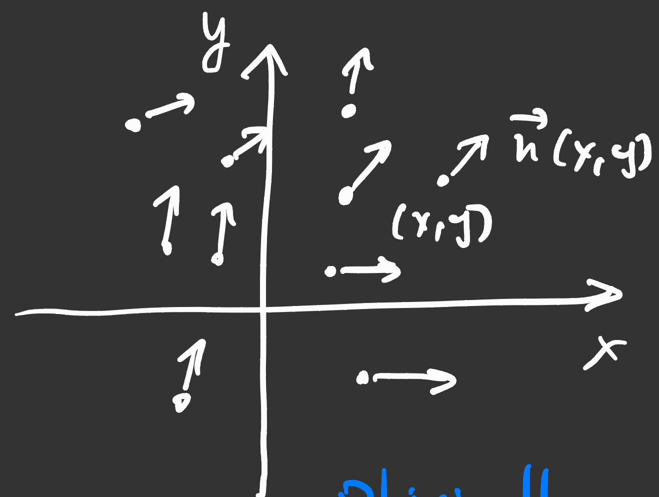
Vector fields



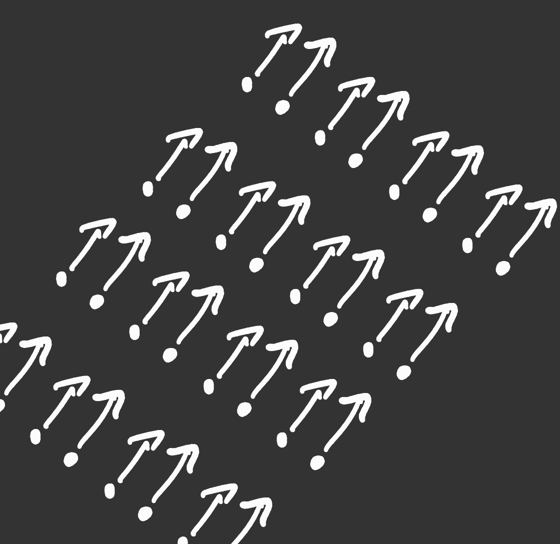
A field is a piece of the plane which at every point grows something (wheat, corn, etc). Here, vectors are grown.

Physically, think of a fluid: at every point there is a velocity.

$$\vec{u}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$$

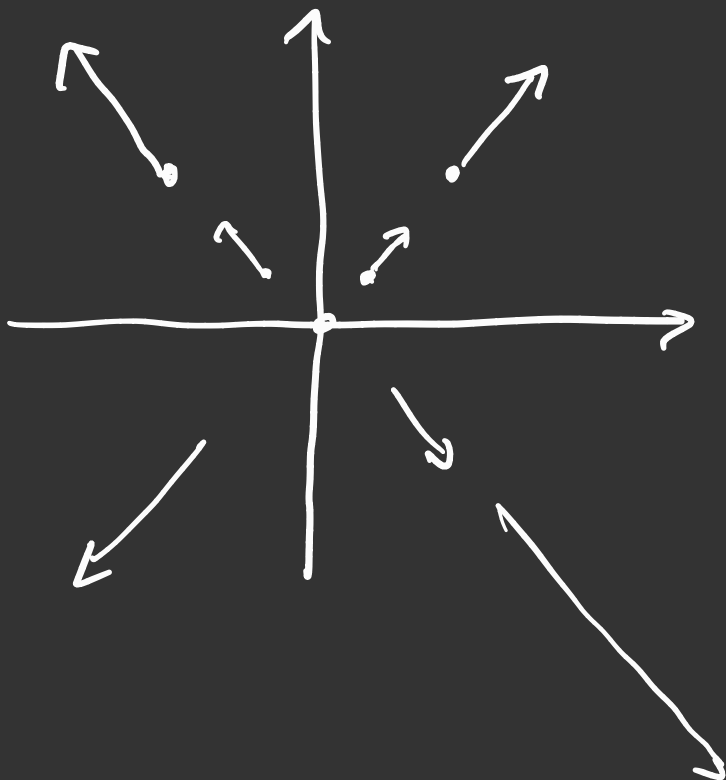
$$= P\,\vec{i} + Q\,\vec{j} \quad \text{(for short)}$$

