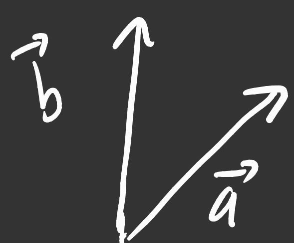
The next operation, <u>cross product</u>, is more interesting. It is defined for vectors in 3 dimensional space, $\vec{a}$ and $\vec{b}$, and their cross product is a third vector $\vec{a} \times \vec{b}$.

To explain this operation, we require several preliminary concepts.

Left and Right pairs of vectors the plane

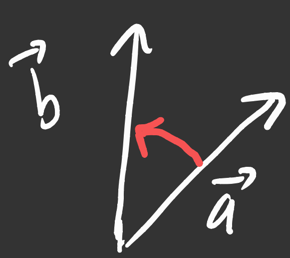


$\leftarrow$ pair of vectors

two different types of pairs. $(\vec{a}, \vec{b})$

Consider the angle between vector $\vec{a}$ and $\vec{b}$ which is smaller than π.

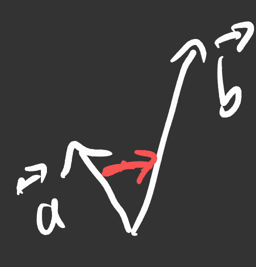
consider which direction should we rotate $\vec{a}$ so that its direction eventually coincides with $\vec{b}$.



$(\vec{a}, \vec{b})$

counterclockwise

<u>right pair</u>



clockwise

<u>left pair</u>

If $(\vec{a}, \vec{b})$ forms a right pair then
$(\vec{b}, \vec{a})$ is a left pair,

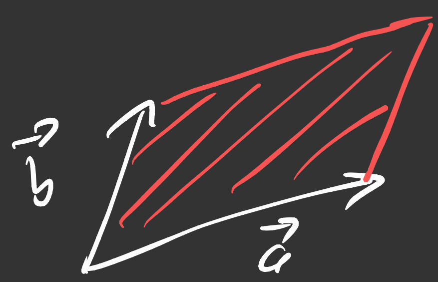
and conversly.

Different pairs of vectors are either right or left, unless the angle is 0 ($\vec{a}$ and $\vec{b}$ have same direction) or angle is 2π ($\vec{a}$ and $\vec{b}$ are antiparallel).

These are degenerate cases.

Signed area of a parallelogram

Say a parallelogram is defined by vectors $\vec{a}$ and $\vec{b}$.



Consider the signed area defined by

$$A(\vec{a}, \vec{b}) = \begin{cases} \text{usual area} & \text{if } (\vec{a}, \vec{b}) \text{ is right} \\ -\text{usual area} & \text{if } (\vec{a}, \vec{b}) \text{ is left} \end{cases}$$

positive area is just area of piece of paper.

negative area is if you make a hole in shape of a parallelogram, you can count it as negative area (it is what is deficient in the whole sheet of paper).

Signed area behaves much more regularly than usual area.

How to find it?

Let us look for the formula.

We will do so step by step by examining the properties of this function.

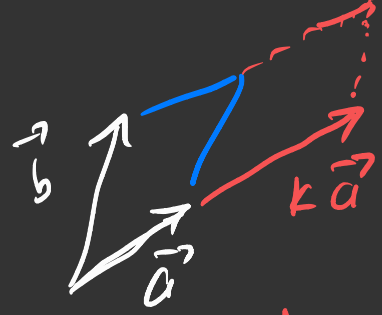
unlike dot product!

Properties of $A(\vec{a},\vec{b})$

(1) $A(\vec{a},\vec{b}) = -A(\vec{b},\vec{a})$ (anti-symmetric)

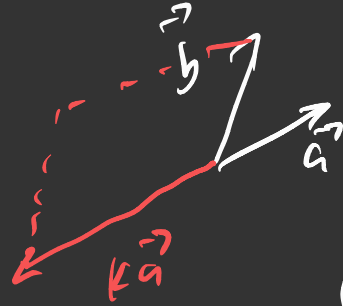
because if you take a right pair and change places, then $(\vec{b},\vec{a})$ is left.

