(1) (a) Consider the Archimedean spiral $\vec{r}(t) := (x(t), y(t)) = (t\cos(t), t\sin(t))$ and the scalar function given by $f(x, y) = x + 2y$. Find the tangent to the curve. Compute the gradient of the function. Find the result, $g(t) = f(x(t), y(t))$, of evaluating the function f along the curve. Take the derivative of g and verify that it is indeed the dot product of the tangent and the gradient evaluated on the curve.

Solution: This is direct computation. The tangent to the curve is $\tau(t) = (\dot{x}(t), \dot{y}(t)) = (\cos t - t\sin t, \sin t + t\cos t)$. The gradient is given by $(\partial f/\partial x, \partial f/\partial y) = (1, 2)$. The results of evaluating the function on the curve is $g(t) = t(\cos t + 2\sin t)$ with derivative $\dot{g}(t) = (\cos t + 2\sin t) + t(2\cos t - \sin t)$. But $\tau \cdot \nabla f = 1(\cos t - t\sin t) + 2(\sin t + t\cos t) = \dot{g}(t)$, as claimed.

(b) Find an example of a non-constant function that is constant on the Archimedean spiral.

Solution: One can take

$$g(x, y) = \sqrt{x^2 + y^2}\sin\left(\sqrt{x^2 + y^2}\right) - y.$$

Substituting in $(x(t), y(t)) = (t\cos(t), t\sin(t))$ shows $g(x(t), y(t)) = 0$.

(c) Find an example of a curve on which the function $f(x, y)$ is constant.

Solution: Consider the curve $(x(t), y(t)) = (2t, -t)$.

(2) Describe and sketch the behavior, as c varies, of the level curves $f(x, y) = c$ for each of these functions

(a) $f(x, y) = x^2 + y^2 + 1$,
(b) $f(x, y) = 1 - x^2 - y^2$,
(c) $f(x, y) = x^3 - x$,
(d) $f(x, y) = x^2 - y^2$.
(e) $f(x, y) = \max\{|x|, |y|\}$.

Solution: (a) The first function has levels which are circles, for any $c > 1$. For $c = 1$ it is a point. For $c < 0$, it is empty.
(b) Levels are likewise circles, but now for $c < 1$. For $c = 1$ it is a point and for $c > 1$ it is empty.
(c) Level sets are sheets for any real c, function curves $x^3 - x = c$, extruded in the y direction.
(d) Level sets are hyperbolas.

(3) The matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ acts on a vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ by rotating it θ degrees counterclockwise. For example,

$$R_{\pi/2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad R_{\pi/2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

(a) Find a function whose level curves are the same as in #2(e), but rotated $\pi/4$ counterclockwise.

The level curves of $f\left(R_{\pi/4}\begin{bmatrix} x \\ y \end{bmatrix}\right)$ will be the same as the level curves of $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ but rotated counterclockwise by $\pi/4$.

$$f\left(R_{\pi/4}\begin{bmatrix} x \\ y \end{bmatrix}\right) = f\left(\begin{bmatrix} \frac{\sqrt{2}}{2}(x - y) \\ \frac{\sqrt{2}}{2}(x + y) \end{bmatrix}\right) = \max\left(\frac{\sqrt{2}}{2}|x - y|, \frac{\sqrt{2}}{2}|x + y|\right)$$

(b) Sketch the level curves of $g(x, y) = x^2 - y^2$ and $h(x, y) = xy$ and explain how the two graphs of g and h are related to each other.

$g(1, 0) = 1 = h(1, 1)$, and $h(1, 1) = 1$. The level curves of h arise from level curves of g dilated by a linear factor of $\sqrt{2}$ linearly and then rotated counterclockwise by 45 degrees. The graph of h is graph of g stretched by a linear factor of $\sqrt{2}$ and then rotated 45 degrees counterclockwise.

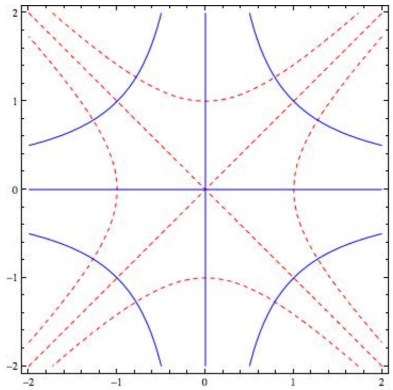

FIGURE 1. *

(4) Find $\partial f/\partial x$, $\partial f/\partial y$ if
- $f(x, y) = xy$
- $f(x, y) = e^{xy}$
- $f(x, y) = x \cos x \cos y$
- $f(x, y) = (x^2 + y^2) \log(x^2 + y^2)$.