Example: $\vec{u} = \text{const}$, e.g. $P = \text{const}$, $Q = \text{const}$

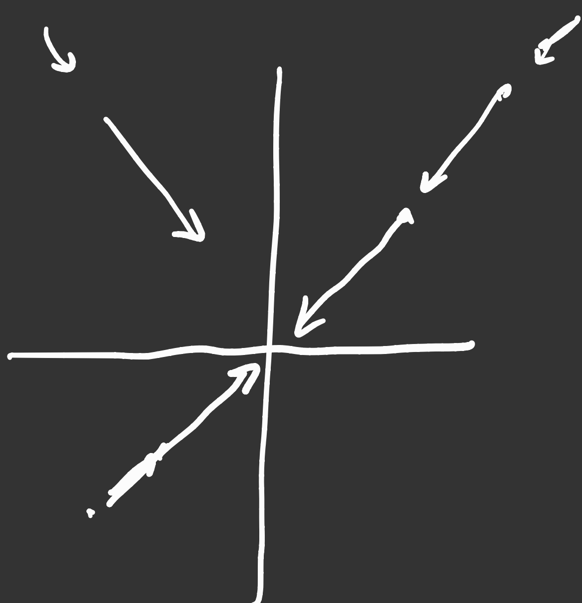


other physical examples
- electric field
- magnetic field
- gravitational force field

Example 2: $u(x,y) = (x,y)$



Example 3 $u(x,y) = u(\vec{r}) = -\dfrac{\vec{r}}{\|\vec{r}\|^3}$



Gravity field

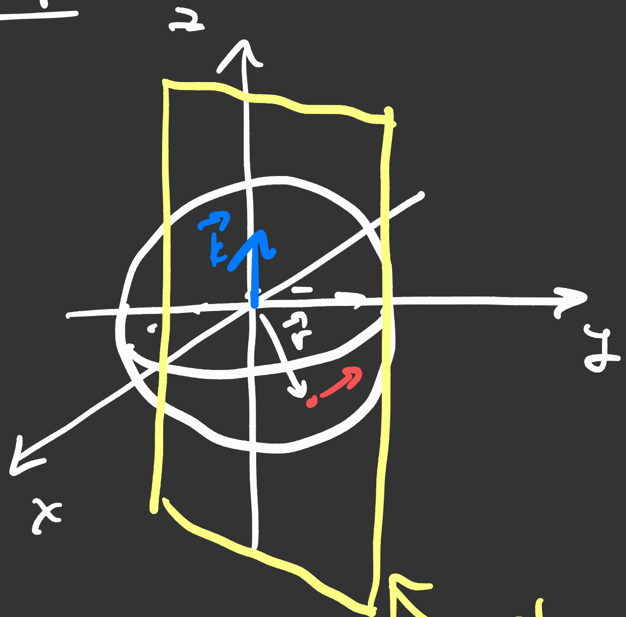
Closer you are to the origin (attracting mass), stronger the pull of gravity.

In 3D, same concept.

$$\vec{u}(\vec{r}) = \vec{u}(x,y,z)$$

$$= P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$$

Example:



rotating solid ball around z-axis $\hat{k}$

- rotates with unit angular velocity, e.g. in 1 second, it turns 1 radian.

plane containing z axis and $\vec{r}$.

The velocity of that point $\vec{r}$ is orthogonal to $\vec{r}$ and orthogonal to $\vec{k}$.

