(1) Find the surface area of the portion of the paraboloid $z = 9 - x^2 - y^2$ that lies over the $z = 0$ plane.

We parametrize the surface as $z = f(x, y)$, $f(x, y) = 9 - x^2 - y^2$ over the xy–plane. The region D over which this graph resides is bounded by the intersection of the surface $z = f(x, y)$ and $z = 0$. In other words, the set of (x, y) such that $9 - x^2 - y^2 = 0$, or a circle of radius 3. Then

$$\iint_S 1dS = \iint_D \sqrt{1 + \partial_x f^2 + \partial_y f^2}dA = \iint_D \sqrt{1 + (-2x)^2 + (-2y)^2}dA = \iint_D \sqrt{1 + 4(x^2 + y^2)}dA.$$

To evaluate this integral, we use polar coordinates. It is

$$\int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}rdrd\theta = 2\pi \int_0^3 \sqrt{1 + 4r^2}rdr = 2\pi \left[\frac{1}{12}(1 + 4r^2)^{3/2}\right]_0^3 = \frac{\pi}{6}(37^{3/2} - 1).$$

(2) (a) Let $f : [1, \infty) \to [0, \infty)$ be a continuously differentiable function. Let S be the surface of revolution obtained by revolving the graph of $y = f(x)$ around the x–axis. Recall that the volume enclosed and surface area are:

$$\text{Vol} = \pi \int_1^\infty f(x)^2 dx, \qquad \text{Area} = 2\pi \int_1^\infty f(x)\sqrt{1 + f'(x)^2}dx.$$

Suppose that $f(x) \leq \mathsf{M}$ for some finite $\mathsf{M} > 0$. Show that, if the surface area is finite, then so is the volume enclosed.

Let A be the surface area of the graph.

$$\text{Vol} = \int_1^\infty f(x)(\pi f(x))dx \leq \frac{\mathsf{M}}{2} \int_1^\infty 2\pi f(x)dx \leq \frac{\mathsf{M}}{2} \int_1^\infty 2\pi f(x)\sqrt{1 + f'(x)^2}dx = \frac{\mathsf{M}}{2}\text{Area}.$$

(b) (Torricelli's trumpet) Let $f(x) = 1/x$ on $[1, \infty)$, revolve the graph of $f(x)$ around the x–axis, we get a trumpet–shaped surface. Find the volume and surface area.

The volume is

$$\text{Vol} = \int_1^\infty f(x)(\pi f(x))dx = \pi \int_1^\infty \frac{dx}{x^2} = \pi.$$

On the other hand, the surface area is

$$\text{Area} = 2\pi \int_1^\infty f(x)\sqrt{1 + f'(x)^2}dx = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}dx > 2\pi \int_1^\infty \frac{dx}{x} = 2\pi \log(x)\Big|_1^\infty = \infty.$$

Torricelli's trumpet is a surface of finite volume and infinite area! You can fill the trumpet with a finite amount of paint, but it requires infinite amount of paint to cover the surface!

(3) Evaluate the integral $\iint_S \vec{F} \cdot dS$ where $\vec{F} = (x, y, 1)$ and S is the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$.

The surface is a graph: $z = f(x, y)$, $f(x, y) = \sqrt{1 - x^2 - y^2}$ sitting above the unit disc D in the xy–plane. Note $f_x = -x(1 - x^2 - y^2)^{-1/2}$, $f_y = -y(1 - x^2 - y^2)^{-1/2}$. Thus

$$\begin{aligned}\iint_S \vec{F} \cdot dS &= \iint_D F(x, y, f(x, y)) \cdot (-f_x, -f_y, 1)dA = \iint_D \left(\frac{x^2 + y^2}{\sqrt{1 - x^2 - y^2}} + 1\right) dA \\ &= \iint_D \frac{x^2 + y^2}{\sqrt{1 - x^2 - y^2}}dA + \pi\end{aligned}$$

since π is the area of the unit disc. We compute the other integral in polar coordinates.

$$\iint_D \frac{x^2+y^2}{\sqrt{1-x^2-y^2}}dA = \int_0^{2\pi}\int_0^1 \frac{r^2}{\sqrt{1-r^2}}rdt = 2\pi\left(\frac{1}{2}\int_0^1 \frac{1-u}{\sqrt{u}}du\right) = \frac{4\pi}{3}.$$

Thus the flux is $\frac{4\pi}{3}+\pi=\frac{7\pi}{3}$.

(4) Let B be the solid ball of radius 1 given by

$$x^2+y^2+z^2 \leq 1.$$

Evaluate the following integrals.

Hint: You can compute each integral independently and in any order. The principle of symmetry may play an important role in each part!

