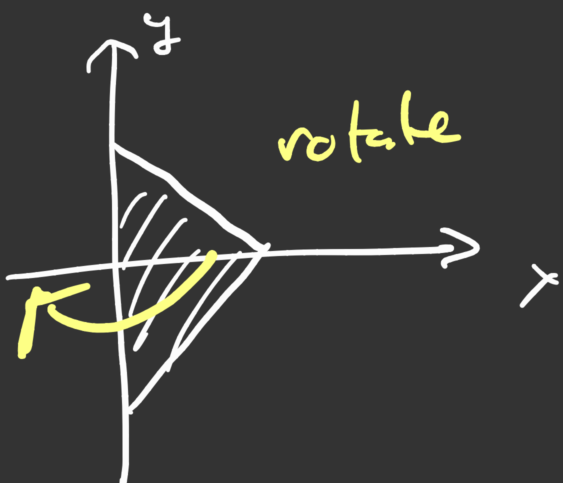
$$= \int_0^1 x^3 (2-2x)\, dx = \int_0^1 (2x^3 - 2x^4)\, dx$$

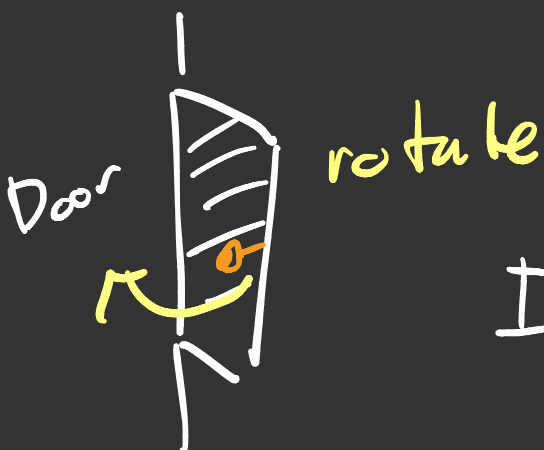
$$= \frac{1}{2} - \frac{2}{5} = \frac{1}{10}.$$

Center of Mass:

$$(\bar{x}, \bar{y}) = \left(\frac{1/10}{1/6}, 0 \right) = (0.6, 0)$$

Moment of Inertia





angular velocity

Kinetic Energy $= \frac{1}{2} I \omega^2$

moment of inertia depends on axis of rotation

$(\text{speed})^2$ for rotation

$$I_x = \iint_D \underbrace{y^2 \rho(x,y)}_{\text{kinetic energy}} \, dA$$

$$I_y = \iint_D x^2 \rho(x,y) \, dA$$

Example:

$$I_x = \iint_D y^2 x^1 \, dA = \int_0^1 dx \int_{-1+x}^{1-x} y^2 x^2 \, dy$$

$$= \int_0^1 dx \left(x^2 \cdot \frac{2}{3} (1-x)^3 \right) = \int_0^1 dx \, \frac{2}{3} \left(x^2 - 3x^3 + 3x^4 - x^5 \right)$$

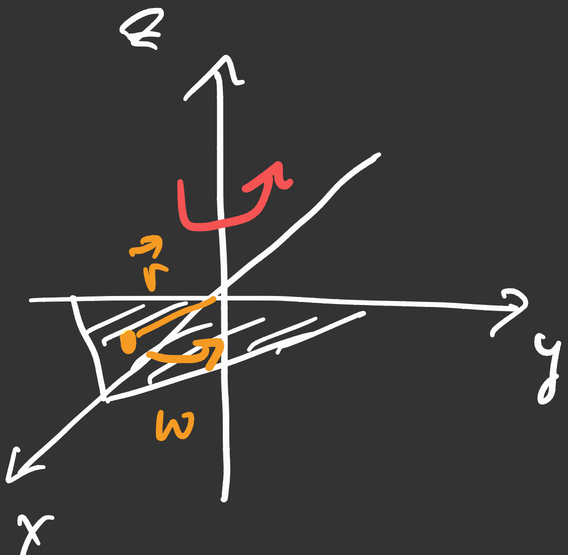
$$= \frac{2}{3} \left(\frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6} \right) = \frac{1}{90}.$$

$$I_y = \iint_D x^2 x^2 \, dA$$

$$= \int_0^1 dx \int_{-1+x}^{1-x} x^4 \, dy = \int_0^1 dx \left((2-2x) x^4 \right) dx$$

$$= \frac{2}{5} - \frac{2}{6} = \frac{2}{30} = \frac{1}{15}$$

$$I_z = I_x + I_y$$



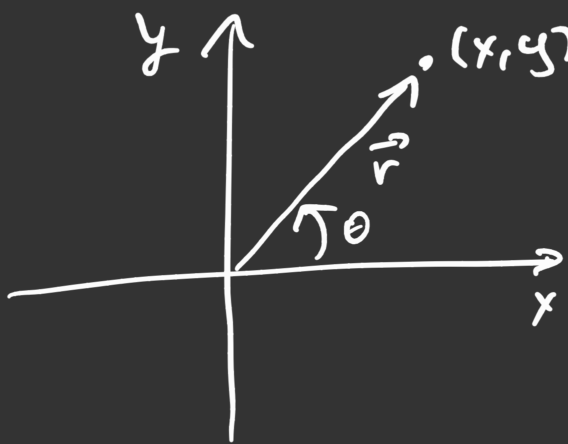
rotating around z-axis
kinetic energy of
each particle is

$$dm \, w |\vec{r}|^2$$
$$= dm \, w^2 (x^2 + y^2)$$

$$I_z = \iint_D (x^2+y^2) \, dA = I_x \; I_y .$$

In our example $\quad I_z = \frac{1}{90} + \frac{1}{15} = \frac{7}{90}$

Double integral in polar coordinates



polar coordinates of (x,y)

