1 Review MAT 307 Multivariable Calculus

All the integral theorems we've encountered can be summarized by the following

$$\int_M d\omega = \int_{\partial M} \omega.$$

Here ω can be a function or a vector field, and d as derivative can be ordinary derivative, grad, curl, or div. M is the object we integrate over, which could be an interval in $\mathbb{R}$, a curve, a surface or a solid; ∂M is the boundary of M with appropriate orientation. The integral symbol can represent sum (0-d case), ordinary integral, line integral, double integral, flux integral, or a triple integral.

- **Fundamental Theorem of Calculus**: 1-variable function $f(x)$,

$$\int_a^b \frac{df}{dx} dx = f(b) - f(a).$$

- **Fundamental Theorem of Line integral**: 2-d vector fields $\vec{F}(x, y) = \nabla f(x, y)$,

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \nabla(f(\vec{r}(t)) \cdot \vec{r}'(t)dt = f(\vec{r}(b)) - f(\vec{r}(a)).$$

- **Green's Theorem**: 2-d vector fields $\vec{F}(x, y) = (P, Q)$,

$$\int\int_D (Q_x - P_y)dxdy = \oint_{C=\partial D} \vec{F} \cdot d\vec{r},$$

- **Divergence Theorem**: 2-d vector fields $\vec{F}(x, y) = (P, Q)$,

$$\int\int_D P_x + Q_y dxdy = \oint_{C=\partial D} \vec{F} \cdot \vec{n} ds.$$

- **Fundamental Theorem of Line integral**: 3-d vector fields $\vec{F}(x, y, z) = \nabla f(x, y, z)$,

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \nabla(f(\vec{r}(t)) \cdot \vec{r}'(t)dt = f(\vec{r}(b)) - f(\vec{r}(a)).$$

- **Stokes' Theorem**: 3-d vector fields $\vec{F}(x, y, z)$,

$$\int\int_\Sigma curl(\vec{F}) \cdot d\vec{S} = \oint_{C=\partial\Sigma} \vec{F} \cdot d\vec{r}.$$

- **Gauss' (Divergence) Theorem**: 3-d vector fields $\vec{F}(x,y,z)$,

$$\int\int\int_W div(\vec{F})dV = \int_{\Sigma=\partial W} \vec{F}\cdot d\vec{S}.$$

$$curl(grad f) = \vec{0}, \qquad div(curl(\vec{F})) = 0.$$

Taking the boundary twice of a surface and of a solid, we get empty set.

Approaches to Evaluating Line Integrals $\int_C \vec{F}\cdot d\vec{r}$

- directly using parametrization of curve:
  C is unit circle oriented counterclockwise,

$$\vec{r}(t) = (\cos t, \sin t), \qquad t\in[0,2\pi], \qquad \vec{F} = (-y^2, x^2)$$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (-\sin^2, \cos^2 t)\cdot(-\sin t, \cos t)dt = \int_0^{2\pi} \sin^3 t + \cos^3 t dt = 0$$

- if $\vec{F} = \nabla f$, use FTLI:
  C is the arc of a unit circle oriented counterclockwise from $\theta = \pi/4$ to $\theta = 3\pi/4$.

$$\vec{r}(t) = (\cos t, \sin t), \qquad t\in[\pi/4, 3\pi/4], \qquad \vec{F} = (y,x) = \nabla f, \qquad f(x,y) = xy$$

$$\int_C \vec{F}\cdot d\vec{r} = f(x,y)\Big|_{\frac{1}{\sqrt{2}}(1,1)}^{\frac{1}{\sqrt{2}}(-1,1)} = -1$$

- if C is a closed curve in the plane, use Green's theorem to evaluate a double integral provided that $\vec{F}$ is defined everywhere inside C:
  C is ellipse $\frac{x^2}{4} + y^2 = 1$ oriented counterclockwise.

$$\vec{r}(t) = (2\cos t, \sin t), \qquad t\in[0,2\pi], \qquad \vec{F} = (-y, x)$$

$$\oint_C \vec{F}\cdot d\vec{r} = \int\int_E 2dxdy = 2A(E) = 4\pi$$

- if C is a closed curve in the plane, $\vec{F}$ is not defined somewhere inside C and $curl(\vec{F}) = \vec{0}$, use Green's theorem to evaluate a line integral along a simpler closed C' that's obtained from C without crossing the points where $\vec{F}$ is undefined:

