(1) Drop the condition that $\vec{v}+\vec{w}=\vec{w}+\vec{v}$ in the vector space axioms, but assume that the additive inverse satisfies $\vec{X}+-\vec{X}=-\vec{X}+\vec{X}=0$. Prove that this condition can be recovered.

Solution: $\vec{v}+\vec{v}+\vec{w}+\vec{w}=(1+1)(\vec{v}+\vec{w})=\vec{v}+\vec{w}+\vec{v}+\vec{w}$.

(2) *True or False.* For each of the following statements, decide if the statement is *always* true $\boxed{T}$ or if the statement is *not always* true $\boxed{F}$. *Give reasons for your answers. If a statement is false, give a counterexample.*

(a) $\boxed{T}$ $\boxed{F}$ Let A be an $m\times n$ matrix, suppose $A\vec{x}=\vec{b}$ has a solution for all $\vec{b}\in\mathbb{R}^m$, then the solution to $A^t\vec{y}=\vec{d}$, when it exists, is unique.

Solution: True. If $A\vec{x}=\vec{b}$ has a solution for all $\vec{b}\in\mathbb{R}^m$, then $\mathcal{C}(A)=\mathbb{R}^m$. It follows that $\text{rank}(A)=m$. Since $\text{rank}(A)=\text{rank}(A^t)$ and $A^t\in M(n,m)$, we must have that $\text{null}(A^t)=m-\text{rank}(A^t)=0$. Therefore, any solution to $A^t\vec{y}=\vec{d}$, if it exists, must be unique.

(b) $\boxed{T}$ $\boxed{F}$ Suppose A is an $m\times n$ matrix, A and A^t have the same nullity.

Solution: False. Let

$$A=\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix},\quad\text{then}\quad A^t=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}.$$

As they are both in echelon form, we see that $\text{null}(A)=1$, but $\text{null}(A^t)=0$.

(c) $\boxed{T}$ $\boxed{F}$ When a matrix A is non-singular, its transpose A^t can be singular.

Solution: False. Recall that A is non-singular if $\text{rref}(A)=I_n$. In particular, A is a square matrix. Since $n=\text{rank}(A)=\text{rank}(A^t)$ we must have that $\text{null}(A^t)=n-\text{rank}(A^t)=0$, so $\text{rref}(A^t)$ has n pivots and must be I_n.

(d) $\boxed{T}$ $\boxed{F}$ Assume that $A,B\in M(5,7)$ both have rank 3 and $\vec{b}\in\mathbb{R}^5$, then $\{\vec{x}\in\mathbb{R}^7\mid A\vec{x}=\vec{b}\text{ and }B\vec{x}=\vec{0}\}$ is always non-empty.

Solution: False. Let

$$A=B=\begin{bmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{bmatrix}\quad\text{and}\quad\vec{b}=\begin{bmatrix}1\\1\\1\\0\\0\end{bmatrix}$$

Then, if $A\vec{x}=\vec{b}$, the we must have $x_1=x_2=x_3=1$. However, if $\vec{x}$ is such that $B\vec{x}=\vec{0}$, then $x_1=x_2=x_3=0$. Therefore, $\{\vec{x}\in\mathbb{R}^7\mid A\vec{x}=\vec{b}\text{ and }B\vec{x}=\vec{0}\}$ is empty.

(e) $\boxed{T}$ $\boxed{F}$ If A and B are two $n\times n$ matrices, if $AB=0$ then $A=0$ or $B=0$.

Solution: False. Consider

$$A=B=\begin{bmatrix}0&1\\0&0\end{bmatrix},\quad\text{then}\quad AB=\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

.

(f) $\boxed{T}$ $\boxed{F}$ If A and B are two $n\times n$ matrices, then $(AB)^2=A^2B^2$.

Solution: False. Consider

$$A=\begin{bmatrix}0&1\\1&0\end{bmatrix}\quad\text{and }B\quad=\begin{bmatrix}1&1\\0&1\end{bmatrix}$$

Then,

$$(AB)^2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \text{while} \quad A^2B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

(g) $\boxed{T}$ $\boxed{F}$ Suppose $A \in M(m,n)$, $B \in M(n,q)$ and the columns of B are linearly dependent, then columns of AB must be linearly dependent.

Solution: True. Let $\vec{v}_1, \dots, \vec{v}_q$ be the columns of B and assume we have a nontrivial relation $c_1\vec{v}_1 + \cdots + c_q\vec{v}_q = \vec{0}$. Then, applying A and using linearity, we have that $c_1A\vec{v}_1 + \cdots + c_qA\vec{v}_q = A\vec{0} = \vec{0}$. By definition, the columns of AB are $A\vec{v}_i$, so they are therefore also linearly dependent.

(h) $\boxed{T}$ $\boxed{F}$ Suppose A and B are matrices such that AB is defined and the rows of B are linearly dependent, then rows of AB must be linearly dependent.

