MAT 307, Multivariable Calculus with Linear Algebra, Fall 2024 — Homework 12

(1) (a) Find the surface area of the portion of the paraboloid $z = 9 - x^2 - y^2$ that lies over the $z = 0$ plane.
(b) Find the volume of the solid that is bounded by the paraboloid $z = 9 - x^2 - y^2$, the xy–plane and the cylinder $x^2 + y^2 = 4$.

(2) (a) Let $f : [1, \infty) \to [0, \infty)$ be a continuously differentiable function. Let S be the surface of revolution obtained by revolving the graph of $y = f(x)$ around the x–axis. Recall that the volume enclosed and surface area are:

$$\text{Vol} = \pi \int_1^\infty f(x)^2 dx, \qquad \text{Area} = 2\pi \int_1^\infty f(x)\sqrt{1 + f'(x)^2} dx.$$

Suppose that $f(x) \leq \mathsf{M}$ for some finite $\mathsf{M} > 0$. Show that, if the surface area is finite, then so is the volume enclosed.

(b) (Torricelli's trumpet) Let $f(x) = 1/x$ on $[1, \infty)$, revolve the graph of $f(x)$ around the x–axis, we get a trumpet–shaped surface. Find the volume and surface area. Is the result surprising?

(3) Evaluate the integral $\iint_S \vec{F} \cdot dS$ where $\vec{F} = (x, y, 1)$ and S is the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$.

(4) Let B be the solid ball of radius 1 given by

$$x^2 + y^2 + z^2 \leq 1.$$

Evaluate the following integrals.
Hint: You can compute each integral independently and in any order. The principle of symmetry may play an important role in each part!
(a) $\int\int\int_B (x^{2025} + y^{2025} + z^{2025})dV$
(b) $\int\int\int_B (x^2 + y^2 + z^2 - xy - yz - xz)dV$
(c) $\int\int\int_B (x^{2n} + y^{2n} + z^{2n})dV$ where n is a positive integer.

(5) Use Stoke's theorem to evaluate the integral $\oint_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = (-xy, -xz, -yz)$ and C is the triangle with vertices $(0, 1, 0)$, $(0, 1, 5)$ and $(3, 1, 0)$ oriented by taking the vertices in that order.

(6) [1] Find the probability that three random numbers chosen uniformly from $[0, 1]$ represent the side lengths of some triangle. (Yes, this is a Calc III question).

Hint: If the three numbers are x, y and z, they are sides of a triangle if and only if

$$x + y \geq z, \qquad x + z \geq y, \qquad \textit{or} \qquad y + z \geq x.$$

Find the probability that $0 \leq x \leq y \leq z \leq 1$ and $x + y \geq z$ first. Note that when $x \geq \frac{1}{2}$, $x + y \geq 1$; and when $x \leq \frac{1}{2}$, $x + y \leq 1$ if $y \leq 1 - x$ and $x + y \geq 1$ if $y \geq 1 - x$. Then consider the other 5 possibilities similarly: $x \leq z \leq y$, $y \leq x \leq z$, $y \leq z \leq x$, $z \leq x \leq y$, and $z \leq y \leq x$.

[1] Not to appear on the exam.