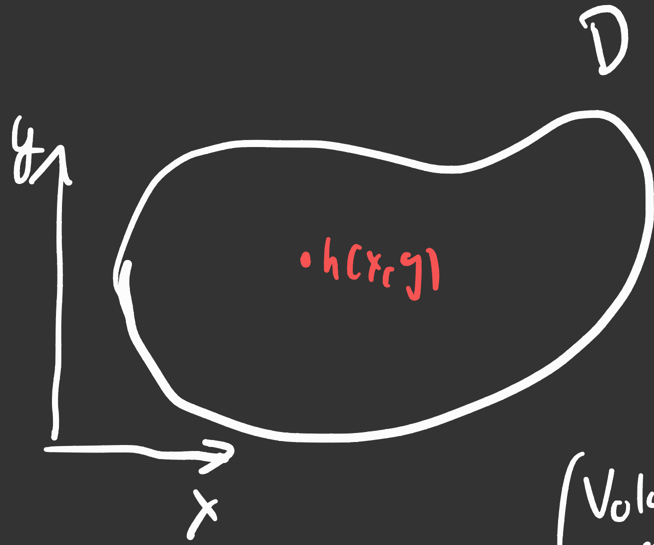
Double Integral

How much paint needed to cover domain D?

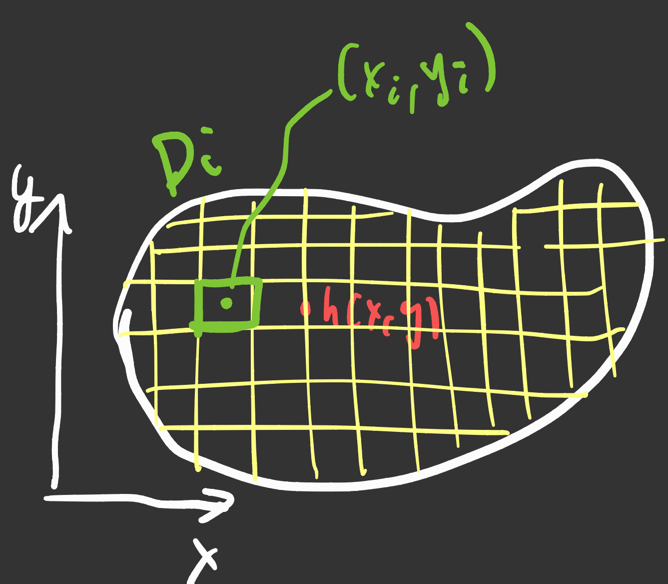
$h(x,y)$ is thickness of a layer of paint covering the domain.



If constant, then

$$\begin{pmatrix}\text{Volume of} \\ \text{paint needed} \\ \text{to cover } D\end{pmatrix} = (\text{thickness}) \cdot (\text{Area of Domain})$$

If $h(x,y)$ is variable, this is a new problem. To solve this problem, we have a device called double integral.



assume $h(x,y) =$ constant in D_i

$$\text{Area}(D_i) \cdot h(x_i, y_i)$$

amount of paint to cover D_i

$$\text{Total Volume} \quad S_N = \sum_{i=1}^{N} h(x_i, y_i) \, \text{Area}(D_i)$$

S_N depends on

- how we broke up D into small pieces
- how we chose point (x_i, y_i) in D_i

But in the limit $n \to \infty$ and

$$\max \text{size}(D_i) \to 0,$$

then $S_N \to S$ exists (independent of above choices)

and represents the amount of paint.

We denote it as:

infinitesimally small area elements.

$$\lim_{N \to \infty} S_N = \iint_D h(x,y)\, dA$$

Properties:

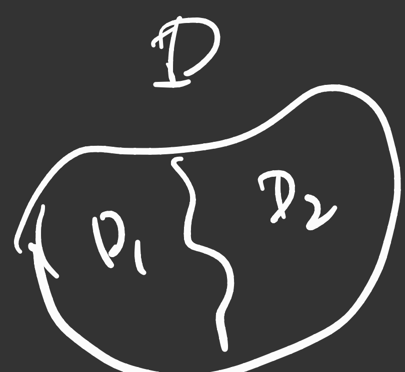
① If $f(x,y) = g(x,y) + h(x,y)$, then

$$\iint_D f\,dA = \iint_D g\,dA + \iint_D h\,dA$$

② $$\iint_D c\,f(x,y)\,dA = c\iint_D f(x,y)\,dA.$$

These are obvious from definition with finite sums. Called linearity.

