(1) Find the point on $x^2 - z^2 = 1$ closest to the origin.

Solution: $x^2 - z^2 = 1$ is a hyperbolic cylinder. Minimize $f(x, y, z) = x^2 + y^2 + z^2 = x^2 + y^2 + x^2 - 1$, $\nabla f(x, y) = (0, 0)$ implies $x = y = 0$ and $z^2 = -1$, not possible. Method fails in this case!
But use Lagrange multiplier,

$$f(x, y, z) = x^2 + y^2 + z^2, \qquad g(x, y, z) = x^2 - z^2 = 1$$

$$\nabla f = \lambda \nabla g, \qquad y = 0, \qquad x = \lambda x, \qquad z = -\lambda z$$

$$\lambda = 1, \qquad z = 0, \qquad x = \pm 1$$

$(\pm 1, 0, 0)$ are the solutions. Geometric considerations show that they are the minimum solutions for f. In fact, let $f(x, y, z) = z^2 + 1 + y^2 + z^2$, the unconstrained optimization method will work.

(2) Prove that the arithmetic mean is always greater than or equal to the geometric mean:

$$(x_1 x_2 \cdots x_n)^{1/n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$$

where $x_i \geq 0$. *Hint: Maximize* $f(y_1, \cdots, y_n) = y_1 \cdots y_n$ *subject to* $g(y_1, \cdots, y_n) = y_1 + \cdots + y_n = 1$ *where* $y_i \geq 0$.

Solution: Maximize $f(y_1, \cdots, y_n) = y_1 \cdots y_n$ subject to $g(y_1, \cdots, y_n) = y_1 + \cdots + y_n = 1$ where $y_i \geq 0$. Let $p = y_1 \cdots y_n$.

$$\nabla f(\vec{y}) = \lambda \nabla g(\vec{y}), \qquad g(\vec{y}) = 1$$

We find that

$$\frac{p}{y_i} = \lambda \qquad \forall i, \qquad y_i = \frac{1}{n}.$$

$p \leq (1/n)^n$. Thus

$$p = y_1 \cdots y_n \leq \left(\frac{1}{n}\right)^n$$

$$(y_1 \cdots y_n)^{1/n} \leq \frac{1}{n} = \frac{y_1 + \cdots + y_n}{n}$$

Note that $\left(\frac{1}{n}\right)^n$ is max for f, as the domain is compact, and f must attain its min and max. On the boundary of the domain, $f(\vec{y}) = 0$ are the minimums. At the "center", it is maximum. Now set

$$y_i = \frac{x_i}{\sum_{i=1}^n x_i} = \frac{x_i}{s}, \qquad s = \sum_{i=1}^n x_i$$

$$(y_1 \cdots y_n)^{1/n} = \left(\frac{x_1 \cdots x_n}{s^n}\right)^{1/n} \leq \frac{y_1 + \cdots + y_n}{n} = \frac{x_1 + \cdots x_n}{sn}$$

$$(x_1 x_2 \cdots x_n)^{1/n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$$

(3) A light ray travels from point A to point B crossing a boundary between two media (say the interface is along $\{y = 0\}$). In the first medium its speed is v_1, and in the second it is v_2. Show that the trip is made in minimum time when Snell's law holds:

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$$

Solution: Single-Variable Unconstrained Optimization Method: Assume points A and B are distance a, b from the medium divider, and A, B are 1 unit apart horizontally.

$$\sin\theta_1 = \frac{c}{\sqrt{a^2+c^2}}, \qquad \sin\theta_2 = \frac{1-c}{\sqrt{b^2+(1-c)^2}}$$

The time it takes for light to travel is

$$f(c) = \frac{\sqrt{a^2+c^2}}{v_1} + \frac{\sqrt{b^2+(1-c)^2}}{v_2}$$

$$f'(c) = \frac{c}{v_1\sqrt{a^2+c^2}} - \frac{1-c}{v_2\sqrt{b^2+(1-c)^2}} = 0$$

implies

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

To verify that Snell's law gives the global minimum, check that

$$f''(c) = \frac{a^2}{v_1(a^2+c^2)^{3/2}} + \frac{b^2}{v_2(b^2+(1-c)^2)^{3/2}} > 0.$$

$f(c)$ is a function that concaves up and has only one critical point, so minimum can not occur at boundary, c is a local min and global min for f.