(a) $\partial_x f = y$, $\partial_y f = x$
(b) $\partial_x f = ye^{xy}$, $\partial_y f = xe^{xy}$
(c) $\partial_x f = \cos x \cos y - x \sin x \cos y$, $\partial_y f = -x \cos x \sin y$
(d) $\partial_x f = 2x \log(x^2 + y^2) + 2x$, $\partial_y f = 2y \log(x^2 + y^2) + 2y$

(5) Suppose that the temperature at the point (x, y, z) in space is $T(x, y, z) = x^2 + y^2 + z^2$. Let a particle follow the helix $\vec{r}(t) = (\cos t, \sin t, t)$ and let $T(t)$ be its temperature at time t.
- What is $T'(t)$?
- Find an approximate value for the temperature at $t = \frac{\pi}{2} + 0.01$.

$T(t) = \cos^2 t + \sin^2 t + t^2$. Thus $T'(t) = -\sin t \cos t + \cos t \sin t + 2t = 2t$. The approximation is $T(\frac{\pi}{2} + 0.01) \approx T(\frac{\pi}{2}) + (0.01) \cdot T'(\frac{\pi}{2}) = 1 + (\pi/2)^2 + 0.01\pi$.

(6) Compute the directional derivative of f in the given directions $\vec{v}$ at the given points P.
- $f(x, y, z) = xy^2 + y^2z^3 + z^3x$, $P = (4, -2, 1)$, $\vec{v} = \frac{1}{\sqrt{14}}(\vec{i} + 3\vec{j} + 2\vec{k})$
- $f(x, y, z) = e^{-z} \sin(x) \sin(y)$, $P = (\pi, \frac{\pi}{2}, 0)$, $\vec{v} = \frac{12}{13}\vec{i} + \frac{3}{13}\vec{j} + \frac{4}{13}\vec{k}$.

$\nabla f = (y^2 + z^3, 2yx + 2yz^3, 3z^2y^2 + 3z^2x)$. Thus $\nabla f|_P = (5, -20, 24)$. $D_{\vec{v}} f|_P = -\sqrt{\frac{7}{2}}$.
$\nabla f = e^{-z}(\cos x \sin y, \sin x \cos y, \sin x \sin y)$. Thus $\nabla f|_P = (-1, 0, 0)$. $D_{\vec{v}} f|_P = -\frac{12}{13}$.

(7) In electrostatics, the force $\vec{P}$ of attraction between two particles of opposite charge is given by $\vec{P} = k\frac{\vec{r}}{\|\vec{r}\|^3}$ (Coulomb's law), where k is a constant and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$. Show that $\vec{P}$ is the gradient of $f = -\frac{k}{\|\vec{r}\|}$.

Note that $\nabla\|\vec{r}\| = \frac{\vec{r}}{\|\vec{r}\|}$. Thus $\nabla\|\vec{r}\|^{-1} = -\frac{1}{\|\vec{r}\|^2}\nabla\|\vec{r}\| = -\frac{\vec{r}}{\|\vec{r}\|^3}$. The result follows.

(8) * Fix two points p_1 and p_2 in $\mathbb{R}^2$. Let

$$d(\vec{x}) = \|\vec{x} - \vec{p}_1\| + \|\vec{x} - \vec{p}_2\|.$$

(a) What are the level lines of the function d?

Solution: They are ellipses.

(b) Show that the normal vector to a level set of d at some point p has the following property: it bisects the the angle between the vectors which point from p to p_1 and to p_2 respectively.

Solution: We compute the gradient

$$\nabla d(\vec{x}) = \frac{\vec{x} - \vec{p}_1}{\|\vec{x} - \vec{p}_1\|} + \frac{\vec{x} - \vec{p}_2}{\|\vec{x} - \vec{p}_2\|}.$$

Thus it is seen that the gradient vector is equal to the diagonal of a rhombus constructed from the the unit vectors of the radius vectors pointing from p to p_1 and to p_2 respectively. Thus, the normal to the ellipse at a bisects the the angle between the two radial vectors. The physical interpretation of this fact is that a light ray coming from one focus enters the other focus after reflecting off an elliptical mirror.

(**Extra**) * *Marsden & Tromba*: §2.3 #9(c), 14, 18, 22, §2.4: #18, 25; §2.5: #3(d), 7, 22; §2.6: #8(a).

*Not to appear on quiz.