(1) Let $\vec{F}(x,y,z) = (\cos x \sin y, \sin x \cos y, 1)$, and C is the line segment from $(1,0,0)$ to $(0,0,3)$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.

$$\vec{r}(t) = (1-t, 0, 3t), \qquad t \in [0,1]$$

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_0^1 (\cos(1-t)\sin 0, \sin(1-t)\cos 0, 1) \cdot (-1,0,3)dt = \int_0^1 3dt = 3$$

(2) If $\vec{r}(t) = (a\cos t, a\sin t)$, $t \in [0, 2\pi]$, and $\vec{F}(x,y) = (-y, x)$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (-a\sin t, a\cos t) \cdot (-a\sin t, a\cos t)dt = 2\pi a^2.$$

(3) If $\vec{r}(t) = (\cos \pi t, \sin \pi t)$, $t \in [0,2]$, and $\vec{F}(x,y) = (x,y)$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^2 (\cos \pi t, \sin \pi t) \cdot (-\pi \sin \pi t, \pi \cos \pi t)dt = 0.$$

(4) Compute $\int_C x^2 dx - xydy + dz$ where C is the parabola $z = x^2$, $y = 0$ from $(-1,0,1)$ to $(1,0,1)$.

We parametrize the curve as follows $(x,y,z) = (t, 0, t^2)$ for $-1 \leq t \leq 1$. Then

$$\int_C x^2 dx - xydy + dz = \int_{-1}^1 (t^2 + 2t)dt = (\frac{t^3}{3} + t^2)\Big|_{-1}^1 = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}.$$

(5) (M&T, # 7.2.6) Let $\vec{r}(t)$ be a parametrization of a curve C.

- Suppose that $\vec{F}$ is perpendicular to $\vec{r}'(t)$ at the point $\vec{r}(t)$. Show $\int_C \vec{F} \cdot d\vec{r} = 0$.

  Being perpendicular $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = 0$ for each t. Then

$$\oint_C \vec{F} \cdot d\vec{r} = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)dt = 0.$$

- Suppose that $\vec{F}$ is parallel to $\vec{r}'(t)$ at the point $\vec{r}(t)$ (e.g. $\vec{F}(\vec{r}(t)) = \lambda(t)\vec{r}'(t)$ for some $\lambda(t) > 0$). Show $\int_C \vec{F} \cdot d\vec{r} = \int_C \|\vec{F}\| ds$.

  Noting that $\|\vec{F}(\vec{r}(t))\| = \lambda(t)\|\vec{r}'(t)\|$ since $\lambda(t) > 0$, we have

$$\oint_C \vec{F} \cdot d\vec{r} = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)dt = \int \lambda(t)\|\vec{r}'(t)\|^2 dt = \int \|\vec{F}(\vec{r}(t))\|\|\vec{r}'(t)\|dt = \int_C \|\vec{F}\|ds.$$

- Suppose L is the length of C and $\|\vec{F}\| \leq M$. Prove $\left|\int_C \vec{F} \cdot d\vec{r}\right| \leq ML$.

  From the above we have

$$\left|\oint_C \vec{F} \cdot d\vec{r}\right| = \left|\int \|\vec{F}(\vec{r}(t))\|\|\vec{r}'(t)\|dt\right| \leq M \int \|\vec{r}'(t)\|dt = ML.$$

(6) Let

$$\vec{F}(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right),$$

and C be a parametrized curve defined, for $t \in [0,1]$, by

$$\vec{r}(t) = \Big(a\cos(2k\pi t), b\sin(2k\pi t)\Big),$$

where k is a positive integer and $0 < b \le a$.

- Note that in polar coordinates, $\tan\theta = \frac{y}{x}$, so $\theta = \arctan\frac{y}{x}$. Show that

  $$\frac{d\theta}{dt} = \vec{F}\cdot(x'(t), y'(t)),$$

  and hence conclude that

  $$\int_0^1 \frac{d\theta}{dt}dt = \oint_C \vec{F}\cdot d\vec{r} = 2k\pi.$$

  In fact, for any smooth closed curve C in $\mathbb{R}^2$ that intersects itself finitely many times and does not pass through the origin, the line integral $\frac{1}{2\pi}\int_C \frac{-ydx+xdy}{x^2+y^2}$ is always an integer (known as the winding number of C around the origin).

  $\nabla \arctan\frac{y}{x} = \vec{F}$. By Chain Rule, $\frac{d\theta}{dt} = \nabla\arctan\frac{y}{x}\cdot(x'(t), y'(t))$.

  $$\int_0^1 \frac{d\theta}{dt}dt = \theta(1) - \theta(0)$$

  by Fundamental Theorem of Calculus, which is $2k\pi$.

- Using the above, evaluate the integral

  $$\int_0^1 \frac{dt}{a^2\cos^2(2k\pi t) + b^2\sin^2(2k\pi t)}.$$

  Hint: In computing $\oint_C \vec{F}\cdot d\vec{r}$, write $x'(t)$ in terms of $y(t)$, and $y'(t)$ in terms of $x(t)$.

  $$\oint_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}\cdot(x'(t), y'(t))dt = \theta(1) - \theta(0) = 2k\pi.$$

  $$2k\pi = \oint_C \vec{F}\cdot d\vec{r} = \int_0^1 \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)\cdot\left(-\frac{2k\pi a}{b}y, \frac{2k\pi b}{a}x\right)dt,$$

  $$= 2k\pi\int_0^1 \frac{\frac{a}{b}y^2 + \frac{b}{a}x^2}{x^2+y^2}dt = 2k\pi\int_0^1 ab\frac{\cos^2(2k\pi t)+\sin^2(2k\pi t)}{a^2\cos^2(2k\pi t) + b^2\sin^2(2k\pi t)}dt.$$

  Hence

  $$\int_0^1 \frac{dt}{a^2\cos^2(2k\pi t) + b^2\sin^2(2k\pi t)} = \frac{1}{ab}.$$