(1) Using the dot product, prove the Pythagorean theorem. Namely, show that if the lengths a, b, c of the sides of a triangle satisfy $a^2 + b^2 = c^2$, then the triangle is a right triangle and visa verse.

Proof: Let $\vec{u}, \vec{v}, \vec{w}$ be vectors representing the sides of the triangle, so that $|\vec{u}| = a$ and so on. Since it is a triangle, $\vec{u} + \vec{v} = \vec{w}$. Then $(\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = c^2$. But this is also $a^2 + b^2 + 2\vec{u} \cdot \vec{v}$. Thus $\vec{u} \cdot \vec{v} = 0$ which happens if and only if $\vec{u}$ and $\vec{v}$ are orthogonal. Thus $a^2 + b^2 = c^2$ if and only if the triangle is right.

(2) Let $\vec{v}$ and $\vec{u}$ be vectors in $\mathbb{R}^2$ and recall that the area of the triangle is defined by $\vec{v}$ and $\vec{u}$ is given by $A(\vec{v}, \vec{u}) = \frac{1}{2}|\vec{v}^{\perp} \cdot \vec{u}|$ where $\vec{v}^{\perp} = (-v_2, v_1)$ denotes counterclockwise rotation of the vector $\vec{v}$ by $90°$.

(a) Show that $(\vec{v}^{\perp} \cdot \vec{u})^2 = \|\vec{v}\|^2\|\vec{u}\|^2 - (\vec{v} \cdot \vec{u})^2$. Explain what this means geometrically.

Solution: We expand and see it:

$$\begin{aligned}(\vec{v}^{\perp} \cdot \vec{u})^2 &= (u_2v_1 - v_2u_1)^2 = u_2^2v_1^2 + v_2^2u_1^2 - 2u_1u_2v_1v_2,\\ \|\vec{v}\|^2\|\vec{u}\|^2 &= (u_1^2 + u_2^2)(v_1^2 + v_2^2) = u_1^2v_1^2 + u_1^2v_2^2 + u_2^2v_1^2 + u_1^2v_2^2,\\ (\vec{v} \cdot \vec{u})^2 &= (u_1v_1 + u_2v_2)^2 = u_1^2v_1^2 + u_2^2v_2^2 + 2u_1u_2v_1v_2.\end{aligned}$$

Geometrically, $\vec{v} \cdot \vec{u}$ is a multiple of the cosine of the angle, ϑ, between the vector $\vec{v}$ and $\vec{u}$

$$\vec{v} \cdot \vec{u} = \|\vec{v}\|\|\vec{u}\| \cos\vartheta.$$

If you rotate the vector $\vec{v}$ counterclockwise $90°$, it will make angle $\vartheta + \frac{\pi}{2}$ with $\vec{u}$. Then

$$\vec{v}^{\perp} \cdot \vec{u} = \|\vec{v}\|\|\vec{u}\| \cos(\vartheta + \tfrac{\pi}{2}) = -\|\vec{v}\|\|\vec{u}\| \sin\vartheta.$$

Indeed, we have

$$\|\vec{v}\|^2\|\vec{u}\|^2 - (\vec{v} \cdot \vec{u})^2 = \|\vec{v}\|^2\|\vec{u}\|^2(1 - \cos^2\theta) = \|\vec{v}\|^2\|\vec{u}\|^2 \sin^2\vartheta = (\vec{v}^{\perp} \cdot \vec{u})^2.$$

(b) Use this fact to establish *Heron's formula* (most probably due to Archimedes), namely that a triangle with sides lengths a, b and c has area

$$\mathsf{Area}(a, b, c) = \sqrt{s(s-a)(s-b)(s-c)}, \qquad s := \frac{1}{2}(a + b + c).$$

To do this, you will need to relate to $(\vec{v} \cdot \vec{u})^2$ with magnitudes of $\vec{v}$, $\vec{u}$ and $\vec{v} - \vec{u}$ (which should describe the sides of the triangle). There is also some algebra.

Solution: Let use assume that the triangle has vertices O, A and B and $a = \|\vec{v}\|$, $b = \|\vec{u}\|$ and $c = \|\vec{v} - \vec{u}\|$. We know $\mathsf{Area}(a, b, c) = \frac{1}{2}|\vec{v}^{\perp} \cdot \vec{u}|$. So

$$\begin{aligned}4|\mathsf{Area}(a, b, c)|^2 &= (\vec{v}^{\perp} \cdot \vec{u})^2 = \|\vec{v}\|^2\|\vec{u}\|^2 - (\vec{v} \cdot \vec{u})^2\\ &= a^2b^2 - \frac{1}{4}\Big(\|\vec{v} - \vec{u}\|^2 - \|\vec{v}\|^2 - \|\vec{u}\|^2\Big)^2\\ &= a^2b^2 - \frac{1}{4}(c^2 - a^2 - b^2)^2\\ &= \frac{1}{4}(2ab - c^2 + a^2 + b^2)(2ab + c^2 - a^2 - b^2)\\ &= \frac{1}{4}((a + b)^2 - c^2)(c^2 - (a - b)^2)\\ &= \frac{1}{4}(a + b + c)(a + b - c)(c - a + b)(c + a - b).\end{aligned}$$

The formula follows.