Thus, it is proportional to $\vec{k} \times \vec{r}$

If the angular velocity is 1, then

$$\vec{u}(\vec{r}) = \vec{k} \times \vec{r}$$

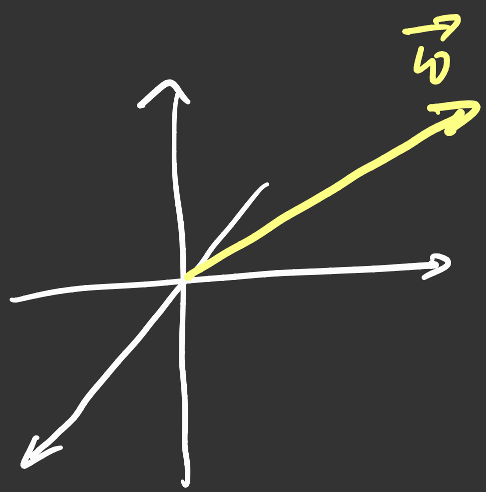
We can imagine a rotation around any other axis.

③

Consider the vector

$$\vec{\omega} = (\omega_1, \omega_2, \omega_2),$$

a vector directed along axis of rotation, so if you look down tip, the rotation is ccw. The length of $\vec{\omega}$ is equal to angular velocity of rotation.



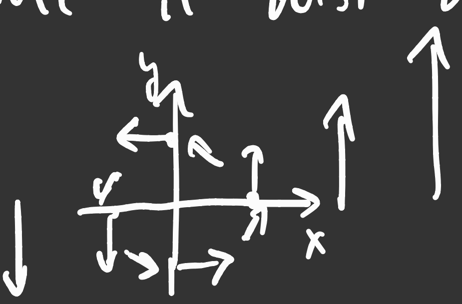
The velocity is then

$$\vec{u}(\vec{r}) = \vec{\omega} \times \vec{r}$$

In particular, if $\vec{\omega} = \vec{k} = (0,0,1)$, then

$$\vec{u}(\vec{r}) = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ x & y & z \end{vmatrix} = (-y, x, 0)$$

Note it doesn't depend on z (as it is rotating along z axis)



← rotating disk in the plane.

Operations on vector fields

Gradient ∇:

$$\nabla f(x,y,z) = (\partial_x f, \partial_y f, \partial_z f) \leftarrow \text{vector field.}$$

Divergence (div):

$$\vec{u}(x,y,z) = \big(P(x,y,z), Q(x,y,z), R(x,y,z)\big)$$

$$\operatorname{div} \vec{u} := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Takes a vector field and results in a scalar.

Ex: $\vec{u}(\vec{r}) = \vec{r} = (x,y,z)$ $\qquad P = x,\ Q = y,\ R = z$

$$\operatorname{div} \vec{u} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

Ex: (solid rotation) $\vec{u}(x,y,z) = (-y, x, 0)$ $\qquad P = -y,\ Q = x,\ R = 0$

$$\operatorname{div} \vec{u} = -\frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} = 0$$

$\uparrow$ we will understand why

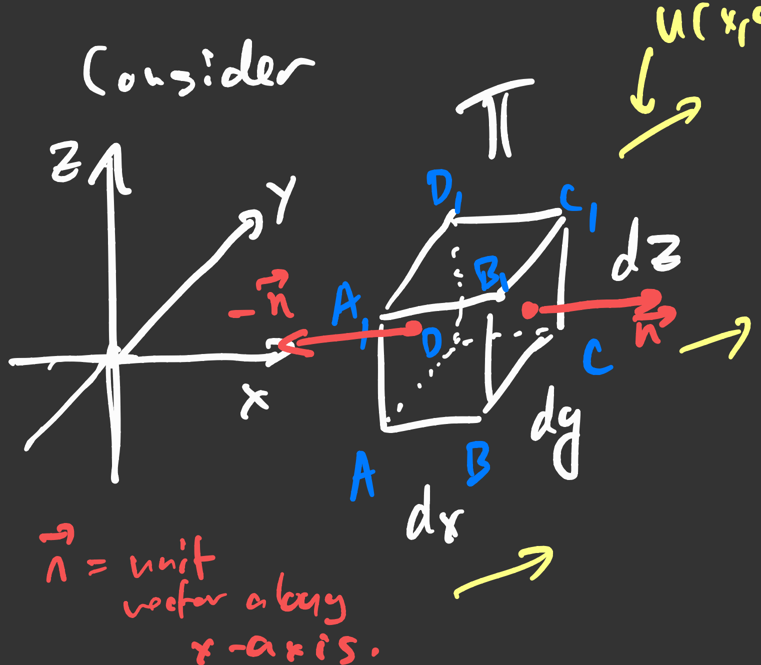
Divergence has many profound applications we will explore.