(2) $A(k\vec{a},\vec{b}) = k\,A(\vec{a},\vec{b})$ for $k \in \mathbb{R}$



if k positive

Since area of parallelogram is proportional to $\|\text{side}\|$, it follows



If pair $(\vec{a},\vec{b})$ is right then $(k\vec{a},\vec{b})$ is left if k negative

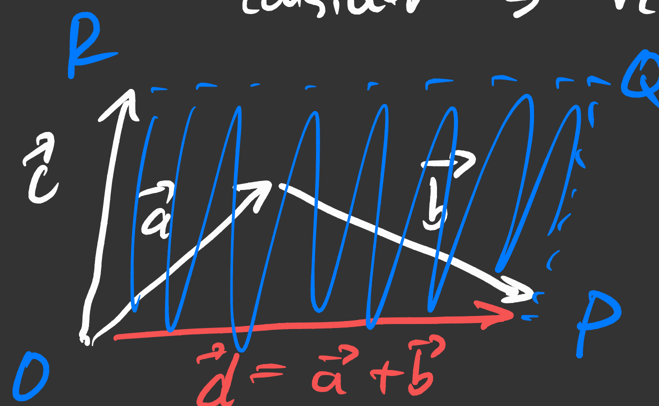
2) $A(\vec{a}, k\vec{b}) = k\, A(\vec{a}, \vec{b})$

3) $A(\vec{a}+\vec{b}, \vec{c}) = A(\vec{a}, \vec{c}) + A(\vec{b}, \vec{c})$

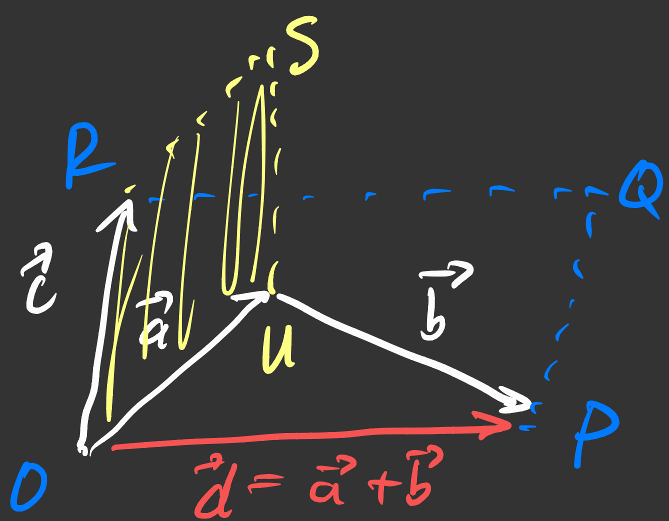
$A(\vec{a}, \vec{b}+\vec{c}) = A(\vec{a}, \vec{c}) + A(\vec{b}, \vec{c})$

distributive property.

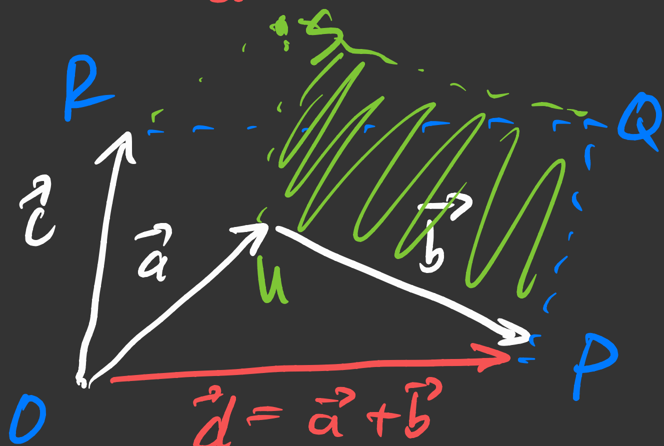
Pf: consider 3 vectors, $\vec{a}, \vec{b}, \vec{c}$