Solution: False. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix}. \text{ Then, } AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(3) Let I_n be the $n \times n$ identity matrix. Suppose A, B are $n \times n$ and A is invertible, show

$$\det(I_n + AB) = \det(I_n + BA).$$

Solution:

$$\det(I_n + AB) = \det(AA^{-1} + AB) = \det(A)\det(A^{-1} + B) = \det(A^{-1} + B)\det(A) = \det(I_n + BA).$$

(4) *Generalized Cross Product*

Given vectors $\vec{v}_1, \dots \vec{v}_{n-1} \in \mathbb{R}^n$, *suppose* there is a vectors $\vec{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ satisfying the property that for all $\vec{w} \in \mathbb{R}^n$,

$$\vec{x} \cdot \vec{w} = \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_{n-1} & \vec{w} \end{bmatrix}. \tag{1}$$

(a) Show that

$$x_i = \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_{n-1} & \hat{e}_i \end{bmatrix} \tag{2}$$

Solution:

$$x_i \overset{\text{by computing}}{=\!=\!=} \vec{x} \cdot \hat{e}_i \overset{\text{by property}}{=\!=\!=} \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_{n-1} & \hat{e}_i \end{bmatrix}$$

(b) Show that $\vec{x}$ as defined in Equation (2) satisfies the property in Equation (1).

Solution: Given the vectors $\vec{v}_1, \dots \vec{v}_{n-1}$, we simply define $\vec{x}$ by Equation (2). To see that it satisfies the desired property, observe that

$$\vec{x} \cdot \vec{w} = \sum_{i=1}^{n} x_i w_i = \sum_{i=1}^{n} w_i \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_{n-1} & \hat{e}_i \end{bmatrix}$$

Since det is multilinear, we have

$$\begin{aligned} \vec{x} \cdot \vec{w} &= \sum_{i=1}^{n} w_i \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots \vec{v}_{n-1} & \hat{e}_i \end{bmatrix} \\ &= \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_{n-1} & \sum_{i=1}^{n} w_i \hat{e}_i \end{bmatrix} \\ &= \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_{n-1} & \vec{w} \end{bmatrix}. \end{aligned}$$

(c) Show that $\vec{x}$ as defined in Equation (2) is the unique vector in $\mathbb{R}^n$ satisfying the property in Equation (1). *Hint: Let $\vec{y}$ be another vector satisfying the property that for all $\vec{w} \in \mathbb{R}^n$,*

$$\vec{y} \cdot \vec{w} = \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_{n-1} & \vec{w} \end{bmatrix}$$

and show $x_i = y_i$ for all i.

Solution: Let $\vec{y}$ be a vector satisfying the property that for all $\vec{w} \in \mathbb{R}^n$,

$$\vec{y} \cdot \vec{w} = \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots \vec{v}_{n-1} & \vec{w} \end{bmatrix}$$

In particular, let $\vec{w} = \hat{e}_i$, then

$$y_i = \vec{y} \cdot \hat{e}_i = \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots \vec{v}_{n-1} & \hat{e}_i \end{bmatrix} = x_i$$

So, applying this for all i gives $\vec{y} = \vec{x}$.

(d) Show that $\vec{x}$ is perpendicular to $\operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_{n-1}\}$. We therefore call $\vec{x}$ the *generalized cross product* of $\vec{v}_1, \ldots, \vec{v}_{n-1} \in \mathbb{R}^n$.

Solution: It suffices to show that for all i,we have $\vec{x} \cdot \vec{v}_i = 0$. Indeed,

$$\vec{x} \cdot \vec{v}_i = \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots \vec{v}_{n-1} & \vec{v}_i \end{bmatrix} \overset{\text{rep. col.}}{=\!=\!=} 0 \quad \text{for} \quad 1 \le i \le n-1.$$

(5) Let $A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$. Compute A^{-1}.

Solution: One can computes $\operatorname{rref}([A \mid I])$ by multiplying on the left by E_i's as follows:

$$\left[\begin{array}{ccc|ccc} -2 & 1 & 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{E_1 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \left[\begin{array}{ccc|ccc} -2 & 1 & 0 & 1 & 0 & 0 \\ 0 & -\frac{3}{2} & 1 & \frac{1}{2} & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix}}$$

$$\left[\begin{array}{ccc|ccc} -2 & 1 & 0 & 1 & 0 & 0 \\ 0 & -\frac{3}{2} & 1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & -\frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right] \xrightarrow{E_3 = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{2}{3} & 0 \\ 0 & 0 & -\frac{3}{4} \end{bmatrix}} \left[\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} \end{array}\right] \rightarrow$$

$$\xrightarrow{E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}} \left[\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & -1 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} \end{array}\right] \xrightarrow{E_5 = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{4} & -\frac{1}{2} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{2} & -1 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} \end{array}\right]$$

