(1) Let $\vec{r}(t)$ be a parametrized curve representing the trajectory of a particle. Let $\vec{v}(t) = \vec{r}'(t)$ be its velocity and $\vec{a}(t) = \vec{r}''(t)$ be its acceleration. Prove that the rate of change of the speed of the particle is the component of its acceleration along its velocity:

$$\frac{d}{dt}\|\vec{v}(t)\| = \mathrm{Comp}_{\vec{v}(t)}\vec{a}(t).$$

Give an example of a particle motion that *does* accelerate, but whose speed never changes.

(2) Consider a particle with position $\vec{r}(t)$ given by the helix

$$\vec{r}(t) = (a\cos\omega t, a\sin\omega t, b\omega t), \qquad a \neq 0.$$

(a) Show the speed of the particle is constant, and the velocity maintains a constant angle with the z-axis.
(b) Find the arclength parametrization of the helix and compute the curvature and principle normal vector. Show that the acceleration vector of the particle is always parallel to the xy plane.
(c) Let $P = \vec{r}(0) = (a, 0, 0)$ and $Q = \vec{r}(\frac{2\pi}{\omega}) = (a, 0, 2\pi b)$. Note $\overrightarrow{PQ}$ is vertical. Show that

$$\vec{r}(\tfrac{2\pi}{\omega}) - \vec{r}(0) = \tfrac{2\pi}{\omega}\vec{r}'(s)$$

cannot hold for any $s \in (0, \frac{2\pi}{\omega})$. Thus, the Mean Value Theorem does not hold for vector functions.

(3) Compute the curvature and draw a (rough) picture of the following spirals:
(a) the Archimedean spiral $\vec{r}(t) = (t\cos(t), t\sin(t))$,
(b) the logarithmic spiral $\vec{r}(t) = (e^t\cos(t), e^t\sin(t))$.
What happens as $t \to \pm\infty$ in both cases? Which do you like better, and why?

(4) Let $\vec{r}(t)$ be the vector from the origin to the position of an object of mass $m > 0$, $\vec{v}(t) := \vec{r}'(t)$ be the velocity and $\vec{a}(t) := \vec{r}''(t)$ be the acceleration. Suppose that $\vec{F}(t) = m\vec{a}$ is the force acting at time t.
(a) Prove that $\frac{d}{dt}(m\vec{r}\times\vec{v}) = \vec{r}\times\vec{F}$. What do you conclude if $\vec{F}$ is parallel to $\vec{r}$?
(b) Prove that a planet (say, of mass m) moving about the Sun (say, of mass M) does so in a fixed plane. Recall, Newton's universal law of gravity says that $\vec{F} = -\frac{GmM}{\|\vec{r}\|^3}\vec{r}$.
(c) Kepler's First Law says that the orbit of a planet around the sun is an ellipse. Derive Kepler's Second Law which states that the rate at which the area swept out by the planet around the sun is constant.*

(5) (a) Let $\vec{r}(t)$ be a differentiable vector valued function of t. Show that, at a local maximum or minimum of $\|\vec{r}(t)\|$, the vector $\vec{r}'(t)$ is perpendicular to $\vec{r}(t)$.
(b) Prove that if the curvature of a curve is identically zero, then the curve is straight.

(6) † Suppose that $\gamma(s) : [0, L] \to \mathbb{R}^d$, $d \geq 2$ is an arc length parametrization of a closed curve Γ of length L. Suppose Γ lies entirely within a sphere of radius R. Prove that average of the absolute curvature $|\kappa(s)| = |\gamma''(s)|$ enjoys the lower bound

$$\frac{L}{R} \leq \int_\Gamma |\kappa(s)|ds.$$

Use this result, together with some typical numbers found searching the internet, to lower bound the net absolute curvature of a strand of your DNA. Hint: use $\gamma'(s)\cdot\gamma'(s) = 1$ and integrate! Bonus: determine precisely which curves saturate the above inequality.

(**Extra**) † *Marsden & Tromba*: §1 (review): # 3, 4, 7, 12, 25, 26; §4.1: # 8; §4.2: # 6, 14, 15, 16.

*Hence, planet travels fast when closer to the sun and slower further away from the sun.
†Not to appear on quiz.