③

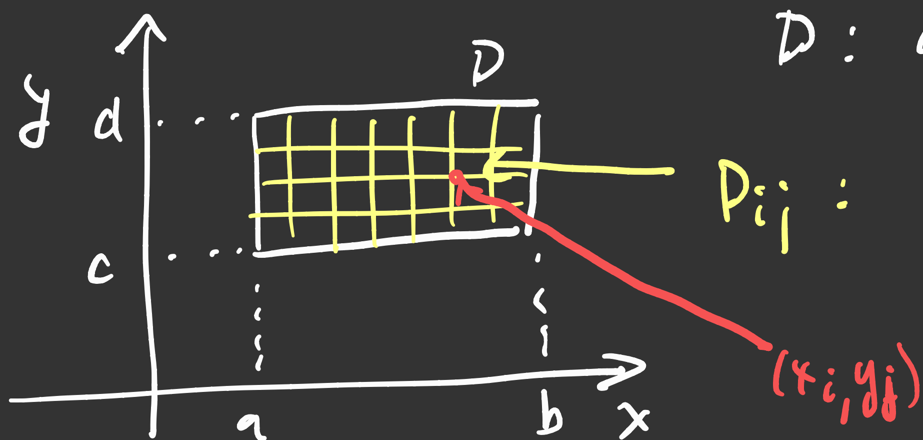


$$\iint_D f\,dA = \iint_{D_1} f_1\,dA + \iint_D f\,dA$$

This property is called additivity.

Iterated integral

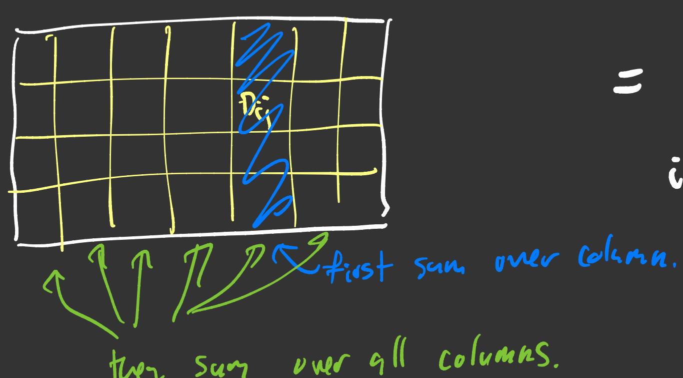
$D: a \le x \le b \quad c \le y \le d$



$P_{ij}: \quad x_i \le x \le x_{i+1}$

$\qquad\quad y_j \le y \le y_{j+1}$

$$S_N = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} f(x_i, y_j)\,\text{Area}(D_{ij})\right)$$



$$= \sum_{i=1}^{m}\left(\sum_{j=1}^{n} f(x_i, y_j)(y_{j+1}-y_j)(x_{i+1}-x_i)\right)$$

$$\approx \sum_{i=1}^{m} \underbrace{\int f(x_i, y)\,dy}_{\substack{\text{Column}(i)\\ F(x_i)}}\,(x_{i+1}-x_i)$$

$$\approx \int_a^b F(x)\,dx$$

Reduce double integral to two single integrals.

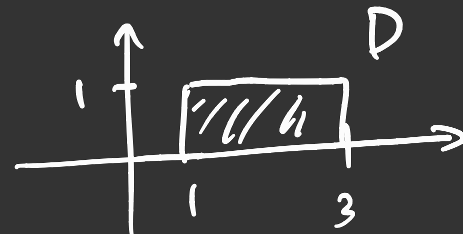
$$= \int_a^b\int_c^d f(x,y)\,dy\,dx.$$

We could have first integrated in x, and then y. This amounts to switching orders of sums.'