(c) Show that, for fixed side lengths a and b, the triangle with the largest area is right.

Solution: Fixing a, b, we seek the maximum of the function $A(c) := \mathsf{Area}(a, b, c)$. A simple calculation (differentiating the square of the area, rather than the area itself) shows

$$A'(c) = \frac{c(a^2 + b^2 - c^2)}{8A(c)}.$$

This is zero for $c^2 = a^2 + b^2$, the Pythagorean relation of a right triangle!

What is more, this allows for a proof of the celebrated Pythagorean theorem: a triangle with side lengths a, b and c is right if and only if $a^2 + b^2 = c^2$. For this, we must prove that $A'(c) = 0$ if and only if the triangle with side lengths a, b and c is right. This follows from the elementary formula for the area of a triangle (which does not use Pythagoras):

$$\mathsf{Area}(a, b, C) = \frac{1}{2}ab\sin C,$$

where a and b are the lengths of any two sides, and C is the included angle. Clearly, this is maximized, as a function of the included angle C, when $C = \frac{\pi}{2}$, namely at the right triangle. Thus, a triangle made with two rigid rods of lengths a, b and an elastic band of variable length c has maximum area when the angle between a and b is right (and visa versa). Thus, we have proved the Pythagorean theorem using calculus!

(d) Show that, for a fixed perimeter, the triangle with the largest area is equilateral.

Solution: Fixing $p = 2s = a+b+c$, we seek the maximum of the function $A(a, b) := \mathsf{Area}(a, b, p-a-b)$. We extremize in both variables

$$\partial_b A(a, b) = \frac{p(p - 2a)(p - 2b - a)}{8A(a, b)}.$$

Since $p > 0$, a critical point $\partial_b A(a, b) = 0$ happens provided either $p = 2a$ or $p = 2b + a$. But the former, $p = 2a$ implies that $a = b + c$, impossible for a triangle (which famously always satisfies this with an inequality). Thus we conclude our interest is the critical point $p = 2b + a$ or $b = \frac{1}{2}(p - a)$. Letting now $A(a) = A(a, \frac{1}{2}(p - a))$, we again extremize

$$A'(a) = \frac{p(p - 3a)}{16A(a)}.$$

We find $A'(a) = 0$ implies $p = 3a$, so that $b = \frac{1}{2}(p - a) = \frac{p}{3}$ and $c = p - a - b = \frac{p}{3}$, an equilateral.

(3) (a) Let $\vec{u}, \vec{v} \in \mathbb{R}^2$ such that they are not parallel, describe the set of vectors $s\vec{u} + t\vec{v}$ where $s + t = 1$. Where are the vectors when s and t are both non-negative?

Solution: First experiment with $\vec{u} = \hat{\imath}$ and $\vec{v} = \hat{\jmath}$, $s = x$ and $t = y$. Can you guess what the set is?

$$s = 1 - t, \qquad s\vec{u} + t\vec{v} = \vec{u} + t(\vec{v} - \vec{u})$$

describes a line through $\vec{u}$ in the direction of $\vec{v} - \vec{u}$. When $t \in [0, 1]$, this is simply the line segment from $\vec{u}$ to $\vec{v}$.

(b) Suppose that $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ do not lie in the same plane. Describe the set of vectors $r\vec{u} + s\vec{v} + t\vec{w}$ where $r, s, t \in \mathbb{R}$ and $r + s + t = 1$. Where are the vectors when r, s and t are all non-negative?

Solution: We have

$$r\vec{u} + s\vec{v} + t\vec{w} = r\vec{u} + s\vec{v} + (1 - r - s)\vec{w} = \vec{w} + r(\vec{u} - \vec{w}) + s(\vec{v} - \vec{w})$$

span the plane determined by the three points $\vec{u}, \vec{v}, \vec{w}$ when $r, s, t \in \mathbb{R}$.
If $r, s, t \geq 0$, then $r + s \leq 1$. Recall from the previous problem that $\vec{w} + r(\vec{u} - \vec{w}) + (1 - r)(\vec{v} - \vec{w})$ consists of the line segment from $\vec{u}$ to $\vec{v}$, we see that $\vec{w} + r(\vec{u} - \vec{v}) + s(\vec{v} - \vec{w})$ is the triangle (convex hull) bounded by the three points $\vec{u}, \vec{v}, \vec{w}$.

