(1) (15 pts) Let $\vec{a}, \vec{b} \in \mathbb{R}^2$ be two vectors. Given the number $\vec{a}^\perp \cdot \vec{b}$ where $\vec{a}^\perp = (-a_2, a_1)$, describe geometrically, in as specific terms possible (length, direction, orientation), the cross product $\vec{A} \times \vec{B}$ of the $\mathbb{R}^3$ vectors $\vec{A} = (\vec{a}, 0)$ and $\vec{B} = (\vec{b}, 0)$, along with the vectors $\vec{A}$ and $\vec{B}$. Draw a picture.

$\vec{A}$ and $\vec{B}$ are vectors that live in the $x - y$ plane in three-space. $\vec{A} \times \vec{B}$ is directed out of the plane, purely in the z-direction and is given by the formula $\vec{A} \times \vec{B} = \vec{a}^\perp \cdot \vec{b}\ \hat{k}$.

- The magnitude of $\vec{a}^\perp \cdot \vec{b}$ equals the length of $\vec{A} \times \vec{B}$, and is the area of the parallelogram on the plane made by $\vec{a}, \vec{b}$.
- The sign determines the orientation. Specifically, if $\vec{a}^\perp \cdot \vec{b} > 0$, then $(\vec{A}, \vec{B}, \vec{A} \times \vec{B})$ is a right triple. Since, in this case, $\vec{A} \times \vec{B}$ would be pointing up, it means $(\vec{a}, \vec{b})$ is a right pair on the plane. If $\vec{a}^\perp \cdot \vec{b} < 0$ it is a left triple, etc.

(2) A particle travels through three-dimensional space with acceleration $\vec{a}(t) = (0, \sqrt{2}, 2t)$, At time $t = 0$ the particle is at $\vec{r}(0) = (1, 0, 0)$ with velocity $\vec{v}(0) = (1, 0, 0)$.

(a) (5 pts) Determine the position $\vec{r}(t)$ of the particle at any time t.

For (a), we have

$$\vec{v}(t) = \int \vec{a}(t) dt = (0, \sqrt{2}t, t^2) + C$$

Evaluating at $t = 0$, the initial condition tells that $(1, 0, 0) = (0, 0, 0) + C$, thus $C = (1, 0, 0)$. Thus

$$\vec{v}(t) = (1, \sqrt{2}t, t^2).$$

Integrating again

$$\vec{r}(t) = \int \vec{v}(t) dt = (t, \tfrac{\sqrt{2}}{2}t^2, t^3/3) + C.$$

Initial condition tells $(1, 0, 0) = (0, 0, 0) + C$ so $C = (1, 0, 0)$, again. Thus $\vec{r}(t) = (t + 1, \frac{\sqrt{2}}{2}t^2, t^3/3)$.

(b) (10 pts) At time $t = 0$, the particle trajectory intersects the helix $\vec{H}(t) = (\cos(t), \sin(t), t)$. What is the angle between the particle's trajectory and the helix at this intersection? (HINT: how do the angles between the curves and angles between their tangent vectors relate?)

For (b), The angle between the curves is the angle between their tangent vectors. We have

$$\vec{r}'(t) = \vec{v}(t) = (1, \sqrt{2}t, t^2).$$

Also $\vec{H}'(t) = (-\sin(t), \cos(t), 1)$. At the time of intersection $t = 0$, the tangent vectors are

$$\vec{r}'(0) = (1, 0, 0), \qquad \text{and} \qquad \vec{H}'(0) = (0, 1, 1).$$

The angle between them is $\theta = \arccos\left(\frac{\vec{r}'(0) \cdot \vec{H}'(0)}{|\vec{r}'(0)||\vec{H}'(0)|}\right) = \arccos(0) = \frac{\pi}{2}$.

(c) (5 pts) How long does it take for the particle to travel 12 units of arc-length along its trajectory after passing through $(1, 0, 0)$?

