(1) Use Green's theorem with $\vec{F} = \frac{1}{2}(-y, x)$ to show that the area of an n–sided polygon D in the xy–plane with vertices $(x_1, y_1), \dots (x_n, y_n)$ is given by

$$\text{Area}(D) = \frac{1}{2}\sum_{i=1}^{n} x_i y_{i+1} - x_{i+1} y_i,$$

where, in the above formula, we use the convention that $(x_{n+1}, y_{n+1}) = (x_1, y_1)$. This is how area is evaluated in computer graphics very efficiently, with no integration involved.

Note that $\text{curl}\vec{F} = 1$. Thus, by Green's theorem

$$\text{Area}(D) = \iint_D \text{curl}\vec{F}\, dA = \oint_C \vec{F} \cdot d\vec{r},$$

where C is the boundary of the polygon D. The line from (x_i, y_i) to (x_{i+1}, y_{i+1}) can be represented by the parametrization

$$\vec{r}(t) = (x_i, y_i) + t(x_{i+1} - x_i, y_{i+1} - y_i), \qquad t \in [0, 1].$$

Thus, along that piece of segment, a computation shows

$$\oint_{C_i} \vec{F} \cdot d\vec{r} = \oint_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt = \frac{1}{2}(x_i y_{i+1} - x_{i+1} y_i).$$

The result follows by summing up all contributions.

(2) Consider a torus, whose radius from the center of the hole to the center of the torus tube is a, and the radius of the tube is b. Consider therefore that $a > b$. Then the equation in Cartesian coordinates for a torus azimuthally symmetric about the z-axis is

$$(\sqrt{x^2 + y^2} - a)^2 + z^2 = b^2.$$

To derive this, one can think of the torus as a surface of revolution generated by rotating the circle $(y - a)^2 + z^2 = b^2$ around the z-axis. Find its surface area in terms of a and b.

You can find the area of the torus using the fact that

$$z = f(x, y) = \pm\sqrt{b^2 - (\sqrt{x^2 + y^2} - a)^2}, \qquad r = \sqrt{x^2 + y^2}, \qquad f_x = -\frac{(r - a)x}{\sqrt{b^2 - (r - a)^2}r}$$

$$\begin{aligned} A(S) &= 2\int\int_D \sqrt{1 + \frac{(r - a)^2}{b^2 - (r - a)^2}} dA \\ &= 2\int_0^{2\pi}\int_{a-b}^{a+b} \frac{b}{\sqrt{b^2 - (r - a)^2}} r dr d\theta \\ &= 4\pi b \int_{-b}^{b} \frac{s + a}{\sqrt{b^2 - s^2}} ds \qquad s = r - a \\ &= 4\pi b \int_{-b}^{b} \frac{a}{\sqrt{b^2 - s^2}} ds \qquad \text{by symmetry} \\ &= 4\pi ab \arcsin \frac{s}{b}\Big|_{-b}^{b} = 4\pi^2 ab. \end{aligned}$$

(3) Find the flux of $\vec{F} = (yz, xz, xy)$ over the surface S which is the graph of $z = x^2 + y^2$ over the unit disk centered at the origin.

$$\int\int_{\Sigma} \vec{F} \cdot d\vec{S} = \int\int_{D} (yz, xz, xy) \cdot (-2x, -2y, 1) dxdy$$
$$= \int\int_{D} (-4xy(x^2+y^2) + xy) dxdy$$
$$= \int_0^{2\pi} \int_0^1 -r^2 \frac{\sin 2\theta}{2}(1-4r^2) r dr d\theta = 0$$

(4) Find the flux of $\vec{F}(x,y,z) = 4x\vec{i} + 4y\vec{j} + 2\vec{k}$ outward (away from the z-axis) through the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 1$.

The surface is defined by the equation $\varphi(x,y,z) = x^2 + y^2 - z = 0$. And the constraint is $z \leq 1$¿. Since $\nabla\varphi = (2x, 2y, -1)$, the normal vector is

$$\vec{n} = \frac{1}{\sqrt{4x^2+4y^2+1}}(2x, 2y, -1).$$

Thus $\vec{F} \cdot \vec{n} = \frac{8x^2+8y^2-2}{\sqrt{4x^2+4y^2+1}}$. Thus the integral to be computed is

$$\iint_S \vec{F} \cdot dS = \iint_{R=\{x^2+y^2\leq 1\}} \frac{8x^2+8y^2-2}{\sqrt{4x^2+4y^2+1}} \sqrt{4x^2+4y^2+1} dxdy$$
$$= \int_0^{2\pi} \int_0^1 (8r^2-2) r dr d\theta = 2\pi.$$

(5) Let S be the surface defined by the portion of the plane $x + y + z = -1$ satisfying $0 \leq x \leq 1$ and $0 \leq y \leq 1$ and oriented so that the normal to the plane is pointing "up". Find the flux of the vector field $\vec{F} = (x, -2y, xz)$ across the surface S.

