(1) Drop the condition that $\vec{v}+\vec{w}=\vec{w}+\vec{v}$ in the vector space axioms, but assume that the additive inverse satisfies $\vec{X}+-\vec{X}=-\vec{X}+\vec{X}=0$. Prove that this condition can be recovered.

(2) *True or False.* For each of the following statements, decide if the statement is *always* true $\boxed{T}$ or if the statement is *not always* true $\boxed{F}$. *Give reasons for your answers. If a statement is false, give a counterexample.*

(a) $\boxed{T}$ $\boxed{F}$ Let A be an $m\times n$ matrix, suppose $A\vec{x}=\vec{b}$ has a solution for all $\vec{b}\in\mathbb{R}^m$, then the solution to $A^t\vec{y}=\vec{d}$, when it exists, is unique.

(b) $\boxed{T}$ $\boxed{F}$ Suppose A is an $m\times n$ matrix, A and A^t have the same nullity.

(c) $\boxed{T}$ $\boxed{F}$ When a matrix A is non-singular, its transpose A^t can be singular.

(d) $\boxed{T}$ $\boxed{F}$ Assume that $A,B\in M(5,7)$ both have rank 3 and $\vec{b}\in\mathbb{R}^5$, then $\{\vec{x}\in\mathbb{R}^7 \mid A\vec{x}=\vec{b} \text{ and } B\vec{x}=\vec{0}\}$ is always non-empty.

(e) $\boxed{T}$ $\boxed{F}$ If A and B are two $n\times n$ matrices, if $AB=0$ then $A=0$ or $B=0$.

(f) $\boxed{T}$ $\boxed{F}$ If A and B are two $n\times n$ matrices, then $(AB)^2=A^2B^2$.

(g) $\boxed{T}$ $\boxed{F}$ Suppose $A\in M(m,n)$, $B\in M(n,q)$ and the columns of B are linearly dependent, then columns of AB must be linearly dependent.

(h) $\boxed{T}$ $\boxed{F}$ Suppose A and B are matrices such that AB is defined and the rows of B are linearly dependent, then rows of AB must be linearly dependent.

(3) Let I_n be the $n\times n$ identity matrix. Suppose A,B are $n\times n$ and A is invertible, show

$$\det(I_n+AB)=\det(I_n+BA).$$

(4) *Generalized Cross Product.* Given vectors $\vec{v}_1,\dots\vec{v}_{n-1}\in\mathbb{R}^n$, *suppose* there is a vector $\vec{x}=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$ satisfying the property that for all $\vec{w}\in\mathbb{R}^n$,

$$\vec{x}\cdot\vec{w}=\det\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_{n-1} & \vec{w}\end{bmatrix}. \tag{1}$$

(a) Show that

$$x_i=\det\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_{n-1} & \hat{e}_i\end{bmatrix}. \tag{2}$$

(b) Show that $\vec{x}$ as defined in Equation (2) satisfies the property in Equation (1).

(c) Show that $\vec{x}$ as defined in Equation (2) is the unique vector in $\mathbb{R}^n$ satisfying the property in Equation (1). *Hint: Let $\vec{y}$ be another vector satisfying the property that for all $\vec{w}\in\mathbb{R}^n$,*

$$\vec{y}\cdot\vec{w}=\det\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_{n-1} & \vec{w}\end{bmatrix}$$

and show $x_i=y_i$ for all i.

(d) Show that $\vec{x}$ is perpendicular to $\operatorname{span}\{\vec{v}_1,\dots,\vec{v}_{n-1}\}$. We therefore call $\vec{x}$ the *generalized cross product* of $\vec{v}_1,\dots,\vec{v}_{n-1}\in\mathbb{R}^n$.

(5) Let $A=\begin{bmatrix}-2 & 1 & 0\\ 1 & -2 & 1\\ 0 & 1 & -2\end{bmatrix}$. Compute A^{-1}.

(6) * Consider a metal rod of unit length and uniform thermal conductivity. Suppose the temperature T at the ends is fixed $T(0) = a$ and $T(1) = b$, and subject to a steady source $f(x)$. This gives the ODE

$$\frac{d^2T}{dx^2} = f(x), \qquad T(0) = a, \qquad T(1) = b, \qquad 0 \leq x \leq 1.$$

For example, if $f(x) = 0$, then $T(x) = a(1-x)+bx$. For a more complicated heat source $f(x)$, we introduce a numerical approximation technique called the *finite difference method*. We discretize the problem by giving a numerical approximation to the solution at a finite set of locations

$$x_0 = 0, x_1 = \frac{1}{N}, \cdots, x_i = \frac{i}{N}, \cdots, x_N = 1$$

for some large integer N. Set $T_i = T\left(\frac{i}{N}\right)$ and $f_i = f\left(\frac{i}{N}\right)$ Let $h = \frac{1}{N}$ and approximate

$$\frac{d^2T}{dx^2}(x_i) \simeq \frac{T_{i+1} - 2T_i + T_{i-1}}{h^2}.$$

Using this approximation, derive a linear system to determine $\vec{T} := (T_1, T_2, \ldots, T_{N-1})$, e.g. express $A\vec{T} = \vec{b}$ for an appropriate matrix A and vector $\vec{b}$ (depending on f_i, boundary conditions a, b and spacing h).

(7) *

(a) Let Δ_n be the determinant of $n \times n$ matrix $\begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ & & \ddots & & & \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & 0 & 0 & 1 & -2 \end{bmatrix}$.

Prove that $\Delta_n - \Delta_{n-1} = -(\Delta_{n-1} + \Delta_{n-2})$ and evaluate Δ_n.

(b) Let Δ_n be the determinant of $n \times n$ matrix $\begin{bmatrix} 1+x^2 & x & 0 & \cdots & \\ x & 1+x^2 & x & 0 & \cdots \\ 0 & x & 1+x^2 & x & \cdots \\ 0 & 0 & x & 1+x^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots \end{bmatrix}$.

Prove that $\Delta_n - \Delta_{n-1} = x^2(\Delta_{n-1} - \Delta_{n-2})$ and evaluate Δ_n.

*Not to appear on quiz.