(a) $\int\int\int_B (x^{2023}+y^{2023}+z^{2023})dV$

Each term is an odd function on B, by symmetry,

$$\int\int\int_B (x^{2023}+y^{2023}+z^{2023})dV = 0$$

(b) $\int\int\int_B (x^2+y^2+z^2-xy-yz-xz)dV$

$$\begin{aligned}\int\int\int_B (x^2+y^2+z^2-xy-yz-zx)dV &= \int\int\int_B (x^2+y^2+z^2)dV - 2\int\int\int_B (xy+yz+zx)dV \\ &= \int\int\int_B (x^2+y^2+z^2)dV = \int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^4\sin\phi d\rho d\phi d\theta = \frac{4}{5}\pi\end{aligned}$$

Note that by symmetry, the integral of the terms xy, yz, xz are all 0.

(c) $\int\int\int_B (x^{2n}+y^{2n}+z^{2n})dV$ where n is a positive integer.

Use symmetry of x, y, z:

$$\begin{aligned}\int\int\int_B (x^{2n}+y^{2n}+z^{2n})dV &= 3\int\int\int_B x^{2n}dV \qquad \text{by symmetry} \\ &= 3\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} x^{2n}dzdydx \\ &= 6\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}x^{2n}dydx \\ &= 6\int_{-1}^1 \frac{\pi(1-x^2)}{2}x^{2n}dx \\ &= 6\pi\left(\frac{1}{2n+1}-\frac{1}{2n+3}\right) = \frac{12\pi}{(2n+1)(2n+3)}.\end{aligned}$$

Alternatively,

$$\begin{aligned}\int\int\int_B (x^{2n}+y^{2n}+z^{2n})dV &= 3\int\int\int_B z^{2n}dV \\ &= 3\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^{2n+2}\cos^{2n}\phi\sin\phi d\rho d\phi d\theta \\ &= \frac{6\pi}{(2n+3)(2n+1)}\cos^{2n+1}\phi\Big|_{\pi}^{0} = \frac{12\pi}{(2n+1)(2n+3)}.\end{aligned}$$

(5) Use Stoke's theorem to evaluate the integral $\oint_C \vec{F}\cdot d\vec{r}$ where $\vec{F}=(-xy,-xz,-yz)$ and C is the triangle with vertices $(0,1,0)$, $(0,1,5)$ and $(3,1,0)$ oriented by taking the vertices in that order.

Let S be the solid triangle bounded by C. Since it is oriented by the order of the points, this gives a normal vector that points along the positive y-axis (draw figure, use right hand rule). Thus $\vec{n} = (0, 1, 0)$. Since $\vec{F}$ is differentiable, Stokes theorem tells that the line integral in equation equals the $\iint_S \text{curl}\vec{F} \cdot \hat{n} dS$. The curl is computed to be $\text{curl}\vec{F} = (x - z, 0, z - x)$. Thus $\text{curl}\vec{F} \cdot \hat{n} = 0$ so $\oint_C \vec{F} \cdot d\vec{r} = 0$.

(6) (Challenge) Find the probability that three random numbers chosen uniformly from $[0, 1]$ represent the side lengths of some triangle. (Yes, this is a Calc III question).

Hint: If the three numbers are x, y and z, they are sides of a triangle if and only if

$$x + y \geq z, \qquad x + z \geq y, \qquad \text{or} \qquad y + z \geq x.$$

Find the probability that $0 \leq x \leq y \leq z \leq 1$ and $x + y \geq z$ first. Note that when $x \geq \frac{1}{2}$, $x + y \geq 1$; and when $x \leq \frac{1}{2}$, $x + y \leq 1$ if $y \leq 1 - x$ and $x + y \geq 1$ if $y \geq 1 - x$. Then consider the other 5 possibilities similarly: $x \leq z \leq y$, $y \leq x \leq z$, $y \leq z \leq x$, $z \leq x \leq y$, and $z \leq y \leq x$.

The probability that $0 \leq x \leq y \leq z \leq 1$ and $x + y \geq z$ is

$$\int_0^{1/2} \int_x^{1-x} \int_y^{x+y} 1 dz dy dx + \int_0^{1/2} \int_{1-x}^1 \int_y^1 1 dz dy dx + \int_{1/2}^1 \int_x^1 \int_y^1 1 dz dy dx$$

$$= \frac{1}{24} + \frac{1}{48} + \frac{1}{48} = \frac{1}{12}$$

Consider the other 5 possibilities similarly: $x \leq z \leq y$, $y \leq x \leq z$, $y \leq z \leq x$, $z \leq x \leq y$, and $z \leq y \leq x$, we have that the three numbers form the three sides of a triangle with probability $\frac{6}{12} = \frac{1}{2}$.