For (c), the arc length parameter relative to $t = 0$ is

$$s(t) = \int_0^t |\vec{v}(\tau)| d\tau = \int_0^t \sqrt{1 + 2\tau^2 + \tau^4} d\tau = \int_0^t \sqrt{(1 + \tau^2)^2} d\tau = \int (1 + \tau^2) d\tau = t + \tfrac{1}{3}t^3.$$

From this we see that $s(t) = 12$ if and only if $t = 3$. It therefore takes 3 time units for the particle to travel 12 units along its trajectory after intersecting the helix.

(3) *True or False.* Answer the following questions concerning systems (i) and (ii) below. Here, x, y, z are the unknowns and $a_i, b_i, c_i, d_i \in \mathbb{R}$. *Explain your answers. If a statement is false, give a counterexample.*

$$\text{(i)}\ \begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \qquad \text{(ii)}\ \begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 + 1 \end{aligned}$$

(a) (10 pts) If (i) has exactly one solution, then the same is true for (ii). **True**. Let

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{bmatrix}, \quad \vec{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}, \quad \vec{d}' = \begin{bmatrix} d_1 \\ d_2 \\ d_3 + 1 \end{bmatrix}$$

If (i) has exactly one solution then $\operatorname{rref}([A \mid \vec{d}])$ has a pivot in in every one of the first three columns. Since $\operatorname{rref}([A \mid \vec{d}])$ and $\operatorname{rref}([A \mid \vec{d}'])$ differ only in the last column, we see that (ii) must have a unique solution. Geometrically, the three row vectors of A define three intersecting planes. A unique solution to (i) means that the first two planes intersect in a line and the third plane intersects that line transversely (i.e. doesn't contain the line). The third plane in (ii) is simply a shift of the third plane in (i) in the direction of $\vec{r}_3$, so the intersection remains a point.

(b) (10 pts) If the solution set of (i) is a line, then the same is true for (ii). **False**. A counterexample is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad \vec{d} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{d}' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where the solution set of (i) is $\operatorname{span}\{[0\ 0\ 1]^t\}$, while (ii) is inconsistent.

(c) (10 pts) If (i) has no solutions, then the same is true for (ii). **False**. A counterexample is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad \vec{d} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{d}' = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

where (i) is inconsistent, while $\operatorname{span}\{[0\ 0\ 1]^t\}$ is the solution set of (ii).

(4) (a) (10 pts) Find the angle between the gradients of the functions

$$u(x, y) = \sqrt{x^2 + y^2}, \qquad v(x, y) = x + y + 2\sqrt{xy},$$

at the point $(1, 1)$. We compute

$$\nabla u(x, y) = \frac{1}{\|\vec{x}\|^{1/2}}(x, y), \qquad \nabla v(x, y) = (1 + \frac{y}{\sqrt{xy}}, 1 + \frac{x}{\sqrt{xy}}).$$

Then $\nabla u(1, 1) = \frac{1}{\sqrt{2}}(1, 1)$, $\nabla v(1, 1) = (2, 2)$. If θ is the angle between the two vectors,

$$\cos\theta = \frac{\nabla u(1, 1) \cdot \nabla v(1, 1)}{\|\nabla u(1, 1)\|\|\nabla v(1, 1)\|} = 1.$$

Thus $\theta = 0$ – the vectors are aligned. (You could just immediately realize and state this).

(b) (10 pts) At the point $(1, 1)$, find the direction of greatest change in the scalar fields u and v, along with the magnitude of that change.

(5) (15 pts) If f and g are functions from $\mathbb{R}^3$ to $\mathbb{R}$, f is differentiable, and $\nabla f(\vec{x}) = g(\vec{x})\vec{x}$. Show spheres centered at the origin are contained in the level set for f, i.e. f is constant on sphere.

Choose a curve $\vec{r}(t)$ on a sphere, it suffices to show that $f(\vec{r}(t))$ is constant. Take derivative,

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) = g(\vec{r}(t))\vec{r}(t) \cdot \vec{r}'(t).$$

Since $\vec{r}(t)$ is on the sphere, $\|\vec{r}(t)\|^2 = R^2$ is constant, and $\frac{d}{dt}\|\vec{r}(t)\|^2 = 2\vec{r}(t) \cdot \vec{r}'(t) = 0$. So $\frac{d}{dt} f(\vec{r}(t)) = 0$, f is constant on $\vec{r}(t)$ and hence on the sphere.