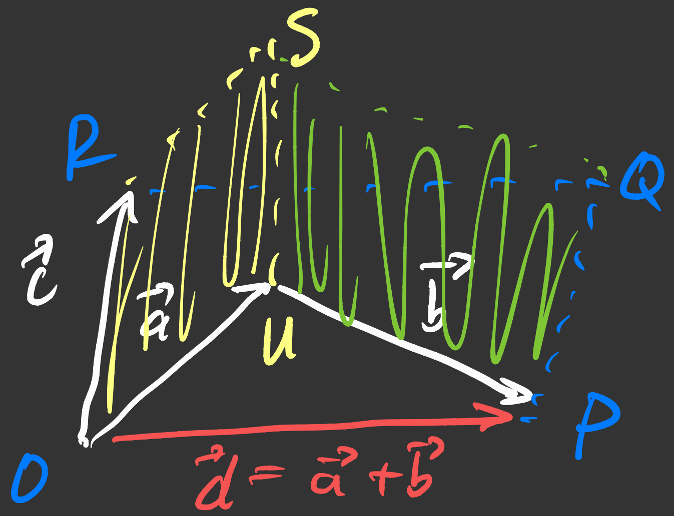
$A(\vec{a}+\vec{b}, \vec{c})$ = signed area of OPQR



$A(\vec{a}, \vec{c})$ = signed area of OUSR



$A(\vec{b}, \vec{c})$ = signed area of UPQS



$A(\vec{a}, \vec{c}) + A(\vec{b}, \vec{c})$

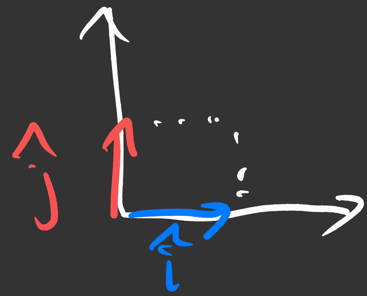
The difference between this area and that of OPQR is that we added $\triangle$ RSQ and subtracted $\triangle$ OUP.

These two triangle are shifts of one another, thus have same signed area.

Here we considered simplest case when all pairs of vectors are right, and so all signed areas are simple areas.

One needs to consider more general configurations, but it is the same (after more work).

4)



$A(\hat{i},\hat{j}) = 1$

since $(\hat{i},\hat{j})$ is a right pair

$A(\hat{j},\hat{i}) = -1$

$A(\hat{i},\hat{i}) = 0$ (degenerate parallelogram has zero width)

$A(\hat{j},\hat{j}) = 0$

Now we may derive the formula.

Suppose $\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2)$

$$
\begin{aligned}
A(\vec{a},\vec{b}) &= A(a_1\hat{i} + a_2\hat{j},\ b_1\hat{i} + b_2\hat{j}) \\
&= A(a_1\hat{i},\ b_1\hat{i}) + A(a_1\hat{i},\ b_2\hat{j}) \\
&\quad + A(a_2\hat{j},\ b_2\hat{j}) + A(a_2\hat{j},\ b_1\hat{i}) \\
&= a_1 b_1 \cancelto{0}{A(\hat{i},\hat{i})} + a_1 b_2 \cancelto{1}{A(\hat{i},\hat{j})} \\
&\quad + a_2 b_2 \cancelto{0}{A(\hat{j},\hat{j})} + a_2 b_1 \cancelto{-1}{A(\hat{j},\hat{i})} \\
&= a_1 b_2 - a_2 b_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}
\end{aligned}
$$

determinant

We have our formula:

$$A(\vec{a},\vec{b}) = a_1 b_2 - a_2 b_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

We can see the origin of <u>determinants</u>, at least in 2d. It is simply the signed area of parallelogram.

The usual area of parallelogram is

$$\left| \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right|$$ absolute value.

