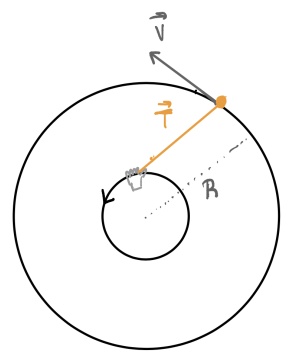
(1) Floating inside an empty room is a perfect triangle (presumably alien technology). There is a force field so you cannot approach it. However, you can shine a light on it and measure its shadow on any of the four walls, or ceiling. How do you the area of the triangle from your measurements.

(2) In ancient times, one of the deadliest weapons was the sling. Indeed, David purportedly slew the Philistine giant Goliath using only a sling. How?

Hint: let's model a sling as follows: the hand of the slinger revolves around a circle, accelerating in such a way so that the sling (drawn in orange) maintains the end of the sling revolves on a larger circle of radius R. The resulting tension force, directed in towards the hand, is denoted $\vec{T}$. Show that the speed evolves according to the tangential acceleration and then relate the tangential acceleration to the centripetal acceleration via the tension. You should end up with an ordinary differential equation on the speed, which can be solved. Its behavior may come as a surprise (or not, if you've ever slung a sling).



(3) Let $d \geq 2$ and $f(x) : \mathbb{R}^d \to \mathbb{R}^d$ be a vector field of unit vectors, e.g. $\|f(x)\| = 1$ for all x.
   (a) For all $x \in \mathbb{R}^2$, then the principle normal vector $N(x)$ and the curvature $\kappa(x)$ of the integral curve passing through x (e.g. a parametrized curve $\gamma(t) : \mathbb{R} \to \mathbb{R}^d$ satisfying $\gamma'(t) = f(\gamma(t))$ and $\gamma(0) = x$), at the point x, are given by
$$\kappa N = f \cdot \nabla f.$$
   (b) If $d = 2$, prove that (up to convention on sign) that the curvature equals the curl of the vector field
$$\kappa = \nabla^{\perp} \cdot f.$$
This problem should shed some light on the geometric meaning of the curl.
   (c) Use part (b) to show that the integral curves of (particle trajectories in) a potential field ∇V of modulus one are straight lines. Give some examples of such a potential.

(4) Suppose that an airplane travels at constant airspeed (speed relative to the air through which it is moving) travels from New York to London, and back (e.g. traverses a closed curve $C \subset \mathbb{R}^3$). Suppose that wind can be described by a time independent, irrotational velocity field $\vec{w}(x)$.
   (a) Show that, if $\vec{v}$, defined along the curve C, denotes the actual velocity of the airplane (relative to the ground), then the time it takes for the airplane to complete its tour of C is
$$T = \oint_C \frac{ds}{\|\vec{v}(C(s))\|},$$
where s is the arclength parameter of C, and $C(s) : [0, L] \to \mathbb{R}^3$ is the corresponding parametrization.
   (b) Prove that the time it takes the airplane to complete its tour of C is always less when there is no wind then when there is wind. E.g. despite the wind speeding the airplane up in some moments during its flight path, the net effect can only slow the airplane down. Hint: Use the Cauchy-Schwarz inequality in the form $\left(\int_C f(s)g(s)ds\right)^2 \leq \int_C |f(s)|^2 ds \int_C |g(s)|^2 ds$ with $f = 1/g = \|\vec{v}(C(s))\|^{1/2}$.

(5) Let $f : \mathbb{R}^n \to \mathbb{R}$ having a single critical point and be such that the level sets $S_c := \{x \in \mathbb{R}^n \mid f(x) = c\}$ are compact surfaces without boundary and foliate $\mathbb{R}^n$ as c varies. The normal to these surfaces is defined everywhere and is given by the formula $\hat{n} = \nabla f / |\nabla f|$. The mean curvature is defined by $H := \operatorname{div} \hat{n}$. Prove that, for any c in the range of f, we have

$$\int_{S_c} H\hat{n}\; dA = 0.$$

(6) Let A be an annulus and $\vec{v}$ a smooth vector field on A which is tangent to the boundaries ∂A_{top} and ∂A_{bot}. Suppose $\operatorname{div} \vec{v} = 0$ and that $\vec{v} \cdot \hat{\tau}|_{\partial A_{\text{top}}} > 0$ and $\vec{v} \cdot \hat{\tau}|_{\partial A_{\text{bot}}} < 0$, where $\hat{\tau}$ is the tangent pointing clockwise around the boundaries.
 (a) Prove $\vec{v}$ vanishes at at least two points of A.
 (b) Give an example of a $\vec{v}$ on A having exactly two critical points.
 (c) Drop the condition of $\operatorname{div} \vec{v} = 0$ and give an example of a $\vec{v}$ on A having exactly one critical point.

(7) Fix $x_0 \in \mathbb{R}^n$ and a twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$. Let $B_\varepsilon(x_0)$ be a ball of radius ε about x_0. Study the limit

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \fint_{B_\varepsilon(x_0)} \Big(f(x) - f(x_0)\Big) \mathrm{d}x,$$

where $\fint_A f \mathrm{d}x = \frac{1}{\mathrm{Vol}(A)} \int_A f(x) \mathrm{d}x$.

(8) Fix a scalar function $f : \mathbb{R}^d \to \mathbb{R}$, and points $a, b \in \mathbb{R}^d$. Consider the functional

$$A[\gamma] := \int_\gamma f \mathrm{d}\ell$$

where, for an interval $I \subset \mathbb{R}$, $\gamma : I \to \mathbb{R}^d$ is any path satisfying connecting a and b. Prove that extremal curves satisfy

$$\mathbf{Q}_\tau \Big(\nabla \ln f(\gamma) - \frac{\ddot{\gamma}}{\|\dot{\gamma}\|^2}\Big) = 0.$$

where τ is the unit tangent vector to the extremal curve, and $\mathbf{Q}_\tau = I - \tau \otimes \tau$ is the orthogonal projector onto the normal space to the curve. That is, if γ is assumed to be parametrized by arclength, we have

$$\ddot{\gamma} = \mathbf{Q}_{\dot{\gamma}} \nabla \ln f(\gamma).$$