(4) Four vectors are erected perpendicularly to the four faces of a general tetrahedron, each vector is pointing outward and has length equal to the area of the face. Show that the sum of these four vectors is $\vec{0}$. This fact is a precursor to Stokes' theorem.

Solution: Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be vectors representing the three edges starting from a fixed vertex. The other three vectors along the edges of the tetrahedron are $\vec{v}_2 - \vec{v}_1, \vec{v}_3 - \vec{v}_2, \vec{v}_3 - \vec{v}_1$. The four perpendiculars are

$$\vec{v}_3 \times \vec{v}_2, \quad \vec{v}_1 \times \vec{v}_3, \quad \vec{v}_2 \times \vec{v}_1, \quad (\vec{v}_2 - \vec{v}_1) \times (\vec{v}_3 - \vec{v}_1).$$

One now checks that their sum is $\vec{0}$.

(5) Let $\vec{u}, \vec{v}, \vec{w}$ be three vectors that are not co-planar, namely

$$\alpha\vec{u} + \beta\vec{v} + \gamma\vec{w} = \vec{0} \qquad \text{if and only if} \qquad \alpha = \beta = \gamma = 0.$$

(a) Show that $\vec{u} \times \vec{v}, \vec{v} \times \vec{w}, \vec{w} \times \vec{u}$ are not co-planar.

Solution: Suppose

$$\alpha\vec{u} \times \vec{v} + \beta\vec{v} \times \vec{w} + \gamma\vec{w} \times \vec{u} = \vec{0}$$

Dotting it with $\vec{u}, \vec{v}, \vec{w}$ respectively, and recalling $V(\vec{u}, \vec{v}, \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w} \neq 0$ is the signed volume of the parallelepiped spanned the $\vec{u}, \vec{v}, \vec{w}$, we get that $\alpha = \beta = \gamma = 0$.

(b) Suppose $a, b, c \in \mathbb{R}$, find the point of intersections of the three planes

$$\vec{u} \cdot (x, y, z) = a, \qquad \vec{v} \cdot (x, y, z) = b, \qquad \vec{w} \cdot (x, y, z) = c.$$

Express the solution (x, y, z) as $(x, y, z) = \alpha\vec{v} \times \vec{w} + \beta\vec{w} \times \vec{u} + \gamma\vec{u} \times \vec{v}$.

Solution: Dotting with $\vec{u}, \vec{v}, \vec{w}$, we have $\alpha = \frac{a}{V(\vec{u},\vec{v},\vec{w})}$, $\beta = \frac{b}{V(\vec{u},\vec{v},\vec{w})}$, $\gamma = \frac{c}{V(\vec{u},\vec{v},\vec{w})}$.

(6) This problem illustrates some counterintuitive features of high dimensional space. Consider a pyramid in $\mathbb{R}^n$ with vertices at the origin O and $\hat{e}_1, \hat{e}_2, \cdots, \hat{e}_n$. The base of the pyramid is the $(n-1)$-dimensional object B defined by

$$B = \left\{ (x_1, x_2, \cdots, x_n) \Big| \sum_{i=1}^{n} x_i = 1, x_i \geq 0 \text{ for all } i \right\}.$$

(a) Find the coordinates of the point (called the centroid of B) C in the base B which is equidistant from each vertex of B and calculate the length $\|\overrightarrow{OC}\|$.

$$C = \frac{1}{n}(1, 1, \cdots, 1).$$

(b) Use the Cauchy-Schwarz inequality to show that C is the closest point in B to the origin O.

Solution: We have, by Cauchy-Schwarz,

$$\|(x_1, x_2, \cdots, x_n)\|^2 \|(1, \cdots, 1)\|^2 \geq |(x_1, \cdots, x_n) \cdot (1, \cdots, 1)|^2 = 1$$

with equality if and only if $(x_1, \cdots, x_n) = \lambda(1, \cdots, 1)$, i.e.

$$x_1 = \cdots = x_n = \frac{1}{n}$$

(c) Calculate the angle θ between $\overrightarrow{OC}$ and any edge $\overrightarrow{OV_i}$, where V_i is the vertex corresponding to $\hat{e}_i$. What happens to this angle θ and the length $\|\overrightarrow{OC}\|$ as $n \to \infty$?

$$\cos\theta = \frac{(1, 0, \cdots, 0) \cdot \frac{1}{n}(1, 1, \cdots, 1)}{\|(1, 0, \cdots, 0)\| \|\frac{1}{n}(1, 1, \cdots, 1)\|} = \frac{\frac{1}{n}}{\frac{1}{\sqrt{n}}} = \frac{1}{\sqrt{n}} \to 0, \qquad \theta \to \frac{\pi}{2}$$