(1) Let $\vec{F}(x,y,z) = (\cos x \sin y, \sin x \cos y, 1)$, and C is the line segment from $(1,0,0)$ to $(0,0,3)$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.

(2) If $\vec{r}(t) = (a\cos t, a\sin t)$, $t \in [0, 2\pi]$, and $\vec{F}(x,y) = (-y, x)$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.

(3) If $\vec{r}(t) = (\cos \pi t, \sin \pi t)$, $t \in [0,2]$, and $\vec{F}(x,y) = (x,y)$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.

(4) Compute $\int_C x^2 dx - xydy + dz$ where C is the parabola $z = x^2$, $y = 0$ from $(-1,0,1)$ to $(1,0,1)$.

(5) Let $\vec{r}(t)$ be a parametrization of a curve C.
- Suppose that $\vec{F}$ is perpendicular to $\vec{r}'(t)$ at the point $\vec{r}(t)$. Show $\int_C \vec{F} \cdot d\vec{r} = 0$.
- Suppose that $\vec{F}$ is parallel to $\vec{r}'(t)$ at the point $\vec{r}(t)$ (e.g. $\vec{F}(\vec{r}(t)) = \lambda(t)\vec{r}'(t)$ for some $\lambda(t) > 0$). Show $\int_C \vec{F} \cdot d\vec{r} = \int_C \|\vec{F}\| ds$.
- Suppose L is the length of C and $\|\vec{F}\| \le M$. Prove $\left|\int_C \vec{F} \cdot d\vec{r}\right| \le ML$.

(6) [1] Let
$$\vec{F}(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right),$$
and C be a parametrized curve defined, for $t \in [0,1]$, by
$$\vec{r}(t) = \Big(a\cos(2k\pi t), b\sin(2k\pi t)\Big),$$
where k is a positive integer and $0 < b \le a$.
- Note that in polar coordinates, $\tan\theta = \frac{y}{x}$, so $\theta = \arctan\frac{y}{x}$. Show that
$$\frac{d\theta}{dt} = \vec{F} \cdot (x'(t), y'(t)),$$
and hence conclude that
$$\int_0^1 \frac{d\theta}{dt} dt = \oint_C \vec{F} \cdot d\vec{r} = 2k\pi.$$
In fact, for any smooth closed curve C in $\mathbb{R}^2$ that intersects itself finitely many times and does not pass through the origin, the line integral $\frac{1}{2\pi}\int_C \frac{-ydx+xdy}{x^2+y^2}$ is always an integer (known as the winding number of C around the origin).
- Using the above, evaluate the integral
$$\int_0^1 \frac{dt}{a^2\cos^2(2k\pi t) + b^2\sin^2(2k\pi t)}.$$
Hint: In computing $\oint_C \vec{F} \cdot d\vec{r}$, write $x'(t)$ in terms of $y(t)$, and $y'(t)$ in terms of $x(t)$.

[1] Not to appear on the quiz.