(1) Set $\vec{r}(x,y,z) = (x,y,z)$, and $r = \sqrt{x^2+y^2+z^2} = \|\vec{r}\|$.
 (a) Show that $\nabla \cdot (r^n \vec{r}) = (n+3)r^n$. In particular, $\nabla \cdot (\vec{r}/r^3) = 0$.

$$\frac{\partial}{\partial x} x r^n = \frac{\partial}{\partial x} x(x^2+y^2+z^2)^{n/2} = r^n + \frac{n}{2} x \cdot (2x) \cdot (x^2+y^2+z^2)^{n/2-1} = r^n + nx^2 r^{n-2}$$

$$\nabla \cdot (r^n \vec{r}) = \frac{\partial}{\partial x} x r^n + \frac{\partial}{\partial y} y r^n + \frac{\partial}{\partial z} z r^n = 3r^n + nr^n$$

When $n = -3$, $\nabla \cdot (r^n \vec{r}) = (-3+3)r^n = 0$.

 (b) Show that $\nabla \times (r^n \vec{r}) = \vec{0}$.

$$\frac{\partial}{\partial z} y r^n - \frac{\partial}{\partial y} z r^n = yzr^{n-2} - zyr^{n-2} = 0$$

The other terms $\frac{\partial}{\partial z} x r^n - \frac{\partial}{\partial x} z r^n$, and $\frac{\partial}{\partial x} y r^n - \frac{\partial}{\partial y} x r^n$ are similarly 0.

(2) The partial differential equation for a smooth function $f(x,t)$

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$$

models the displacement of a 1-dimensional vibrating string from its equilibrium position with wave velocity c. Set $u = x + ct$ and $v = x - ct$ so that $(x,t) = \left(\frac{u+v}{2}, \frac{u-v}{2c}\right)$. Define

$$F(u,v) = f(x,t) = f\left(\frac{u+v}{2}, \frac{u-v}{2c}\right).$$

Show that

$$\frac{\partial^2 F}{\partial u \partial v} = 0.$$

Deduce that $F(u,v) = g(u) + h(v)$ for some differentiable functions g and h over any rectangle in the uv-plane.

$$F_v = f_x x_v + f_t t_v = f_x \frac{1}{2} - f_t \frac{1}{2c}$$

$$F_{vu} = \frac{1}{2}(f_{xx} x_u + f_{xt} t_u) - \frac{1}{2c}(f_{tx} x_u + f_{tt} t_u)$$

$$= \frac{1}{2}\left(f_{xx}\frac{1}{2} + f_{xt}\frac{1}{2c}\right) - \frac{1}{2c}\left(f_{tx}\frac{1}{2} + f_{tt}\frac{1}{2c}\right) = \frac{1}{4}(f_{xx} - \frac{1}{c^2} f_{tt}) = 0$$

$F_{vu} = 0$ implies that F_v is independent of u, hence it is a function of v alone. Let $\int F_v dv = h(v)$, and the constant of integration can be an arbitrary function of u. $F_{uv} = F_{vu}$. $F_{uv} = 0$ implies that $\int F_u du = g(u)$ with a function of v as its integration constant.

$$F(u,v) = g(u) + h(v)$$

Therefore, over a suitable region, $F(u,v) = f(x,t) = g(x+ct) + h(x-ct)$ for some smooth functions g and h. The general solution to the wave equation is the superposition of two traveling waves, one moving to the right and one to the left, both with speed c.

(3) Classify the critical points of $f(x,y) = \sin(xy)$.

$$\nabla f(x,y) = \langle y\cos(xy), x\cos(xy)\rangle$$

Critical points occur at $(0,0)$ or on curves where $xy = \frac{\pi}{2} + k\pi$, where k is any integer.

$$H(f)(x,y) = \begin{bmatrix} -y^2\sin(xy) & \cos(xy) - xy\sin(xy) \\ \cos(xy) - xy\sin(xy) & -x^2\sin(xy) \end{bmatrix}$$

$$H(f)(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad D(f)(0,0) = -1 < 0$$

So $(0,0)$ is a saddle. Along curves where $xy = \frac{\pi}{2} + 2k\pi$,

$$H(f)(x,y) = \begin{bmatrix} -y^2 & -xy \\ -xy & -x^2 \end{bmatrix}, \qquad D(f) = 0$$

Along curves where $xy = \frac{\pi}{2} + (2k+1)\pi$,

$$H(f)(x,y) = \begin{bmatrix} y^2 & xy \\ xy & x^2 \end{bmatrix}, \qquad D(f) = 0$$

Note that $\sin(t)$ is a local max when $t = \frac{\pi}{2} + 2k\pi$, so in the vicinity of points (x,y) where $xy = \frac{\pi}{2} + 2k\pi$, $\sin(xy)$ is a local max. Similarly, $\sin(xy)$ is a local min when $xy = \frac{\pi}{2} + (2k+1)\pi$.