Note, we may introduce notation $a^\perp = (-a_2, a_1)$ for counterclockwise rotation of a vector by $90°$. Then

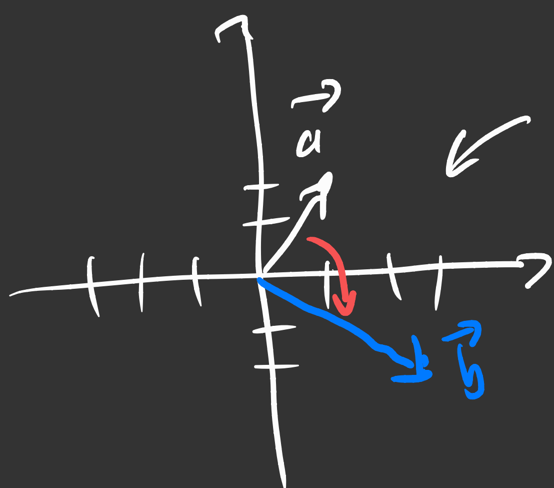
$$\boxed{A(\vec{a},\vec{b}) = a^\perp \cdot b}$$

Example $\quad \vec{a} = (1,2) \qquad \vec{b} = (3,-2)$

$$A(\vec{a},\vec{b}) = 1\cdot(-2) - 2\cdot 3$$

$$= -8.$$

Since $A(\vec{a},\vec{b})$ is negative, it means that $\vec{a}$ and $\vec{b}$ form a left pair.

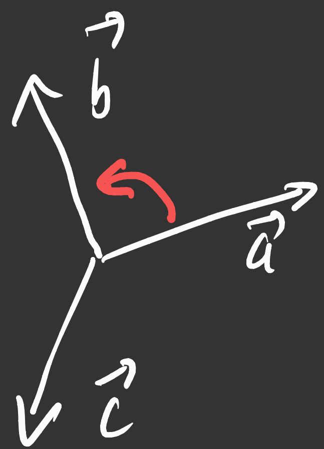


rotate clockwise, forms a left pair.

This is the concept leading to the cross product.

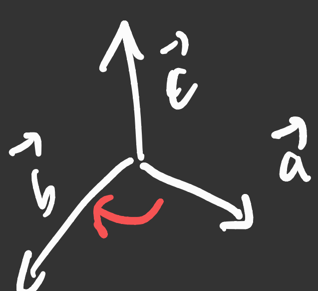
Signed volume of a parallelepiped

Consider three vectors $\vec{a}, \vec{b}, \vec{c}$ not lying in the same plane.



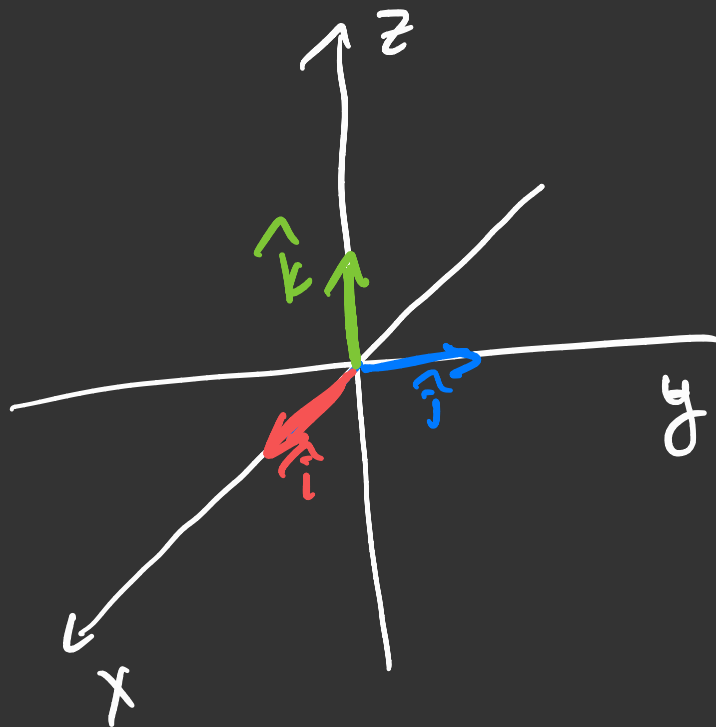
Def: These vectors form a right triple

If you look at $\vec{a}, \vec{b}$ from the view of vector $\vec{c}$, and you have to rotate $\vec{a}$ counterclockwise to align with $\vec{b}$, then $(\vec{a}, \vec{b}, \vec{c})$ is a right triple.

