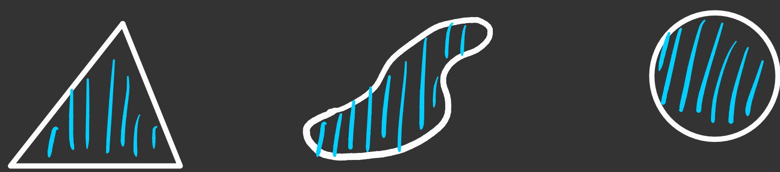
Isoperimetric inequality

Question: Among all planar domains with equal area, which has maximal area?



Answer: Disk!

Theorem: Let C be a simple closed curve, enclosing area A. Then, if $L = \text{length}(C)$, then

$$L^2 \geq 4\pi A.$$

Equality holds if and only if C is a circle.

Proof: First, by Green's theorem, $\frac{1}{2}\int_C x^\perp \cdot ds = \frac{1}{2}\int_A \nabla^\perp \cdot x^\perp \, dA = A.$

Now, parametrize C by $\gamma : [0, L) \to C$ with $|\dot{\gamma}(s)| = 1$. Then

$$A = \frac{1}{2}\int_C x^\perp \cdot ds = \frac{1}{2}\int_0^L \gamma^\perp(t) \cdot \dot{\gamma}(t)\, dt \leq \frac{1}{2}\int_0^L |\gamma(t)|\, dt.$$ (Cauchy-Schwarz)

Equality in Cauchy-Schwarz holds iff $\gamma = \gamma_*$ where $\dot{\gamma}_*(t) = \gamma_*^\perp(t)/|\gamma_*(t)|$. Call this $|\gamma_*(0)| = 1/\omega$.

Note then that $|\gamma_*(t)| = |\gamma_*(0)|$ for all t.

Then $\gamma(t) = R_{\omega t}\,\gamma(0)$. ← rotation matrix. Since $\gamma(t)$ must have period L, $\omega = \frac{2\pi}{L}$.

$$A \leq \frac{1}{2}\int_0^L |\gamma_*(t)|\, dt = \frac{L^2}{4\pi}$$

In case of circle of radius R, $A = \pi R^2$, $L = 2\pi R$, so $A = \frac{L^2}{4\pi}$. ∎