(4) Calculate $1.98^{2.01}$. This of this value as $z = x^y$ with $x = 1.98$ and $z = 2.01$. Is the value of z more sensitive to small changes in x or in y?

$$f(x,y) = x^y = e^{y \ln x}, \qquad f_x(x,y) = x^y \frac{y}{x}, \qquad f_y(x,y) = x^y(\ln x)$$

$$df = f_x dx + f_y dy$$

$$x_0 = y_0 = 2, \qquad f_x(2,2) = 4, \qquad f_y(2,2) = 4\ln 2 = 2.8$$

$$df = f(1.98, 2.01) - f(2,2) \simeq 4 * (-0.02) + 2.8 * 0.01 = -0.08 + 0.028 = -0.052$$

$$L(x,y) = f(2,2) + f_x(2,2)(x-2) + f_y(2,2)(y-2) = f(2,2) + df = 3.948$$

$$1.98^{2.01} = 3.9473 \simeq 3.948$$

$$f(2 + \Delta x, 2) - f(2,2) \simeq 4\Delta x$$

$$f(2, 2 + \Delta y) - f(2,2) \simeq 4(\ln 2)\Delta y = 2.8\Delta y$$

So z is more sensitive to change in x than in y.

(5) Find the quadratic Taylor approximation for $f(x,y) = \sin(x^2 + y) + y$ near $P = (0, \pi)$.

$$f_x = 2x\cos(x^2+y), \qquad f_x(P) = 0, \qquad f_y = \cos(x^2+y) + 1, \qquad f_y(P) = 0$$

$$f_{xx} = 2\cos(x^2+y) - 4x^2\sin(x^2+y), \qquad f_{xy} = f_{yx} = -2x\sin(x^2+y), \qquad f_{yy} = -\sin(x^2+y)$$

$$f_{xx}(P) = -2, \qquad f_{xy}(P) = f_{yx}(P) = 0, \qquad f_{yy}(P) = 0$$

$$Q(x,y) = f(0,\pi) - x^2 = \pi - x^2$$

(6) Consider the function $f(x,y) = \ln(\sqrt{x^2+y^2} + y)$.

- Determine the domain of f and sketch it in the xy-plane.

  We can only take the logarithm of nonnegative numbers, so

$$\sqrt{x^2+y^2} + y > 0 \implies \sqrt{x^2+y^2} > -y.$$

  This is satisfied automatically if $y > 0$. If $y \le 0$, both sides of the inequality are positive and we may safely square to get

$$x^2 + y^2 > y^2 \qquad \implies \qquad x^2 > 0 \qquad \implies \qquad x \neq 0.$$

  Therefore the domain excludes all points along the negative y-axis (the origin is excluded as well). Note that there are no domain restrictions from the square root term $\sqrt{x^2+y^2}$ since $x^2 + y^2$ is never negative.

- What is the linearization of f at $(3,-4)$?

  We know that the linearization is

$$L(x,y) = f_x(3,-4)(x-3) + f_y(3,-4)(y+4) + f(3,-4).$$

  First note $f(3,-4) = \ln(1) = 0$. Next, we compute

$$f_x(x,y) = \frac{1}{\sqrt{x^2+y^2}+y} \cdot \left(\frac{1}{2}\frac{1}{\sqrt{x^2+y^2}} \cdot 2x\right) = \frac{x}{(\sqrt{x^2+y^2}+y)\sqrt{x^2+y^2}},$$

$$f_y(x,y) = \frac{1}{\sqrt{x^2+y^2}+y} \cdot \left(\frac{1}{2}\frac{1}{\sqrt{x^2+y^2}} \cdot 2x + 1\right) = \frac{y}{(\sqrt{x^2+y^2}+y)\sqrt{x^2+y^2}} + 1.$$

  Therefore

$$f_x(3,-4) = \frac{3}{(5-4)\cdot 5} = \frac{3}{5},$$

$$f_y(3,-4) = \frac{-4}{(5-4)\cdot 5} + 1 = \frac{1}{5}$$

  and hence

$$L(x,y) = \frac{3}{5}(x-3) + \frac{1}{5}(y+4).$$