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

Example: D: $1 < x < 3$, $0 < y < 1$

$$I = \iint_D xy\,dA$$



$$= \int_1^3 \int_0^1 xy\,dy\,dx = \int_1^3 \frac{x}{2}\,dx = \frac{1}{4}(3^2 - 1)$$
$$= 2.$$

Example D: $-1 \le x \le 3$ $0 \le y \le 2$

$$\iint_D (x^2 - 3y^2)\, dA = \int_{-1}^{3}\int_{0}^{2} (x^2 - 3y^2)\, dy\, dx$$

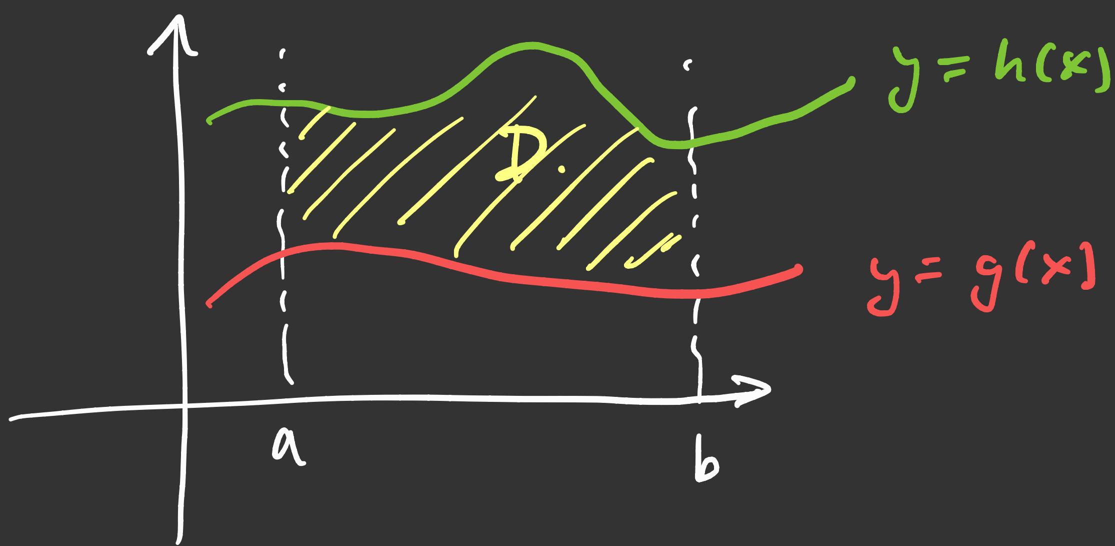
$$= \int_{-1}^{3} \left[2x^2 - 8 \right] dx = \frac{2}{3}x^3 \Big|_{-1}^{3} - 8x \Big|_{-1}^{3}$$

$$= 18 + \frac{2}{3} - 24 - 8$$

$$= -14 + \frac{2}{3}.$$

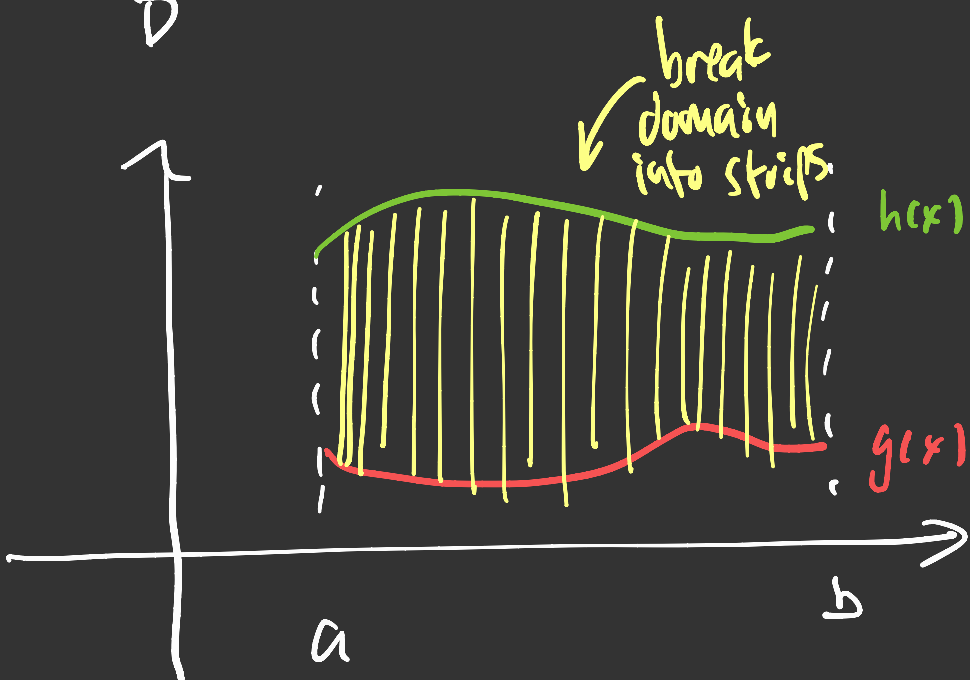
These integrals are as easy or as hard as integrals of functions of 1 variable. Sometimes easy, sometimes transendentally hard.