C is ellipse $\frac{x^2}{4}+y^2=1$ oriented counterclockwise, C_1 is unit circle oriented counterclockwise.

$$\vec{r}_1(t)=(\cos t,\sin t),\qquad t\in[0,2\pi],\qquad \vec{F}=\left(-\frac{y}{x^2+y^2},\frac{x}{x^2+y^2}\right)$$

$$\oint_C \vec{F}\cdot d\vec{r}=\oint_{C_1}\vec{F}\cdot d\vec{r}=\int_0^{2\pi}(-\sin t,\cos t)\cdot(-\sin t,\cos t)dt=2\pi$$

- if C is a closed curve in $\mathbb{R}^3$, use Stokes' theorem to evaluate a flux integral. Always choose a simple surface for $curl(\vec{F})$ that fills in the curve:
  C is the curve intersection of $x^2+y^2=1$ and $z=1$ oriented counterclockwise when viewed from above. Let D be the unit disk given by $x^2+y^2\le 1$ and $z=1$.

$$\vec{r}(t)=(\cos t,\sin t,1),\qquad t\in[0,2\pi],$$

$$\vec{F}=(-y^2+e^{x^2},\cos(yz)+x,\sin(x^2\cos yz))$$

$$\oint_C \vec{F}\cdot d\vec{r}=\int\int_D curl\vec{F}\cdot d\vec{S}=\int\int_D curl\vec{F}\cdot(0,0,1)dS=\int\int_D(1+2y)dS=\pi$$

- if C is not closed and $\vec{F}$ not conservative, we can either
  - add another curve to get a closed curve and then apply Green or Stokes, and then subtract the line integral over the added curve:
    C is half circle from $(1,0)$ to $(-1,0)$, L is line segment from $(-1,0)$ to $(1,0)$,

$$\vec{r}(t)=(\cos t,\sin t),\qquad t\in[0,\pi],\qquad \vec{F}=(3x^2y+\cos y-y,x^3-x\sin y)$$

$$\vec{l}(t)=(-1+2t,0),\qquad t\in[0,1]$$

$$\int_{C+L}\vec{F}\cdot d\vec{r}=\int\int_D(Q_x-P_y)dS=\int\int_D 1dS=\frac{\pi}{2}$$

$$\int_C \vec{F}\cdot d\vec{r}=\frac{\pi}{2}-\int_L\vec{F}\cdot d\vec{r}=\frac{\pi}{2}-\int_0^1(1,\star)\cdot(2,0)dt=\frac{\pi}{2}-2$$

  - or find $\vec{G}$ such that $\vec{F}-\vec{G}=\nabla f$ for some f, where it is easier to evaluate $\int_C\vec{G}\cdot d\vec{r}$ than the original line integral.
    C is half circle from $(1,0)$ to $(-1,0)$,

$$\vec{r}(t)=(\cos t,\sin t,1),\qquad t\in[0,\pi],\qquad \vec{F}=(3x^2y+\cos y-y,x^3-x\sin y),$$

$$\vec{G}=(-y,0),\qquad \vec{F}-\vec{G}=\nabla f,\qquad f(x,y)=x^3y+x\cos y$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{G}\cdot d\vec{r} + \int_C \nabla f \cdot d\vec{r}$$

$$= \int_0^{\pi} (-\sin t, 0)\cdot(-\sin t, \cos t)dt + f(x,y)\Big|_{(1,0)}^{(-1,0)} = \frac{\pi}{2} - 2$$

Approaches to Evaluating Double Integrals $\int\int_D f dA$

- directly by changing the order of integration or change of variables formula.
- write $f = Q_x - P_y$ for some 2-d vector field (P, Q) and use Green's theorem to compute a line integral. This is very handy when the region of integration lies within some parametrized curve.

Approaches to Evaluating Flux Integrals $\int\int_\Sigma \vec{F}\cdot d\vec{S}$

- directly using appropriate parametrization of surface:
  Σ is graph of $f(x,y) = x^2 + y^2$ over unit disk D given by $x^2 + y^2 \leq 1$, $\vec{F} = (yz, xz, xy)$.

  $$\Phi(x,y) = (x, y, x^2+y^2), \qquad \Phi_x \times \Phi_y = (-2x, -2y, 1)$$

  $$\int\int_\Sigma \vec{F}\cdot d\vec{S} = \int\int_D (yz, xz, xy)\cdot(-2x, -2y, 1)dxdy = \int\int_D xy(1-4x^2-4y^2)dxdy = 0$$

  by symmetry!