Thus

$$A^{-1} = -\frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

(6) Consider a metal rod of unit length and uniform thermal conductivity. Suppose the temperature T at the ends is held fixed $T(0) = a$ and $T(b)$, and subject to a time independent source $f(x)$. This gives the ODE

$$\frac{d^2T}{dx^2} = f(x), \qquad T(0) = a, \qquad T(1) = b, \qquad 0 \le x \le 1.$$

For example, if $f(x) = 0$, then $T(x) = a(1-x)+bx$. For a more complicated heat source $f(x)$, we introduce a numerical approximation technique called the *finite difference method*. We discretize the problem by giving a numerical approximation to the solution at a finite set of locations

$$x_0 = 0, x_1 = \frac{1}{N}, \cdots, x_i = \frac{i}{N}, \cdots, x_N = 1$$

for some large integer N. Set $T_i = T\left(\frac{i}{N}\right)$ and $f_i = f\left(\frac{i}{N}\right)$ Let $h = \frac{1}{N}$ and approximate

$$\frac{d^2T}{dx^2}(x_i) \simeq \frac{T_{i+1} - 2T_i + T_{i-1}}{h^2}.$$

Using this approximation, derive a linear system to determine $\vec{T} := (T_1, T_2, \ldots, T_{N-1})$, e.g. express $A\vec{T} = \vec{b}$ for an appropriate matrix A and vector $\vec{b}$ (depending on f_i, boundary conditions a, b and spacing h).

Solution: This gives a linear system of equations

$$T_{i+1} - 2T_i + T_{i-1} = h^2 f_i$$

Additionally, we have the boundary conditions $T_0 = a$ and $T_N = b$. We must determine the $(n-1)\times(n-1)$ matrix A and vector $\vec{b}$. Note that $T_0 = a$ and $T_N = b$, so there are $N-1$ linear equations:

$$T_2 - 2T_1 + a = h^2 f_1, \quad b - 2T_{N-1} + T_{N-2} = h^2 f_{N-1},$$

$$T_{i+1} - 2T_i + T_{i-1} = h^2 f_i \quad \text{for} \quad i = 2, \ldots, N-2.$$

Moving a and b to the other sides of the first and last equation gives:

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ & & \ddots & & & \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & 0 & 0 & 1 & -2 \end{bmatrix}, \quad \vec{T} = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{N-1} \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} h^2 f_1 - a \\ h^2 f_2 \\ \vdots \\ h^2 f_{N-2} \\ h^2 f_{N-1} - b \end{bmatrix}.$$

(7) (a) Let Δ_n be the determinant of $n \times n$ matrix $\begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ & & \ddots & & & \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & 0 & 0 & 1 & -2 \end{bmatrix}$.

Prove that $\Delta_n - \Delta_{n-1} = -(\Delta_{n-1} + \Delta_{n-2})$ and evaluate Δ_n.

Solution: Note that

$$\Delta_1 = -2, \qquad \Delta_2 = 3, \qquad \Delta_3 = -4$$

Laplace expand along the first column, we have

$$\Delta_n = -2\Delta_{n-1} - \Delta_{n-2}, \qquad \Delta_n + \Delta_{n-1} = -(\Delta_{n-1} + \Delta_{n-2})$$

$$\Delta_2 + \Delta_1 = 1, \qquad \Delta_3 + \Delta_2 = -1, \qquad \Delta_n + \Delta_{n-1} = (-1)^n$$

$\Delta_1 = -2$, and suppose $\Delta_{n-1} = (-1)^{n-1} n$, it follows by induction that

$$\Delta_n = -\Delta_{n-1} + (-1)^n = (-1)^n n + (-1)^n = (-1)^n (n+1).$$

(b) Let Δ_n be the determinant of $n \times n$ matrix $\begin{bmatrix} 1+x^2 & x & 0 & \cdots & \\ x & 1+x^2 & x & 0 & \cdots \\ 0 & x & 1+x^2 & x & \cdots \\ 0 & 0 & x & 1+x^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots \end{bmatrix}$.

Prove that $\Delta_n - \Delta_{n-1} = x^2(\Delta_{n-1} - \Delta_{n-2})$ and evaluate Δ_n.

Solution: Use Laplace expansion along first column, and then first row:

$$\Delta_n = (1+x^2)\Delta_{n-1} - x^2\Delta_{n-2}$$

Set $f(n) = \Delta_n - \Delta_{n-1}$. Then, $f(n) = x^2 f(n-1)$. We can compute that

$$f(2) = 1 + x^2 + x^4 - (1 + x^2) = x^4$$

So, it should be easy to see that $f(n) = x^{2n}$. Then,

$$\begin{aligned}
\Delta_n &= f(n) + \Delta(n-1) \\
&= f(n) + f(n-1) + \Delta(n-2) \\
&= \cdots = f(n) + f(n-1) + \cdots + f(2) + \Delta_1 \\
&= x^{2n} + \cdots + x^2 + 1 \\
&= \frac{x^{2n+2} - 1}{x^2 - 1}.
\end{aligned}$$