Domains of Type 1

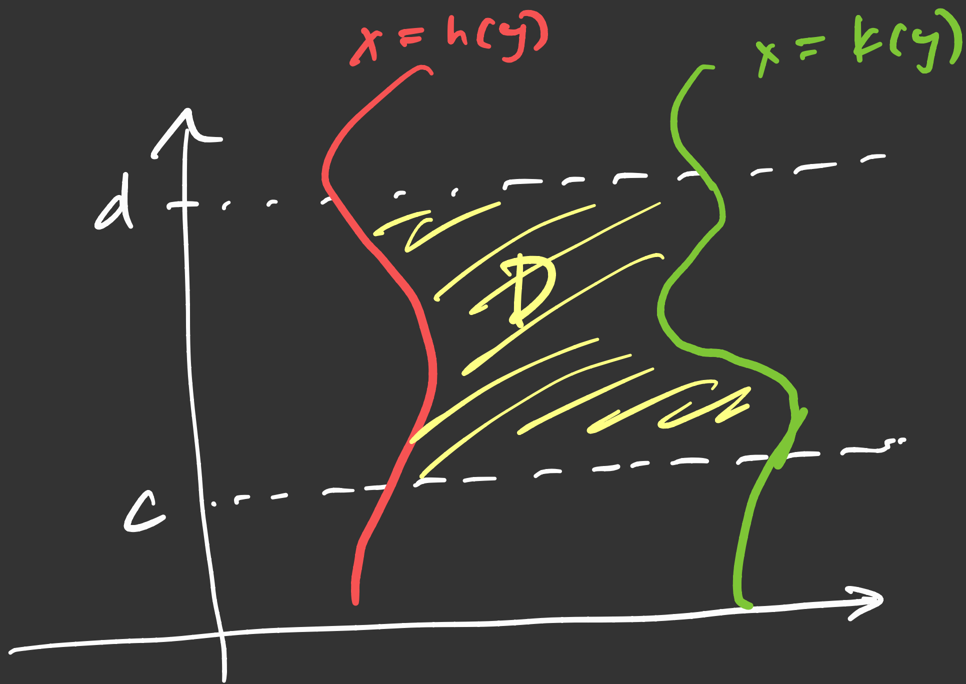


$$D: \quad a \le x \le b, \quad g(x) \le y \le h(x)$$

$$I = \iint_D f(x,y)\,dA = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\,dy \right) dx.$$



Domain of Type 2



$$D: \quad c \le y \le d, \quad h(y) \le x \le k(y)$$

$$\boxed{I = \iint_D f(x,y)\,dA = \int_c^d \left(\int_{h(y)}^{k(y)} f(x,y)\,dx \right) dy}$$

There are domains both of Type 1 and 2.