(length of $\vec{r}$, angle θ)

$r = \|\vec{r}\|$

$x = r\cos\theta$

$y = r\sin\theta$

$\Longleftrightarrow$

$\vec{r} = \sqrt{x^2+y^2}$

$\theta = \arctan\left(\frac{y}{x}\right)$

works in the right half plane

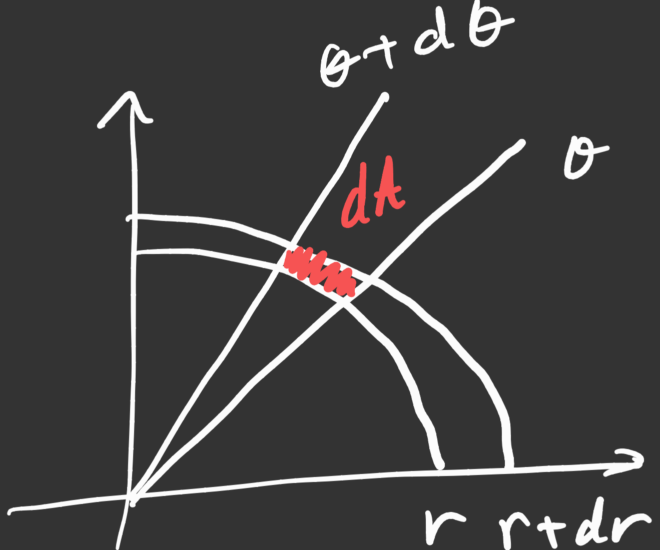
modify to $\operatorname{arccot}\left(\frac{x}{y}\right)$ in upper half plane, etc.

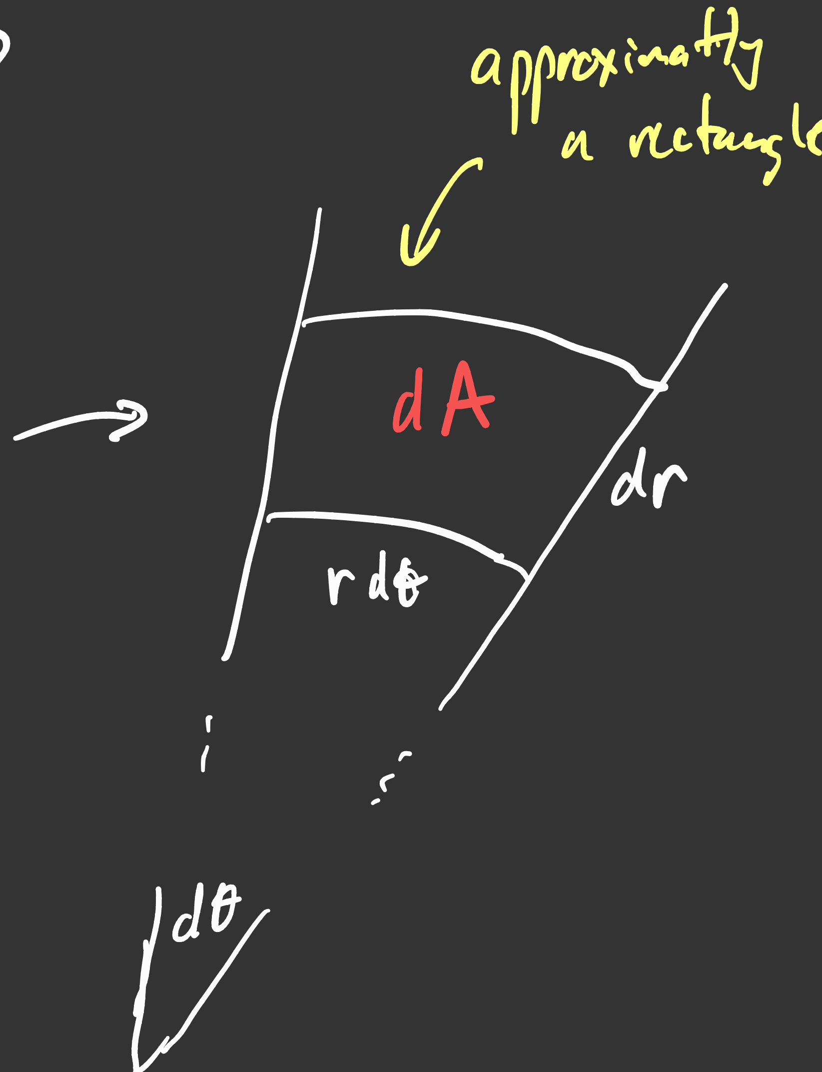
In cartesian coordinates, simplest domains are rectangles.

