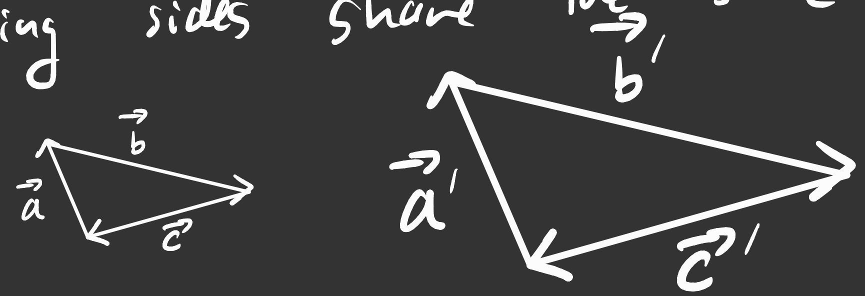
MAT 307: Euclidean Geometry with vectors

Theorem: If corresponding sides of two triangles are parallel, then the lengths of the corresponding sides share the same ratio.



Proof: Since the sides are parallel

$$\vec{a} = \lambda_1 \vec{a}', \quad \vec{b} = \lambda_2 \vec{b}', \quad \vec{c} = \lambda_3 \vec{c}'$$

Since they each form a triangle:

$$\vec{a} + \vec{b} + \vec{c} = 0 \qquad \text{and} \qquad \vec{a}' + \vec{b}' + \vec{c}' = 0$$

$$\Downarrow$$

$$\lambda_1 \vec{a}' + \lambda_2 \vec{b}' + \lambda_3 \vec{c}' = 0$$

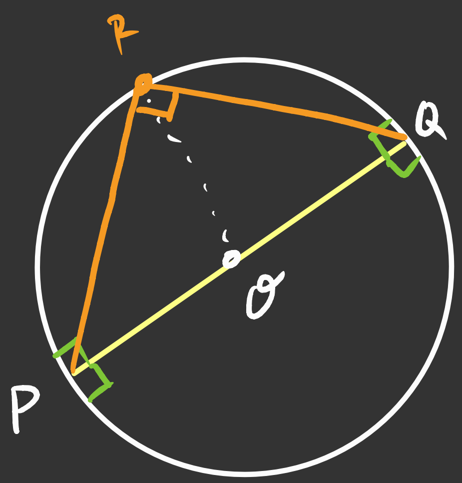
$$\Longrightarrow (\lambda_2 - \lambda_1)\vec{b}' + (\lambda_3 - \lambda_1)\vec{c}' = 0$$

So if $\lambda_2 \neq \lambda_1$ then $\vec{b}' = -\dfrac{\lambda_3 - \lambda_1}{\lambda_2 - \lambda_1}\vec{c}'$.

contradiction (would be degenerate!)

Thus $\lambda_2 = \lambda_1$. Similarly $\lambda_3 = \lambda_1$. Thus we conclude $\lambda_1 = \lambda_2 = \lambda_3 \equiv \lambda$, the common ratio.

Theorem: The angle substended by the diameter of a circle on the circumference is always $90°$.



Proof: Let P, Q be end points of the diameter on the circumference. O is the center. R is a point on the circumference

$$\text{Now} \quad \vec{PR} = \vec{PO} + \vec{OR}$$
$$\vec{QR} = \vec{QO} + \vec{OR}$$