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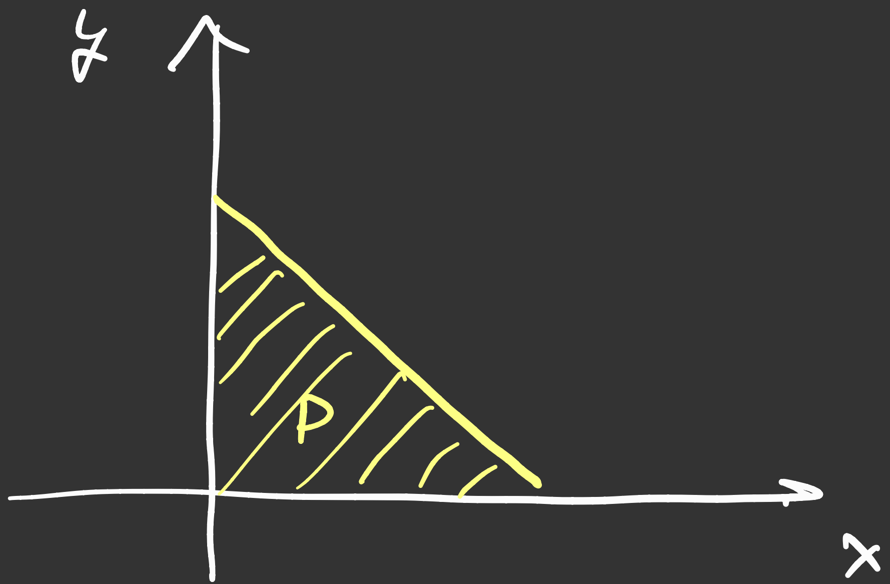
e.g.

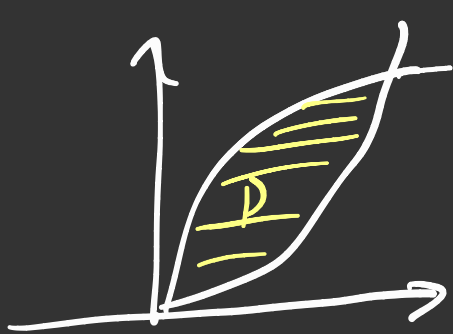


$D: 0 \leq x \leq 1, \quad 0 \leq y \leq 1-x$ Type 1

$D: 0 \leq y \leq 1, \quad 0 \leq x \leq 1-y$ Type 2.

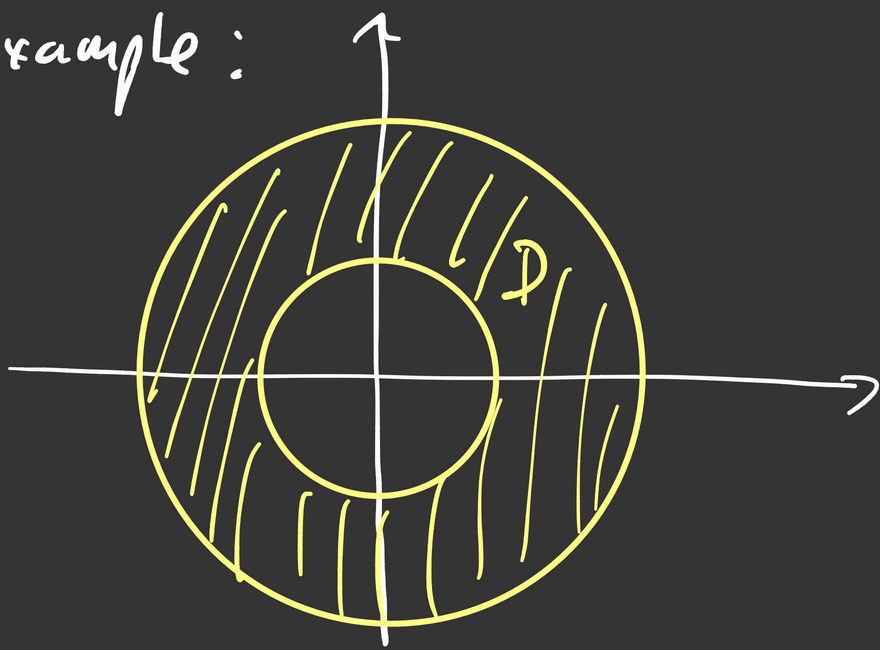
This gives a freedom — sometimes easier to to computations with one Type.

