M&T Sections: 5.3, 6.2, 8.1

(1) Find the volume of the region boundary by $x^2+y^2=1$, $x=z$ and $z=0$. This region is known as the *hoof of Archimedes*.
The floor of hoof is the half disk D with $x^2+y^2\le 1$ and $x\ge 0$ and roof is $z=x$. Then
$$\iint_D x\,dA=\int_{-\pi/2}^{\pi/2}\int_0^1 r^2\cos\theta drd\theta=\frac{1}{3}\int_{-\pi/2}^{\pi/2}\cos\theta d\theta=\frac{2}{3}.$$

(2) Evaluate the following integrals using polar coordinates
- $I_1=\int_0^1\int_0^{\sqrt{1-y^2}}(x^2+y^2)dxdy$,
- $I_2=\int_{\sqrt{2}}^2\int_{\sqrt{4-y^2}}^y dxdy$,
- $I_3=\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{2}{(1+x^2+y^2)^2}dydx$.

$$I_1=\int_0^1\int_0^{\sqrt{1-y^2}}(x^2+y^2)dxdy=\int_0^{\pi/2}\int_0^1 r^3drd\theta=\frac{\pi}{8}$$
$$I_2=\int_{\sqrt{2}}^2\int_{\sqrt{4-y^2}}^y dxdy=\int_{\pi/4}^{\pi/2}\int_2^{2\csc\theta}rdrd\theta=\int_{\pi/4}^{\pi/2}(2\csc^2\theta-2)d\theta=(-2\cot\theta-2\theta)\Big|_{\pi/4}^{\pi/2}=2-\frac{\pi}{2}$$
$$I_3=\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{2}{(1+x^2+y^2)^2}dydx=4\int_0^{\pi/2}\int_0^1\frac{2r}{(1+r^2)^2}drd\theta=\pi.$$

(3) Use Green's theorem to compute $\oint_C\vec F\cdot d\vec r$ if $\vec F=(\arctan(y/x),\log(x^2+y^2))$ along the curve C given by the boundary of the region defined by the polar inequalities $1\le r\le 2$ and $0\le\theta\le\pi/2$, oriented counterclockwise.

Note that $\partial_x\log(x^2+y^2)=\frac{2x}{x^2+y^2}$, and $\partial_y\arctan(y/x)=\frac{x}{x^2+y^2}$ Thus
$$\oint_C\vec F\cdot d\vec r=\iint_D\left(\frac{2x}{x^2+y^2}-\frac{x}{x^2+y^2}\right)dxdy=\int_0^{\pi/2}\int_1^2 r\cos\theta drd\theta=\int_0^{\pi/2}\cos\theta d\theta=1.$$

(4) Find the work done by the force $\vec F=(4x-2y,2x-4y)$ on a particle going counterclockwise around the circle C: $(x-2)^2+(y-2)^2=4$.

Let $P(x,y)=4x-2y$ and $Q(x,y)=2x-4y$. Then $\partial_yP=-2$ and $\partial_xQ=2$. Thus
$$\oint_C\vec F\cdot d\vec r=\iint_D(2-(-2))dA=4\text{Area(circle of radius 2)}=16\pi.$$

(5) Show that the value of $\oint_C xy^2dx+(x^2y+2x)dy$ around any square C depends only on the area enclosed and not on its location in the plane.

Let D be the interior of any given square C. Let $P(x,y)=xy^2$ and $Q(x,y)=x^2y+2x$. Then $\partial_yP=2xy$ and $\partial_xQ=2xy+2$. Thus we find the integral is proportional to the area enclosed only:
$$\oint_C\vec F\cdot d\vec r=\iint_D(2xy+2-2xy)dA=4\text{Area(square)}.$$