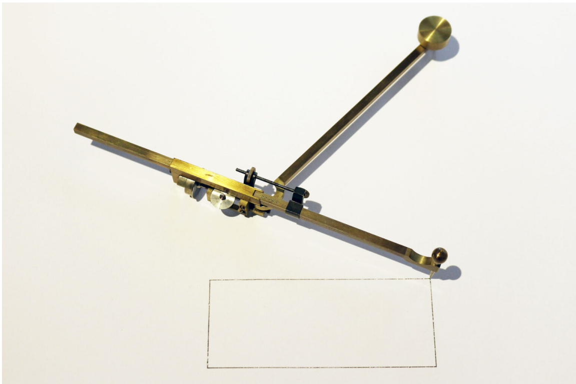
We can write this piece of the plane as a graph over the xy–plane. $z = f(x,y) := -1 - x - y$ for $(x,y) \in [0,1] \times [0,1]$. This parametrization orients the surface with an upward pointing normal. The normal is

$$\vec{n} = \frac{(-f_x, -f_y, 1)}{\sqrt{f_x^2 + f_y^2 + 1}} = \frac{(1,1,1)}{\sqrt{3}}$$

The flux is

$$\iint_S F \cdot dS = \iint_D (x, -2y, xf(x,y)) \cdot (1,1,1) dxdy = -\frac{19}{12}.$$

(6) Archimedes famously found a way to determine the volume of an irregular solid; over two thousand years elapsed before a method to find the area of an irregular region was discovered. A **planimeter** is a mechanical device based on Green's Theorem for measuring the area of a region in the plane. A planimeter



has the shape of a ruler with two arms. One arm of length c is anchored at the origin $(0,0)$ and ends at a point (a, b); a second arm of length c connect (a, b) to (x, y), a point on the boundary C of the region R whose area we want to measure. (a, b) is the elbow of the planimeter, and its position depends on (x, y). We require that the angle between the two arms to be less than π, and hence (a, b) is the unique intersection of the two circles of radius c centered at $(0, 0)$ and (x, y) respectively. The position of (a, b) depends on the position (x, y). The measurement consists of tracing the end point of the second arm (x, y) along the boundary C of region R. A wheel is attached to the second arm that measures the motion of (x, y) in the direction perpendicular to the arm from (a, b) to (x, y): the axis of the wheel is parallel to $\langle x - a, y - b\rangle$, and the turning of the wheel measure motion perpendicular to $\langle x - a, y - b\rangle$. After completing the path along C, the total wheel rotation indicates the area of the region R. As the planimeter traces out the curve C, the wheel integrates along C the vector field perpendicular to $\langle x - a, y - b\rangle$.

Define the planimeter vector field

$$\vec{F}(x, y) = \frac{1}{c} \langle -y + b(x, y), x - a(x, y)\rangle ,$$

which is a unit length vector field. Note that the rate at which the wheel turns doesn't depend on the distance from (x, y) to the origin. The wheel rotation evaluates the line integral

$$\oint_C \vec{F} \cdot d\vec{r}.$$

This line integral can be used to find the area of the region R as follows:

(a) (a, b) is the unique intersection of the two circles of radius c centered at $(0, 0)$ and (x, y) rspt., i.e.

$$a^2 + b^2 = c^2, \qquad (x - a)^2 + (y - b)^2 = c^2.$$

Differentiate these two equations with respect to x and y to show that

$$\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} = 1.$$

Note that c is constant,

$$2aa_x + 2bb_x = 0, \qquad 2(x - a)(1 - a_x) - 2(y - b)b_x = 0$$
$$2aa_y + 2bb_y = 0, \qquad -2(x - a)a_y + 2(y - b)(1 - b_y) = 0$$

Eliminate a_y and b_x, we have

$$a_y = -\frac{b}{a} b_y, \qquad b_x = \frac{x - a}{y - b}(1 - a_x)$$

$$2aa_x + 2b\frac{x - a}{y - b}(1 - a_x) = 0, \qquad 2\frac{b}{a} b_y(x - a) + 2(y - b)(1 - b_y) = 0$$

$$a_x + b_y = \frac{b\frac{x-a}{y-b}}{b\frac{x-a}{y-b} - a} + \frac{y - b}{(y - b) - \frac{b}{a}(x - a)} = \frac{b(x - a) - a(y - b)}{b(x - a) - a(y - b)} = 1$$

(b) Now use Green's Theorem to show that

$$\oint_C \vec{F} \cdot d\vec{r} = \frac{1}{c} Area(R).$$

The dial at the planimeter is calibrated to adjust for the factor $\frac{1}{c}$.

By Green's theorem,

$$\oint_C \vec{F} \cdot d\vec{r} = \int\int_R curl(\vec{F}) dA = \frac{1}{c}\int\int_R (2 - a_x - b_y)\, dA = \frac{1}{c} Area(R).$$