Another example

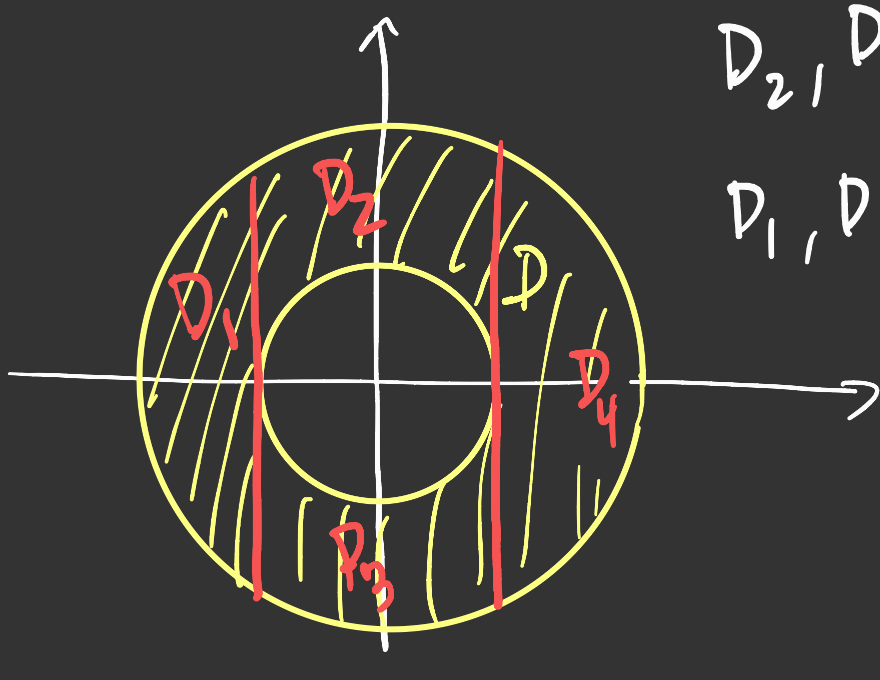


But, there are domains that are neither type 1 nor type 2.

For example:



In this domain, we can subdivide into domains of Type 1/2.



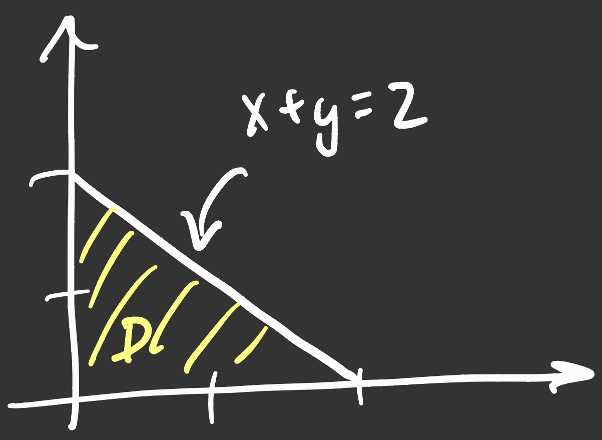
D_2, D_3 – type 1

D_1, D_4 – type 1 or type 2.

Any reasonable domain can be subdivided in this way.