- using properties of dot product:
  Σ is the unit sphere $x^2 + y^2 + z^2 = 1$, and $\vec{F} = \frac{\vec{r}}{r^3}$ where $r = \|\vec{r}\| = \|(x,y,z)\| = \sqrt{x^2+y^2+z^2}$.

  $$\int\int_\Sigma \vec{F}\cdot d\vec{S} = \int\int_\Sigma \frac{\vec{r}}{r^3}\cdot\frac{\vec{r}}{r}dS = \int\int_\Sigma \frac{\vec{r}\cdot\vec{r}}{r^4}dS = \int\int_\Sigma \frac{1}{r^2}dS = \int\int_\Sigma 1dS = 4\pi$$

- if Σ is closed, apply divergence (Gauss') theorem to compute a triple integral of $div(\vec{F})$ as long as $\vec{F}$ is defined everywhere inside Σ:
  Σ is the unit sphere $x^2 + y^2 + z^2 = 1$, and $\vec{F} = (x^2, y^2, z^2)$.

  $$\int\int_\Sigma \vec{F}\cdot d\vec{S} = \int\int\int_W div(\vec{F})dV = 2\int\int\int_W (x+y+z)dV = 0$$

  by symmetry!

- if Σ is closed, and $\vec{F}$ is not defined somewhere inside Σ and $div(\vec{F}) = 0$, perturb Σ to a simpler Σ' without passing through the places where $\vec{F}$ is undefined and compute the flux integral over Σ' instead:
  Σ is the ellipsoid $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$, and $\vec{F} = \frac{\vec{r}}{r^3}$ where $r = \|\vec{r}\| = \|(x, y, z)\| = \sqrt{x^2 + y^2 + z^2}$. $div(\vec{F}) = 0$. Let Σ' be the unit sphere $x^2 + y^2 + z^2 = 1$.

$$\int\int_{\Sigma} \vec{F}\cdot d\vec{S} = \int\int_{\Sigma'} \frac{\vec{r}}{r^3}\cdot\frac{\vec{r}}{r} dS = \int\int_{\Sigma'} \frac{\vec{r}\cdot\vec{r}}{r^4} dS = \int\int_{\Sigma'} \frac{1}{r^2} dS = \int\int_{\Sigma'} 1 dS = 4\pi$$

- if the surface Σ is not closed,
  - either find $\vec{G}$ where $\vec{F} = curl(\vec{G})$, and use Stokes to compute a line integral or find the flux integral through a easier surface with the same boundary:
    Σ is the upper hemisphere $x^2 + y^2 + z^2 = 1$ with $z \geq 0$ oriented upward.

$$\vec{F} = (-y, -z, -x) = curl(xy, yz, zx), \qquad \vec{r}(t) = (\cos t, \sin t, 0), \qquad t \in [0, 2\pi]$$

$$\int\int_{\Sigma} \vec{F} \cdot d\vec{S} = \int_{C} \vec{G} \cdot d\vec{r} = \int_{0}^{2\pi} (\cos t \sin t, 0, 0) \cdot (-\sin t, \cos t, 0) dt = 0$$

  - or close off Σ with another surface Σ' and apply divergence theorem and subtract the flux integral over Σ', assume the new flux integral is easy to handle:
    Σ is the upper hemisphere $x^2 + y^2 + z^2 = 1$ with $z \geq 0$ oriented upward, $\vec{F} = (-y, -z, -x)$. Let D the unit disk $x^2 + y^2 \leq 1$ in the xy-plane oriented downward.

$$\int\int_{\Sigma+D} \vec{F} \cdot d\vec{S} = \int\int\int_{W} div(\vec{F}) dV = 0$$

$$\int\int_{\Sigma} \vec{F}\cdot d\vec{S} = -\int\int_{D} \vec{F}\cdot d\vec{S} = \int\int_{D} (-y, -z, -x)\cdot(0, 0, -1) dS = \int\int_{D} x dS = 0$$

    by symmetry.

Approaches to Evaluating Triple Integrals $\int\int\int_W f dV$

- directly by changing the order of integration or change of variables formula.
- find $\vec{F}$ such that $f = div(\vec{F})$, f and $\vec{F}$ are defined everywhere in W, use divergence theorem to compute a flux integral $\int\int_{\partial W} \vec{F}\cdot d\vec{S}$ oriented outward instead.