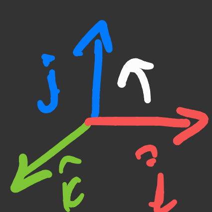


← if clockwise, then $(\vec{a}, \vec{b}, \vec{c})$ is a left triple.

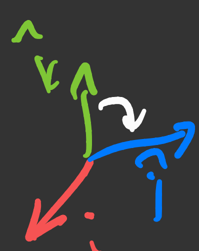
E.g.



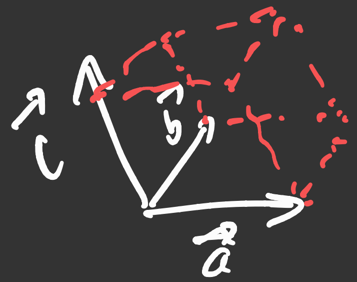
$(\hat{i}, \hat{j}, \hat{k})$ form a right triple



$(\hat{k}, \hat{j}, \hat{i})$



form a left triple



Π = parallelepiped

Def: The signed volume of Π defined by vectors $\vec{a}, \vec{b}, \vec{c}$ is

$$V(\vec{a},\vec{b},\vec{c}) = \begin{cases} \text{usual volume of } \Pi & \text{if } (\vec{a},\vec{b},\vec{c}) \text{ is right triple} \\ -\text{usual volume} & \text{if left triple} \end{cases}$$

What properties?

1) $V(\vec{a},\vec{b},\vec{c}) = -V(\vec{b},\vec{a},\vec{c})$

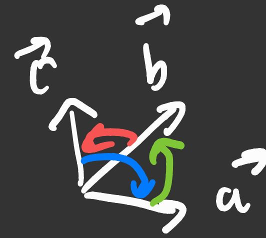
$= -V(\vec{a},\vec{c},\vec{b})$

$= -V(\vec{c},\vec{b},\vec{a})$

Since $(\vec{a},\vec{b},\vec{c})$ is right triple, then $(\vec{b},\vec{a},\vec{c})$, $(\vec{a},\vec{c},\vec{b})$ and $(\vec{c},\vec{b},\vec{a})$ are left triples.

On other hand,

$$V(\vec{a}, \vec{b}, \vec{c}) = V(\vec{c}, \vec{a}, \vec{b})$$
$$= V(\vec{b}, \vec{c}, \vec{a})$$



↑ all right (or left) triples.

There are 6 total permutations of $(\vec{a}, \vec{b}, \vec{c})$. 3 are right and 3 are left.

Volume is always the same.

3 cases signed volume is positive

3 cases " " is negative

2) $V(k\vec{a},\vec{b},\vec{c}) = k\,V(\vec{a},\vec{b},\vec{c})$

$V(\vec{a},k\vec{b},\vec{c}) = k\,V(\vec{a},\vec{b},\vec{c})$

$V(\vec{a},\vec{b},k\vec{c}) = k\,V(\vec{a},\vec{b},\vec{c})$.

3) $V(\vec{a}+\vec{d},\vec{b},\vec{c}) = V(\vec{a},\vec{b},\vec{c}) + V(\vec{d},\vec{b},\vec{c})$

Proved in the same way as in 2D.

draw the two parallelpipeds $\vec{a},\vec{b},\vec{c}$ and $\vec{d},\vec{b},\vec{c}$.

Their difference is some prism.