Example

$D: \quad x \geq 0, \quad y \geq 0, \quad x + y < 2$



$$I = \iint_D xy \, dA$$

$$= \int_0^2 \left(\int_0^{2-x} xy \, dy \right) dx$$

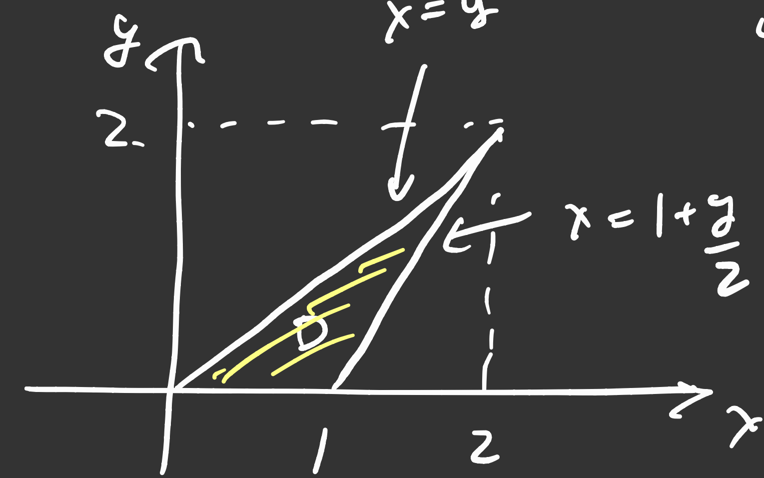
$$= \int_0^2 \frac{x(2-x)^2}{2} \, dx$$

$$= \frac{1}{2} \int_0^2 x \, (4 - 4x + x^2) \, dx$$

$$= \frac{1}{2} \int_0^2 (4x - 4x^2 + x^3) \, dx = \left[x^2 - \frac{2}{3} x^3 + \frac{x^4}{8} \right]_0^2$$

$$= 2^2 - \frac{2}{3} \cdot 2^3 + \frac{2^4}{8} = 4 - \frac{16}{3} + 2 = \ldots$$

__Ex__



D is type 2 domain.

$0 \leq y \leq 2$

$y \leq x \leq 1 + \frac{y}{2}$.

$$I = \iint_D e^{x+y}\, dA = \int_0^2 \int_y^{1+y/2} e^x e^y \, dx \, dy$$

$$= \int_0^2 e^y \int_y^{1+y/2} e^x \, dx \, dy$$

$$= \int_0^2 e^y \left(e^{1+y/2} - e^y \right) dy$$

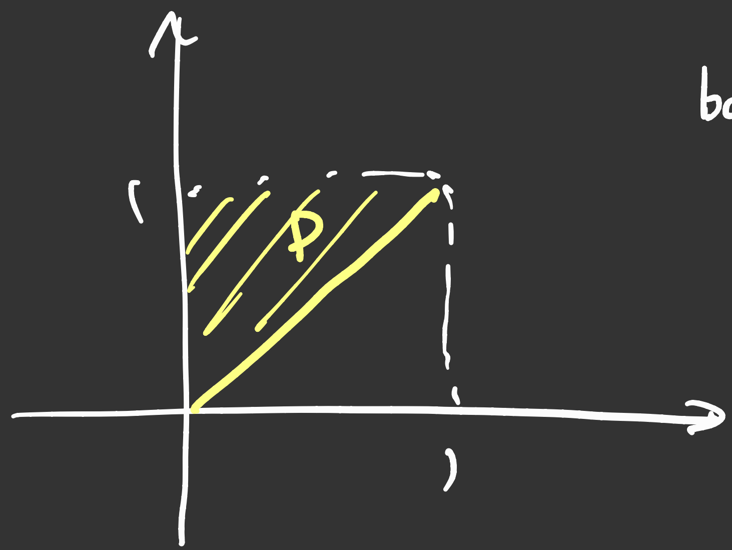
$$= \int_0^2 \left(e^{1+\frac{3}{2}y} - e^{2y} \right) dy = e\left(\frac{2}{3} e^{\frac{3}{2}y} \Big|_0^2 - \frac{1}{2} e^{2y} \Big|_0^2 \right)$$

$$= \frac{2}{3} e \left(e^3 - 1 \right) - \frac{1}{2} \left(e^4 - 1 \right)$$

Ex

$D: \; x \geq 0, \; y \geq 0, \; y \geq x$

both of type 1 and 2.



looks hard...

$$I = \iint_D \frac{1}{1+y^4}\, dA = \int_0^1 dx \left(\int_x^1 \frac{dy}{1+y^4} \right)$$

change order (regard D as type 2)

$$I = \int_0^1 dy \left(\int_0^y \frac{1}{1+y^4}\, dx \right) = \int_0^1 \frac{y}{1+y^4}\, dy$$

Let $t = y^2 \qquad dt = 2y\,dy$

45° or $\frac{\pi}{4}$.

$$= \frac{1}{2} \int_0^1 \frac{dt}{1+t^2} = \frac{1}{2} \arctan(t) \Big|_0^1 = \frac{1}{2} \arctan(1)$$

$$= \frac{\pi}{4} \cdot \frac{1}{2} = \frac{\pi}{8}.$$