(1) Consider the vector field $F(x,y) = (2x/y, (1-x^2)/y^2)$ for $y > 0$.

(a) Check that F is potential (conservative) and find a potential function

Note $\partial_y\left(\frac{2x}{y}\right) = -\frac{2x}{y^2}$, $\partial_x\left(\frac{1-x^2}{y^2}\right) = -\frac{2x}{y^2}$. Hence $\vec{F}$ is conservative. We must now find a scalar function f which is a solution of

$$\partial_x f = \frac{2x}{y}, \qquad \partial_y f = \frac{1-x^2}{y^2}.$$

From the first equation we find $f(x,y) = \frac{x^2}{y} + g(y)$ for some yet unknown free function g. Plugging into the second equation we find

$$-\frac{x^2}{y^2} + g'(y) = \frac{1-x^2}{y^2} = \frac{1}{y^2} - \frac{x^2}{y^2} \implies g'(y) = \frac{1}{y^2}.$$

Thus, $g(y) = -\frac{1}{y} + c$ where c is an arbitrary constant. Therefore, our potential is

$$f(x,y) = \frac{x^2-1}{y} + c.$$

(b) Let C be the curve $(x-3)^5 + y^2 = 3$, from $(2,-2)$ to $(2,2)$. Compute $\int_C \vec{F}\cdot d\vec{r}$.

Since F is conservative we have

$$\int_C \vec{F}\cdot d\vec{r} = f(2,2) - f(2,-2) = 3.$$

(2) Evaluate $\int_C F\cdot dr$ where

$$F(x,y,z) = \left(\frac{1}{y^2+1},\ -\frac{2xy}{(y^2+1)^2} + ze^{yz},\ ye^{yz} + 2z\right)$$

where C is part of the helix $r(t) = \langle\cos(t), \sin(t), t\rangle$ from $(1,0,0)$ to $(1,0,2\pi)$.

We seek a potential for this vector field. Indeed it is

$$f(x,y,z) = \frac{x}{y^2+1} + e^{yz} + z^2$$

Thus

$$\int_C \vec{F}\cdot d\vec{r} = f(1,0,2\pi) - f(1,0,0) = 2 + 4\pi^2 - 2 = 4\pi^2.$$

(3) Find the integral of $f(x,y) = x^2 + y^2$ on the domain

$$D := \{(x,y)\in\mathbb{R}^2 \ :\ 0 \le x \le 1,\ x^2 \le y \le x\}.$$

Sketch the region of integration.

D can be viewed as a type 1 domain with lower boundary $y = x^2$ and upper boundary $y = x$. Then

$$I = \iint_D f(x,y)dxdy = \int_0^1\int_{x^2}^x (x^2+y^2)dydx = \int_0^1 (x^2(x-x^2) + \frac{1}{3}(x^3 - x^6))dx = \frac{3}{35}.$$

(4) Find the limits of integration of $\iint_D f(x,y)dxdy$ if

$$D := \{(x,y) \in \mathbb{R}^2 \ : \ \frac{x^2}{9} + \frac{y^2}{4} \leq 1\}$$

when D is considered first as a type I and then as a type II domain.

As a type 1 domain, the lower boundary is $y = -2\sqrt{1-\frac{x^2}{9}}$ and top boundary is $y = 2\sqrt{1-\frac{x^2}{9}}$. As a type 2 domain, the left boundary is $x = -3\sqrt{1-\frac{y^2}{4}}$ and right boundary is $x = 3\sqrt{1-\frac{y^2}{4}}$.

(5) Sketch the region bounded by the curves $y = \log(x)$, $y = 2\log(x)$ and $x = e$ in the first quadrant. Then express the region's area as an iterated double integral and evaluate.

To compute the area of this region we will write the integral using vertical cross-sections. The projection of the region D to the x-axis is the segment $(1, e)$ (because we are only interested in the region contained in the first quadrant). The vertical segment above x is the vertical segment connecting $(x, \log(x))$ and $(x, 2\log(x))$. The area of the region is

$$\int_1^2 \int_{\log(x)}^{2\log(x)} dydx = \int_1^e \log(x)dx.$$

A primitive for $\log(x) = x\log(x) - x$. Finally, the area enclosed is: $\int_1^2 \int_{\log(x)}^{2\log(x)} dydx = 1$.

(6) Consider $\iint_D f dA = \int_0^3 \int_{-2\sqrt{1-(x/3)^2}}^{2(1-x/3)} f(x,y)dydx$.

(a) Sketch the region of integration.

The limits in x are $0 \leq x \leq 3$. The limits in y are $-2\sqrt{1-(x/3)^2} \leq y \leq 2(1-x/3)$, so, a part of the ellipse $x^2/3^2 + y^2/2^2 = 1$. The upper region is a triangle.

(b) Switch the order of integration in the above integral.

If we are to integrate first in x an then in y, we must split the integral at $y = 0$. In the interval $-2 \leq y \leq 0$, the lower limit in x is 0 and the upper limit comes from $x^2/3^2 + y^2/2^2 = 1$, i.e. $x \leq 3\sqrt{1-(y/2)^2}$. On the other hand, in the interval $0 \leq y \leq 2$, the lower limit in x is again 0 but the upper limit comes from $y = 2(1-x/3)$, i.e. $x = c$. We conclude

$$\iint_D f dA = \int_{-2}^0 \int_0^{3\sqrt{1-(y/2)^2}} f(x,y)dxdy + \int_0^2 \int_0^{3-3y/2} f(x,y)dxdy.$$

(c) Compute the integral $\iint_D f dA$ if $f(x,y) = xy$.

We compute

$$\begin{aligned}\iint_D f dA &= \int_0^3 \int_{-2\sqrt{1-(x/3)^2}}^{2(1-x/3)} f(x,y)dydx \\ &= \frac{1}{2}\int_0^3 4x\left[(1-x/3)^2 - (1-x^2/3^2)\right] = \frac{4}{3^2}\int_0^3 (x^3 - 3x^2)dx = -3.\end{aligned}$$

We can also compute

$$\iint_D f dA = \int_{-2}^0 \int_0^{3\sqrt{1-(y/2)^2}} xydxdy + \int_0^2 \int_0^{3-3y/2} xydxdy = -3.$$

(7) Suppose f is a continuous invertible function from $[0,1]$ to $[0,1]$ such that $f(0)=0, f(1)=1$ and $\int_0^1 f(x)dx = 1/6$. Let g denote the inverse function. Evaluate $\int_0^1 g(x)dx$.

Solution: 5/6. Consider the double integral $\int_0^1 \int_0^{f(x)} dy\, dx$ and change the order of integration.