Same property for other slots:

$V(\vec{a},\vec{b}+\vec{d},\vec{c}) = V(\vec{a},\vec{b},\vec{c}) + V(\vec{a},\vec{d},\vec{c})$

$V(\vec{a},\vec{b},\vec{c}+\vec{d}) = V(\vec{a},\vec{b},\vec{c}) + V(\vec{a},\vec{b},\vec{d})$

4) $V(\hat{i}, \hat{j}, \hat{k}) = 1$ (unit cube)

$V(\hat{k}, \hat{i}, \hat{j}) = 1$ right triple

$V(\hat{j}, \hat{k}, \hat{i}) = 1$

$V(\hat{j}, \hat{i}, \hat{k}) = -1$

$V(\hat{k}, \hat{j}, \hat{i}) = -1$ left triples

$V(\hat{i}, \hat{k}, \hat{j}) = -1$

degenerate

If any two are same, volume is zero e.g. $V(\hat{i}, \hat{i}, \hat{i}) = 0$

$\vec{a} = (a_1, a_2, a_3)$ $\quad$ $\vec{b} = (b_1, b_2, b_3)$

$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ $\quad$ $= b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = (c_1, c_2, c_3) = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$V(\vec{a}, \vec{b}, \vec{c}) =$

$V(a_1\hat{i} + a_2\hat{j} + a_3\hat{k},\ b_1\hat{i} + b_2\hat{j} + b_3\hat{k},\ c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$

$$V(\vec{a}, \vec{b}, \vec{c}) =$$

$$V(a_1\hat{i} + a_2\hat{j} + a_3\hat{k},\ b_1\hat{i} + b_2\hat{j} + b_3\hat{k},\ c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$$

$$= 27 \text{ terms} \quad (3 \times 3 \times 3)$$

$$= V(a_1\hat{i}, b_1\hat{i}, c_1\hat{i}) +$$

$$V(a_1\hat{i}, b_1\hat{j}, c_1\hat{i}) +$$

$$\vdots$$

$$V(a_3\hat{k}, b_3\hat{k}, c_3\hat{k})$$

$$= a_1b_1c_1 V\overset{0}{\cancel{(\hat{i}, \hat{i}, \hat{i})}} + \cdots + a_3b_3c_3 V\overset{0}{\cancel{(\hat{k}, \hat{k}, \hat{k})}}$$

$$= a_1b_2c_3 V\overset{1}{\cancel{(\hat{i}, \hat{j}, \hat{k})}} + a_1b_3c_2 V\overset{-1}{\cancel{(\hat{i}, \hat{k}, \hat{j})}}$$

$$+ a_2b_1c_3 V\overset{-1}{\cancel{(\hat{j}, \hat{i}, \hat{k})}} + a_2b_3c_1 V\cancel{(\hat{j}, \hat{k}, \hat{i})}$$

$$+ a_3b_1c_2 V\overset{1}{\cancel{(\hat{k}, \hat{i}, \hat{j})}} + a_3b_2c_1 V\overset{-1}{\cancel{(\hat{k}, \hat{j}, \hat{i})}}$$

$$= a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$$

$V(\vec{a}, \vec{b}, \vec{c})$

$= a_1 b_2 c_3 + a_3 b_1 c_2 + a_2 b_3 c_1 - a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1$

This funny formula turns out simply to be the determinant!

$$\boxed{V(\vec{a}, \vec{b}, \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}$$

Let us write

$$= c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$$

expand determinant relative to last row

$$- c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}$$

$$+ c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

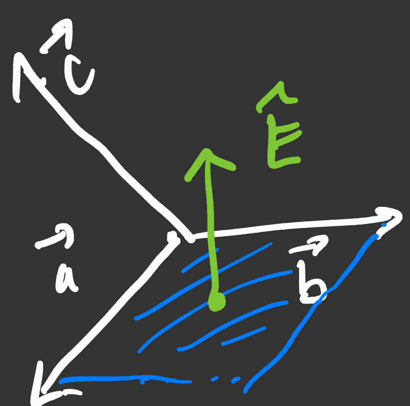
Let us define a vector

$$\vec{F}(\vec{a},\vec{b}) = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)$$

With this, we have the formula

$$V(\vec{a},\vec{b},\vec{c}) = \vec{F}(\vec{a},\vec{b}) \cdot \vec{c}$$

But what is the volume geometrically?