Physical meaning of divergence

$$\vec u = P(x,y,z)\,\vec i + Q(x,y,z)\,\vec j + R(x,y,z)\,\vec k$$

Imagine $\vec u$ is the velocity field of a fluid.

Consider



How much fluid enters, and how much exits?

$$\text{Flux} := \begin{cases} \text{positive if fluid goes out} \\ \text{negative if fluid comes in} \end{cases}$$

Flux through faces ADD_1A_1 and BCC_1B_1

→ note second and third component ($\vec j$ and $\vec k$) are tangent to face, so no contribution to flux.

negative as $\vec i \cdot (-\vec n) = -1$ positive as $\vec i \cdot \vec n = 1$

$$\rightarrow \underbrace{dy\,dz}_{\text{area of face}} \Big(-P(x,y,z) + P(x+dx, y, z) \Big) \approx P_x(x,y,z)\,dx\,dy\,dz .$$

Similarly,

Flux through ABB_1A_1 and DCC_1D_1

$$\approx Q_y \, dx\,dy\,dz$$

Flux through $ABCD$ and $A_1B_1C_1D_1$

$$\approx R_z \, dx\,dy\,dz$$

In total, the flux through the surface Π is

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \operatorname{div} \vec{u}$$

If $\operatorname{div} \vec{u} > 0$, the fluid is somehow generated inside Π (e.g. from nothing, or sources)

Recall $\vec{u} = \vec{r}$

For any Π, the amount of fluid that leaves Π is more than amount that enters.

Not in contradiction with cons law (if no source or sinks, as much enters, as much leaves)

MAT 307: Advanced Multivariable Calculus

∇ (Nabla) $\quad \nabla = (\partial_x, \partial_y, \partial_z)$

$$\nabla f = (\partial_x f, \partial_y f, \partial_z f) = (f_x, f_y, f_z) \quad \text{Gradient}$$

scalar function

$$\nabla f = \text{grad} f$$

vector field $\quad \vec{u}(\vec{r}) = (P(\vec{r}), Q(\vec{r}), R(\vec{r}))$

$$\nabla \cdot \vec{u} = \partial_x P(\vec{r}) + \partial_y Q(\vec{r}) + \partial_z R(\vec{r})$$

Divergence

$$= \text{div}\, \vec{u}$$

notation for same thing

What is $\nabla \times \vec{u}$?

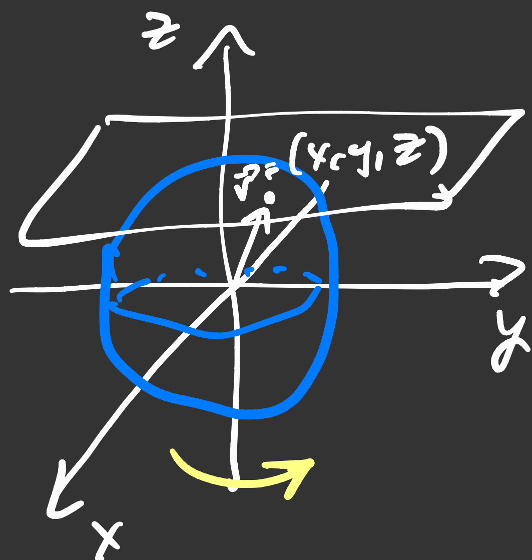
$$\nabla \times \vec{u} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}$$