(4) Find the shortest distances between the points of the line $x + y = 8$ and the ellipse $x^2 + 2y^2 = 6$. *Hint: Pick a point (u, v) on the line and a point (x, y) on the ellipse, and minimize the distance between the two points.*

Solution: Let $f(x, y, u, v) = (x-u)^2 + (y-v)^2$, minimize f subject to the constraints $g_1(x, y) = x^2 + 2y^2 = 6$ and $g_2(u, v) = u + v = 8$. Lagrange multiplier method shows that

$$2(x-u) = \lambda_1 2x, \qquad 2(y-v) = \lambda_1 4y, \qquad -2(x-u) = \lambda_2, \qquad -2(y-v) = \lambda_2.$$

We have $x - u = y - v$ and hence $x = 2y$. g_1 implies that $y = \pm 1$.

When $y = 1$, we have $x = 2$, and $u - v = 1$, so g_2 implies $u = 4.5$ and $v = 3.5$. The distance between (x, y) and (u, v) is $\frac{5}{2}\sqrt{2}$.

When $y = -1$, $x = -2$, $u - v = -1$, so $u = 3.5$ and $v = 4.5$. The distance between the two points is $\frac{11}{2}\sqrt{2}$.

The global min is $\frac{5}{2}\sqrt{2}$ since the distance between the line and ellipse goes to infinity at extreme ends of the line.

Alternatively, use the fact that the shortest distance must be realized by a segment normal to the line $x + y = 8$. So need to find points on the ellipse where the gradient $(2x, 4y)$ is parallel to the normal given by $(1, 1)$, so $x = 2y$, and $y = \pm 1$.

(5) A die shows k with probability p_k for $k = 1, 2, \cdots, 6$. Consider the vector $\vec{p} = (p_1, p_2, p_3, p_4, p_5, p_6)$, where $\sum_{i=1}^{6} p_i = 1$. The **entropy** of the die is defined as

$$f(\vec{p}) = -\sum_{i=1}^{6} p_i \log p_i$$

Find the distributions $\vec{p}$ that minimizes and maximizes the entropy.

Solution:

$$g(\vec{p}) = \sum_{i=1}^{6} p_i = 1$$

$$\nabla f = (-1 - \log p_1, \cdots, -1 - \log p_6), \qquad \nabla g = (1, 1, \cdots 1)$$

$$\nabla f = \lambda \nabla g$$

$$p_i = e^{-(\lambda+1)}, \qquad 1 = \sum_{i=1}^{6} exp(-(\lambda+1))$$

$$\lambda = -\log(1/6) - 1, \qquad p_i = 1/6$$

The fair dice has maximal entropy, and least information content.

$$f(1/6, 1/6, \cdots, 1/6) = -6\frac{1}{6}\log\frac{1}{6} = \log 6 > 0$$

Note again that $p_i \in [0,1]$, and f attains global min/max on the closed and bounded region.

$f(\vec{p}) \geq 0$ as $\log p_i \leq 0$ for all i. f attains global min when $p_i = 1$ for some i and $p_j = 0$ for $j \neq i$. Use L'Hôpital's Rule,

$$\lim_{h\to 0} h\log h = \lim_{h\to 0}\frac{\log h}{1/h} = \lim_{h\to 0}\frac{1/h}{-1/h^2} = \lim_{h\to 0} h = 0$$

On the boundary, where $1 \leq a < 6$ of the p_i's sum to 1, while the other $6-a$ p_j's are 0. WLOG, assume

$$g_1(p_1, \cdots, p_6) = p_1 + \cdots + p_a = 1, \qquad g_2(p_1, \cdots, p_6) = p_{a+1} + \cdots + p_6 = 0$$

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 = \lambda_1(1, \cdots, 1, 0, \cdots, 0) + \lambda_2(0, \cdots, 0, 1, \cdots, 1)$$

$$\nabla f = (\lambda_1, \cdots, \lambda_1, \lambda_2, \cdots, \lambda_2)$$

where there are a λ_1's and $6-a$ λ_2's.

$$p_i = e^{-(\lambda_i+1)}, \qquad 1 = \sum_{i=1}^{a} exp(-(\lambda_1+1)) = a exp(-(\lambda_1+1))$$

$$\lambda_1 = \log a - 1, \qquad p_1 = \cdots = p_a = \frac{1}{a}$$

$p_{a+1}, \cdots, p_6$ must all be 0, as the $p_i \geq 0$ and $p_{a+1} + \cdots + p_6 = 0$. We again find the Lagrange multiplier solution to be

$$f(\frac{1}{a}, \cdots, \frac{1}{a}, 0, \cdots 0) = \log a$$

Hence, $\log 6$ is a global max.

Indeed, when each $p_i = \frac{1}{6}$, the dice is most random; when one $p_i = 1$ and the rest 0, there is no randomness at all.

(**Extra**) [1] *Marsden & Tromba*: §3.2 #6, 9; §3.3 #12, 23, 29, 42; §3.4 #4, 12, 30.

[1]Not to appear on quiz.