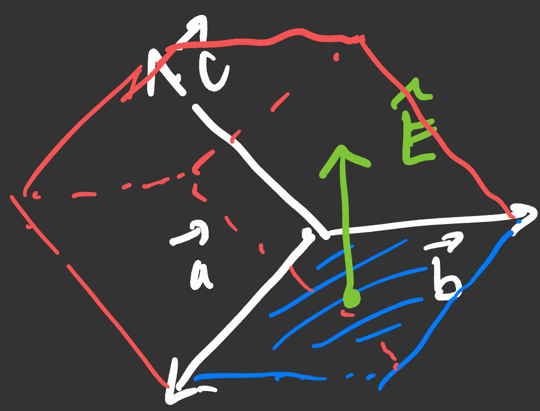


$\hat{E}$ is vector normal to the plane of parallelogram $(\vec{a},\vec{b})$, call it Π.

Moreover

- $\|\hat{E}\| = |\text{area}(\Pi)|$
- direction is so that $(\vec{a},\vec{b},\hat{E})$ form a right triple

Now, we see that

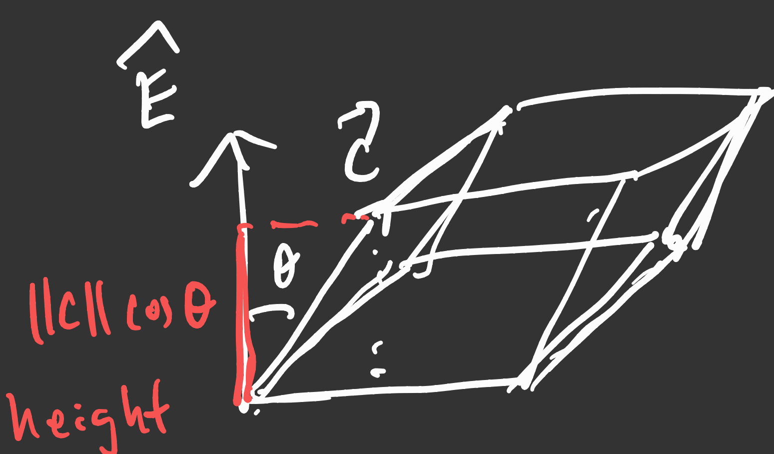


Parallelpiped defined by $(\vec{a}, \vec{b}, \vec{c})$

$$V(\vec{a}, \vec{b}, \vec{c}) = (\text{area of base}) \cdot (\text{height})$$

$$= \vec{E} \cdot \vec{c}$$

$$= \|\vec{E}\| \, \|\vec{c}\| \cos\theta$$



Thus

$$V(\vec{a},\vec{b},\vec{c}) = \vec{F}\cdot\vec{c} = \vec{E}\cdot\vec{c}$$

This holds for any $\vec{c}$. Thus

$$\boxed{\vec{E} = \vec{F}}$$

$\vec{E}$ is a vector orthogonal to plane spanned by $\vec{a}, \vec{b}$ whose length is equal to the area of the parallelogram.

We shall name this quantity:

$$\vec{F}(\vec{a},\vec{b}) = \vec{a} \times \vec{b}$$

<u>Cross product</u>

The formula is

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{i} + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k}$$

(symbolically)

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$V(\vec{a}, \vec{b}, \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

Example: $\vec{a} = (1, 2, 3)$, $\vec{b} = (3, 2, -1)$, $\vec{c} = (0, 2, -3)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 3 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} \hat{i} - \begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix} \hat{j} + \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} \hat{k}$$

$$= (-2-6)\hat{i} - (-1-9)\hat{j} + (2-6)\hat{k}$$

$$= -8i + 10\hat{j} - 4k = (-8, 10, -4)$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = (-8, 10, -4) \cdot (0, 2, -3) = 20 + 12 = 32$$

Since $V(\vec{a}, \vec{b}, \vec{c}) > 0$, $(\vec{a}, \vec{b}, \vec{c})$ is right triple.