In polar coordinates simplest domains are $a \le r \le b$ $\alpha \le \theta \le b$

Consider $\iint_D f(r,\theta)\, dA$.

What is dA?



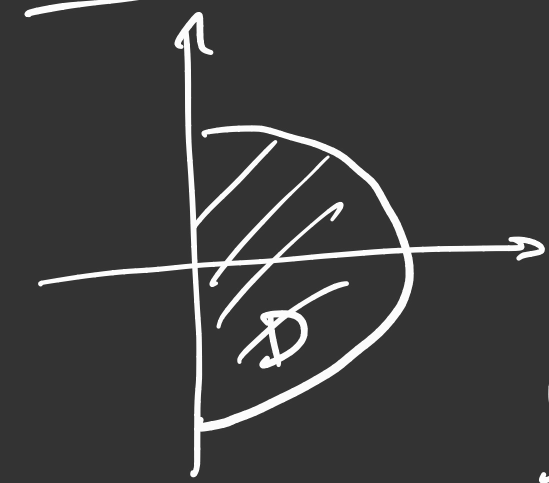


$$dA = r\,dr\,d\theta.$$

Thus

$$\iint_D f(r,\theta)\, dA = \int_\alpha^\beta \int_a^b f(r,\theta)\, r\,dr\,d\theta.$$

Example



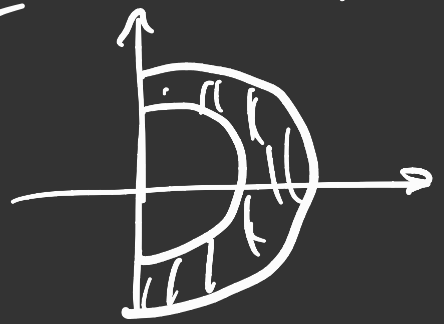
$$D: \quad 0 \le r \le 1$$
$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\iint_D x\,dA = \iint_D r\cos\theta \, r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left(\int_0^1 r\cos\theta\, r\,dr\right)d\theta = \int_{-\pi/2}^{\pi/2}\cos\theta \int_0^1 r^2\,dr\,d\theta$$

$$= \frac{1}{3}\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta = \frac{2}{3}.$$

Ex Horse shoe



$$D = \begin{matrix} a \le r \le b \\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \end{matrix}$$

$$\rho(r,\theta) = 1$$

Center of mass of horseshoe $(\bar{x}, \bar{y})$

$$M = \iint_D \rho \, dA = \int_{-\pi/2}^{\pi/2} \int_a^b r \, dr \, d\theta$$

$$= \pi \int_a^b r \, dr = \frac{\pi}{2}(b^2 - a^2)$$

$$M_x = \iint_D \rho y \, dA = \int_{-\pi/2}^{\pi/2} \int_a^b r \sin\theta \, r \, dr \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \sin\theta \int_a^b r^2 \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \sin\theta \left(\frac{b^3 - a^3}{3}\right) d\theta$$

$$= \frac{b^3 - a^3}{3} \int_{-\pi/2}^{\pi/2} \sin\theta \, d\theta = 0$$

$$M_y = \iint_D \rho x \, dA = \int_{-\pi/2}^{\pi/2} \int_a^b r\cos\theta \, r \, dr \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \cos\theta \left(\frac{b^3 - a^3}{3} \right) d\theta$$

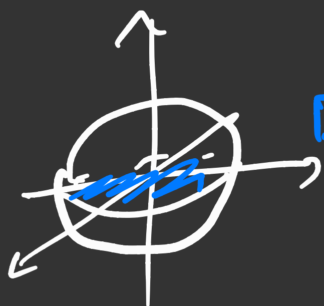
$$= \frac{2}{3}\left(b^3 - a^3\right)$$

$$\bar{x} = \frac{M_y}{M} = \frac{4}{3} \frac{b^3 - a^3}{b^2 - a^2}$$

$$\bar{y} = 0$$

Volume of ball

$B: \quad x^2+y^2+z^2 \leq 1$



$D: \quad x^2+y^2 \leq 1$

in polar $D: \begin{array}{l} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{array}$

$$z = \pm\sqrt{1-x^2-y^2}$$

$$V(B) = \iint_D 2\sqrt{1-x^2-y^2}\, dA$$

$$= \int_0^{2\pi}\int_0^1 2\sqrt{1-r^2}\; r\, dr\, d\theta$$

$$= 4\pi \int_0^1 \sqrt{1-r^2}\; r\, dr$$

$s = r^2$

$ds = 2r\,dr$

$$= 2\pi \int_0^1 \sqrt{1-s}\; ds = 2\pi \int_0^1 \sqrt{s}\, ds$$

$$= -2\pi \; \frac{2}{3}(1-s)^{3/2}\Big|_0^1 = \frac{4\pi}{3}.$$

$$V(\text{Ball of radius } R) = \frac{4\pi}{3} R^3.$$