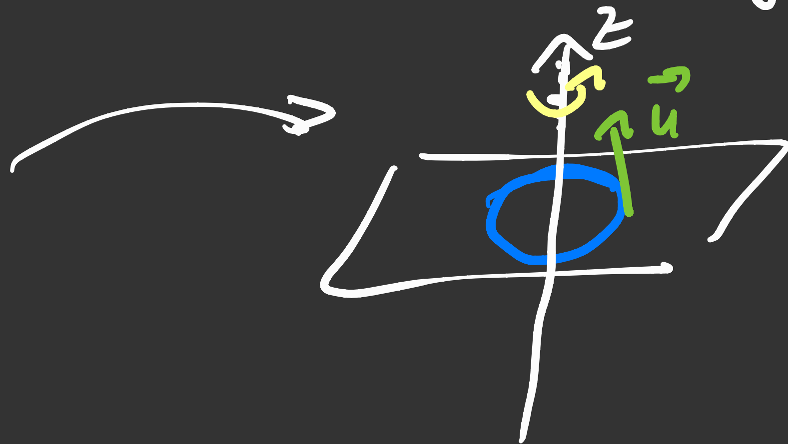
$$= (\partial_y R - \partial_z Q)\vec{i} - (\partial_x R - \partial_z P)\vec{j} + (\partial_x Q - \partial_y P)\vec{k}$$

$$= (R_y - Q_z)\vec{i} + (P_z - R_x)\vec{j} + (Q_x - P_y)\vec{k}$$

$$\boxed{\nabla \times \vec{u} = \operatorname{curl} \vec{u}}$$

<u>Example:</u> Velocity field of rotation of a solid body





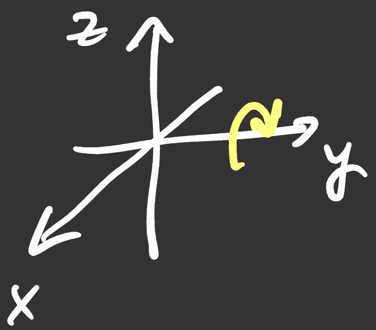
$$u = \vec{k} \times \vec{r}$$

$$\vec{k} = (0, 0, 1)$$

$$\vec{u} = \vec{k} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ x & y & z \end{vmatrix} = (-y, x, 0)$$

$$\operatorname{curl} \vec{u} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = (0, 0, 2) = 2\vec{k}$$

If it rotates around a different axis



$$\vec{u}(\vec{r}) = \vec{j} \times \vec{r}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ x & y & z \end{vmatrix} = (z, 0, -x)$$

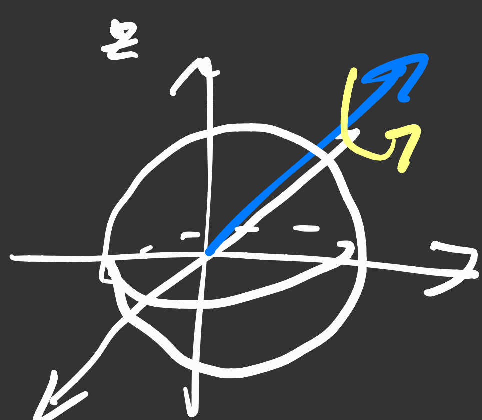
$$\operatorname{curl} \vec{u}(\vec{r}) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ z & 0, & -x \end{vmatrix} = (0, 2, 0)$$

$$= 2\vec{j}$$

Around x-axis:

$$\vec{u}(\vec{r}) = \vec{i} \times \vec{r} \qquad \operatorname{curl} \vec{u} = 2\vec{i}.$$

Axis of rotation in direction $w = (w_x, w_x, w_z)$



with angular rotation speed

$\|w\|$

Then the velocity is

$$\vec{u}(\vec{r}) = \vec{w} \times \vec{r}$$

and

$$\operatorname{curl} \vec{u}(\vec{r}) = 2\vec{w}.$$

check yourselves!

