(1) For what values of a and b does the following system of equations

$$x + 2y + 3z = 4, \qquad x + 4y + 9z = 16, \qquad x + 8y + az = b$$

(a) have a unique solution?
(b) have no solution?
(c) have infinitely many solutions?

(2) Find a matrix A representing the linear transformation T in the following two cases:

$$\text{(a)} \quad T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}, \quad T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}4\\5\end{bmatrix} \qquad\qquad \text{(b)} \quad T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}2x+3y-7z\\ y-z\end{bmatrix}.$$

(3) A square matrix A is called *nilpotent* if $A^k = 0$ for some positive integer k.
(a) Compute the determinant of any nilpotent matrix.
(b) Find a 3×3 matrix A such that $A^2 = 0$ but $A \neq 0$.
(c) Find a 3×3 matrix A such that $A^3 = 0$ but $A^2 \neq 0$.

(4) *True or False.* Answer the following questions concerning systems (i) and (ii) below. Here, x, y, z are the unknowns and $a_i, b_i, c_i, d_i \in \mathbb{R}$. *Explain your answers. If a statement is false, give a counterexample.*

$$\text{(i)} \begin{array}{rcl} a_1x + b_1y + c_1z &=& d_1 \\ a_2x + b_2y + c_2z &=& d_2 \\ a_3x + b_3y + c_3z &=& d_3 \end{array} \qquad \text{(ii)} \begin{array}{rcl} a_1x + b_1y + c_1z &=& d_1 \\ a_2x + b_2y + c_2z &=& d_2 \\ a_3x + b_3y + c_3z &=& d_3 + 1 \end{array}$$

(a) If (i) has exactly one solution, then the same is true for (ii).

(b) If the solution set of (i) is a line, then the same is true for (ii).

(c) If (i) has no solutions, then the same is true for (ii).

(5) * Let $A = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$ and $f_0 = 0$, $f_1 = 1$, $f_2 = 1$ and, for $n \geq 0$, $f_{n+2} = f_{n+1} + f_n$ be the Fibonacci sequence.

(a) Show that $A^2 = \begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}$ and prove, using induction, that

$$A^{2n} = \begin{pmatrix} f_{2n-1} & f_{2n} \\ f_{2n} & f_{2n+1}\end{pmatrix}, \qquad n = 1, 2, 3, \ldots$$

(b) Compute the determinant of A^2, and use it, together with the formula for A^{2n} (relating the determinants), to prove the identity $f_{2n-1}f_{2n+1} - f_{2n}^2 = 1$ for $n = 1, 2, 3, \ldots$.

(6) * Stony Brook University's Board of Trustees, which consists of 20 members, recently had to elect a President. There were three candidates on the shortlist (A, B, and C). On each ballot, the three candidates were listed in order of preference, with no abstentions. The vote outcome is as follows:

- 11 members, a majority, preferred A to B, thus 9 preferred B to A
- 12 members preferred C to A.

Given this, it was suggested that B should withdraw, to enable a direct comparison between A and C. However, B's proponents objected. It turned out that 14 members preferred B to C. Suppose *every possible order* of A, B, and C appeared on at least one ballot, how many members voted for B as their first choice? *Argue how, by eliminating by first comparing a given two of the candidates head to head, and then comparing the remaining two, you could "fairly" elect either A, B or C in this election. (!!)*

(**Extra**) * *Marsden & Tromba*: §1.3: # 1, 2, 3, 4, 8, 19, 42, 45

*Not to appear on quiz.