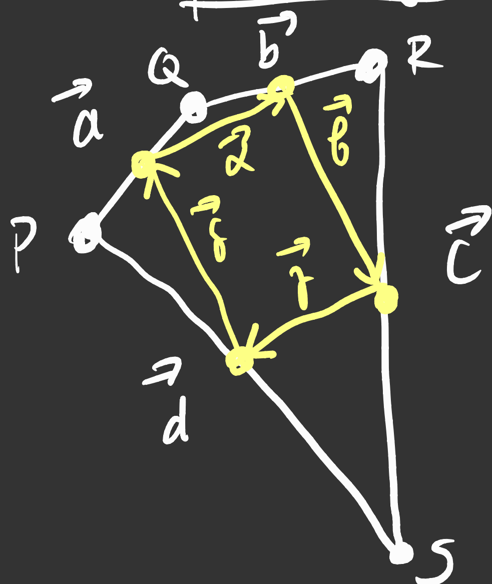
Note $\vec{QO} = -\vec{PO}$. Thus

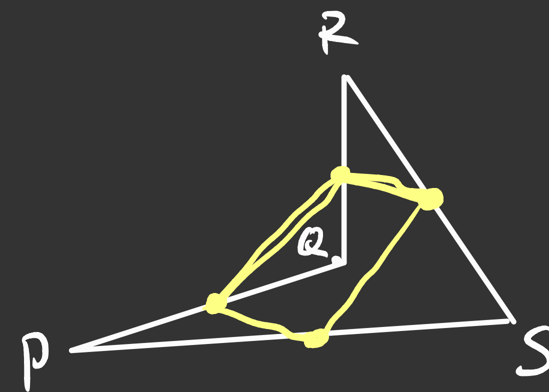
$$\vec{PR} \cdot \vec{QR} = (\vec{PO} + \vec{OR}) \cdot (\vec{QO} + \vec{OR})$$
$$= (\vec{PO} + \vec{OR})(-\vec{PO} + \vec{OR})$$
$$= -\|\vec{PO}\|^2 + \|\vec{OR}\|^2 = \rho^2 - \rho^2 = 0$$

radius of circle

Thus the angle is right. $\blacksquare$

Theorem: Let $PQRS$ be *any* quadrilateral. Consider the figure obtained by connecting the midpoints of all sides. This figure is a *parallelogram*.





Proof: Let

$$\vec{a} = \overrightarrow{PQ}, \quad \vec{b} = \overrightarrow{QR}, \quad \vec{c} = \overrightarrow{RS}, \quad \vec{d} = \overrightarrow{SP}$$

Let $\vec{\alpha}$ = vector connecting midpoint of $\overrightarrow{PQ}$ to that of $\overrightarrow{QR}$

$\vec{\beta}$ = " " $\overrightarrow{QR}$ to $\overrightarrow{RS}$

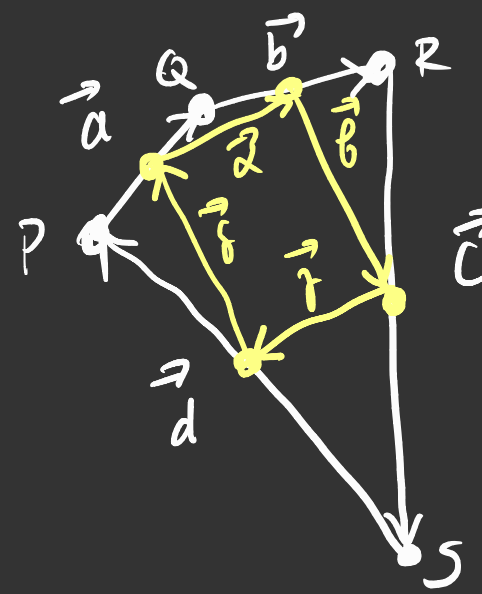
$\vec{\gamma}$ = " " $\overrightarrow{RS}$ to $\overrightarrow{SP}$

$\vec{\delta}$ = " " $\overrightarrow{SP}$ to $\overrightarrow{PQ}$

Then $\vec{\alpha}$ satisfies

$$\frac{1}{2}\vec{a} + \vec{\alpha} = \vec{a} + \frac{1}{2}\vec{b} \quad \Rightarrow \quad \vec{\alpha} = \frac{1}{2}(\vec{a} + \vec{b})$$

Similarly, $\vec{\gamma}$ satisfies



$$-\frac{1}{2}\vec{d} - \vec{\gamma} = -\vec{d} - \frac{1}{2}\vec{c}$$

so

$$\vec{\gamma} = \frac{1}{2}(\vec{d} + \vec{c}).$$

We claim that $\vec{\alpha} = -\vec{\gamma}$. This follows since

$\vec{a} + \vec{b} = \vec{PR}$ and $-\vec{d} - \vec{c} = \vec{PR}$.

Thus two sides are parallel. Similar reasoning shows that the other two sides are parallel.

Thus, the shape is a parallelogram. $\boxtimes$

Note, same statement is true of quadrilateral with vertices dividing each side by any fixed ratio, say 2:1.