(1) Using the dot product, prove the Pythagorean theorem. Namely, show that if the lengths $a, b, c$ of the sides of a triangle satisfy $a^2 + b^2 = c^2$, then the triangle is a right triangle and visa verse.

(2) Let $\vec{v}$ and $\vec{u}$ be vectors in $\mathbb{R}^2$ and recall that the area of the parallelogram defined by $\vec{v}$ and $\vec{u}$ is given by $A(\vec{v}, \vec{u}) = \frac{1}{2}|\vec{v}^\perp \cdot \vec{u}|$ where $\vec{v}^\perp = (-v_2, v_1)$ denotes counterclockwise rotation of the vector $\vec{v}$ by $90°$.

(a) Show that $(\vec{v}^\perp \cdot \vec{u})^2 = \|\vec{v}\|^2\|\vec{u}\|^2 - (\vec{v} \cdot \vec{u})^2$. Explain what this means geometrically.

(b) Use this fact to establish *Heron's formula* (most probably due to Archimedes*), namely that a triangle with sides lengths $a, b$ and $c$ has area

$$\mathsf{Area}(a, b, c) = \sqrt{s(s-a)(s-b)(s-c)}, \qquad s := \frac{1}{2}(a + b + c).$$

To do this, you will need to relate to $(\vec{v} \cdot \vec{u})^2$ with magnitudes of $\vec{v}$, $\vec{u}$ and $\vec{v} - \vec{u}$ (which should describe the sides of the triangle). There is also some algebra.

(c) Show that, for fixed side lengths $a$ and $b$, the triangle with the largest area is right.

(d) † Show that, for a fixed perimeter, the triangle with the largest area is equilateral.

(3) (a) Let $\vec{u}, \vec{v} \in \mathbb{R}^2$ such that they are not parallel, describe the set of vectors $s\vec{u} + t\vec{v}$ where $s + t = 1$. Where are the vectors when $s$ and $t$ are both non-negative?

(b) Suppose that $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ do not lie in the same plane. Describe the set of vectors $r\vec{u} + s\vec{v} + t\vec{w}$ where $r, s, t \in \mathbb{R}$ and $r + s + t = 1$. Where are the vectors when $r, s$ and $t$ are all non-negative?

(4) Four vectors are erected perpendicularly to the four faces of a general tetrahedron, each vector is pointing outward and has length equal to the area of the face. Show that the sum of these four vectors is $\vec{0}$. This fact is a precursor to Stokes' theorem.

(5) (a) Let $\vec{u}, \vec{v}, \vec{w}$ be three vectors that are not co-planar, namely $\alpha\vec{u} + \beta\vec{v} + \gamma\vec{w} = \vec{0}$ only if $\alpha = \beta = \gamma = 0$. Show that $\vec{u} \times \vec{v}, \vec{v} \times \vec{w}, \vec{w} \times \vec{u}$ are not co-planar.

(b) Suppose $a, b, c \in \mathbb{R}$, find the point of intersections of the three planes

$$\vec{u} \cdot (x, y, z) = a, \qquad \vec{v} \cdot (x, y, z) = b, \qquad \vec{w} \cdot (x, y, z) = c.$$

Express the solution $(x, y, z)$ as $(x, y, z) = \alpha\vec{v} \times \vec{w} + \beta\vec{w} \times \vec{u} + \gamma\vec{u} \times \vec{v}$ for some $\alpha, \beta, \gamma$.

(6) This problem illustrates some counterintuitive features of high dimensional space. Consider a pyramid in $\mathbb{R}^n$ with vertices at the origin $O$ and $\hat{e}_1, \hat{e}_2, \cdots, \hat{e}_n$. The base of the pyramid is the $(n-1)$-dimensional object $B$ defined by

$$B = \left\{ (x_1, x_2, \cdots, x_n) \middle| \sum_{i=1}^{n} x_i = 1, x_i \geq 0 \text{ for all } i \right\}.$$

(a) Find the coordinates of the point (called the centroid of $B$) $C$ in the base $B$ which is equidistant from each vertex of $B$ and calculate the length $\|\overrightarrow{OC}\|$.

(b) Use the Cauchy-Schwarz inequality to show that $C$ is the closest point in $B$ to the origin $O$.

(c) Calculate the angle $\theta$ between $\overrightarrow{OC}$ and any edge $\overrightarrow{OV_i}$, where $V_i$ is the vertex corresponding to $\hat{e}_i$. What happens to this angle $\theta$ and the length $\|\overrightarrow{OC}\|$ as $n \to \infty$?

(**Extra**) † *Marsden & Tromba: §1.1: #6, 14, 18, 34; §1.2: #6, 20, 26, 38; §1.R: #14, 20.*

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*See Vol. II, 321 of T. Heath. History of Greek Mathematics. Oxford University Press, 1921.

†Not to appear on quiz.