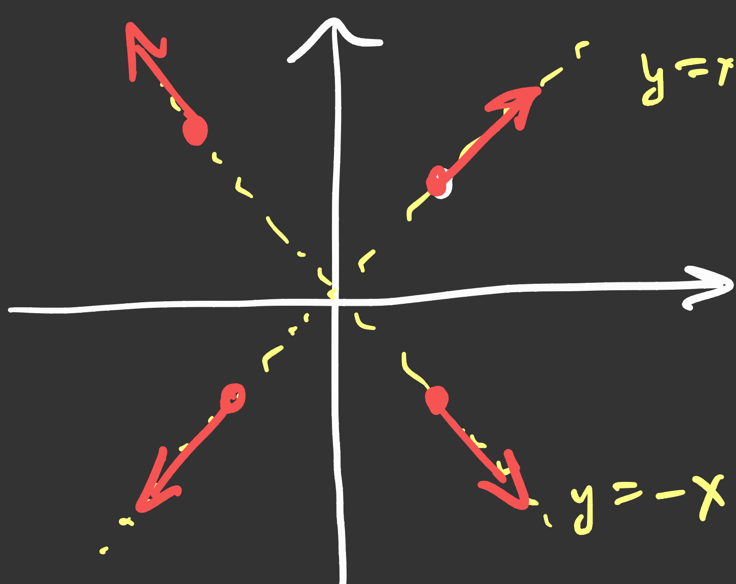
Consider now

$$\vec{u}(\vec{r}) = (y, x, 0)$$

$$\nabla \times \vec{u} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y & x & 0 \end{vmatrix} = (0, 0, 1-1)$$

$$= (0, 0, 0)$$

This vector field has curl zero!



Consider a function $f(\vec{r}) = f(x,y,z)$
and its gradient $\nabla f(\vec{r}) = (f_x, f_y, f_z)$
$= \text{grad } f.$

Let's find its curl.

$$\text{curl grad } f = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \partial_x f & \partial_y f & \partial_z f \end{vmatrix}$$

$$= \left(\partial_y\partial_z f - \partial_z\partial_y f,\ -(\partial_x\partial_z f - \partial_z\partial_x f),\ \partial_x\partial_y f - \partial_y\partial_x f\right)$$

$$= (0,\ 0,\ 0)$$

Since partials commute for functions with continuous second derivatives

Thus

$$\underset{\nabla \times \nabla f}{\overset{\shortparallel}{\text{curl grad } f}} = 0$$

for all functions f!

In particular, our previous example:

$$\vec{u} = (y, x, 0)$$

$$f = xy$$

$$\partial_x f = y, \quad \partial_y f = x, \quad \partial_z f = 0$$

Thus $\vec{u} = \nabla f$, and $\operatorname{curl} \vec{u} = 0$ by our general result.

Much more interesting and important is that the reverse is true:

$$\boxed{\text{If } \vec{u} \text{ is a vector field with } \operatorname{curl} \vec{u} = 0, \text{ then } \vec{u} = \operatorname{grad} f \text{ for some } f.}$$

We'll see this later...

A connection between div and curl:

$$\vec{u} = (P, Q, R)$$

$$\text{curl}\,\vec{u} = \nabla \times \vec{u}$$

What is $\text{div}\,\text{curl}\,\vec{u}$?

First

$$\text{curl}\,\vec{u} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}$$

$$= (R_y - Q_z,\ P_z - R_x,\ Q_x - P_y)$$

$$\text{div}\,\text{curl}\,\vec{u} = \partial_x(R_y - Q_z) + \partial_y(P_z - R_x) + \partial_z(Q_x - P_y)$$

$$= R_{yx} - Q_{xz} + P_{yz} - R_{yx} + Q_{zx} - P_{zy}$$

$$= 0!$$

Thus $\text{div}\,\text{curl}\,\vec{u}$ for any vector field $\vec{u}$.

Basic principles of hydrodynamics. There $\vec{u}$ is velocity field. If $\text{div}\,\vec{u} = 0$, fluid is incompressible.

$w = \text{curl}\,\vec{u}$ vorticity  ← w directed along column