APRIL 20, 2022
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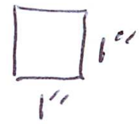
NOW WE TURN TO AREA.

FIRST WHAT DOES AREA MEAN?

1. DECIDE ON SOME BASIC UNIT, ~~AND DECLARE~~
EG 1 SQ INCH, AND DECLARE
A UNIT TO HAVE AREA 1.



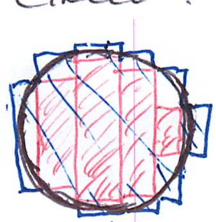
2. WE CAN THEN USE SOME BASIC IDEAS TO
GET AREAS OF RECTANGLES AND ~~OTH~~ POLYGONAL
FIGURES:
- IF $R^\circ \cap S^\circ = \emptyset$, $\text{AREA}(R \cup S) = \text{AREA}(R) + \text{AREA}(S)$.
- ~~WE~~ IF WE HAVE AN ISOMETRY (TRANSLATION, ROTATION, REFLECTION) IT DOESN'T CHANGE AREA.

(A LITTLE WORRY ABOUT BOUNDARIES...)

IF $R \subseteq S$, THEN $\text{AREA}(R) \leq \text{AREA}(S)$

SO ANY POLYGONAL FIGURE IS OK.

WHAT ABOUT A CIRCLE? ACTUALLY TRICKY.



- TILE THE INSIDE TO GET A LOWER BOUND
- COVER WITH TILES TO GET AN UPPER BOUND

REFINE THE TILINGS REPEATEDLY TO GET
BETTER & BETTER APPROXIMATIONS ... IE, TAKE A LIMIT.

NOTE THAT ALL POSSIBLE INNER TILINGS HAVE A COMMON
UPPER BOUND, IE $\sup(|\text{INNER}|) =$ "INNER CONTENT"

ALSO, ALL POSSIBLE OUTER TILINGS HAVE A
COMMON LOWER BOUND $\inf(|\text{OUTER}|) =$ "OUTER CONTENT"

IF THESE ARE EQUAL, THAT IS THE AREA.

DOES EVERY FIGURE HAVE AREA?

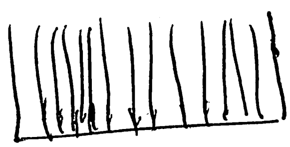
IN 1903, OSGOOD CONSTRUCTED A CONTINUOUS CLOSED CURVE THAT DOES NOT INTERSECT ITSELF (A JORDAN CURVE) WITH INNER CONTENT STRICTLY SMALLER THAN OUTER CONTENT.

HERE'S A SIMPLER EXAMPLE.

LET $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$

LET F BE THE REGION $F = \left\{ (x,y) \,\middle|\, \begin{matrix} 0 \le x \le 1 \\ 0 \le y \le f(x) \end{matrix} \right\}$

WHAT IS THE AREA OF F?



<u>THM</u>: A CLOSED, BOUNDED REGION WITH PIECEWISE SMOOTH BOUNDARY HAS AREA, i.e.

INNER CONTENT = OUTER CONTENT.

<u>DEF</u>: THE <u>MESH</u> OF A GRID IS THE MAXIMUM OF THE DIAMETERS OF THE RECTANGLES IN THE GRID.

THM: LET ~~$\mathcal{R}$~~ $\mathcal{R}$ ~~BE~~ BE A CLOSED, BOUNDED REGION WHICH HAS AREA. LET G_n BE A SEQUENCE OF GRIDS COVERING $\mathcal{R}$ SO THAT $\operatorname{mesh}(G_n) \to 0$ AS $n \to \infty$. THEN THE AREAS OF THE INNER OR OUTER POLYGONS OF G_n CONVERGE TO THE AREA OF $\mathcal{R}$.

NOW LETS TURN TO AREA UNDER CURVES. THE IDEA IS THE SAME.

DEF GIVEN AN INTERVAL $[a,b]$, A PARTITION OF $[a,b]$ IS A FINITE SET OF POINTS $x_0, \dots, x_n$ WITH $x_0 = a$, $x_n = b$, $x_0 < x_1 < x_2 < \dots < x_n$.

GIVEN $f : [a,b] \to \mathbb{R}$ AND A PARTITION OF $[a,b]$, FOR EACH k, DEFINE

$$m_k = \inf \{ f(x) \mid x \in [x_{k-1}, x_k] \}$$
$$M_k = \sup \{ f(x) \mid x \in [x_{k-1}, x_k] \}$$

THEN THE LOWER SUM OF f WRT P IS

$$L(f,P) = \sum_{k=1}^{n} m_k (x_k - x_{k-1})$$

AND THE UPPER SUM OF f WRT P IS

$$U(f,P) = \sum_{k=1}^{n} M_k (x_k - x_{k-1}).$$

(~~[illegible] THE~~ INNER

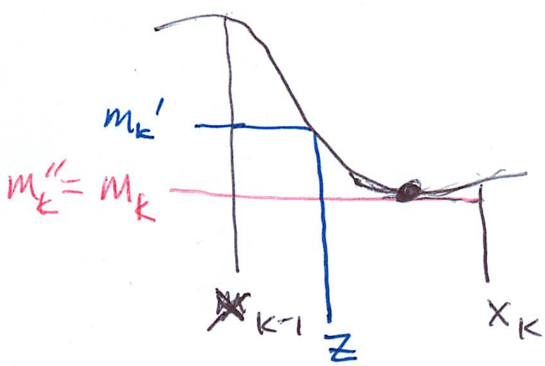
DEF A PARTITION Q ~~of P~~ IS A REFINEMENT OF P IF Q CONTAINS ALL POINTS OF P; IE $P \subseteq Q$

LEMMA IF $P \subseteq Q$, THEN $L(f,P) \leq L(f,Q)$
$U(f,P) \geq U(f,Q)$

(LOWER SUMS GET BIGGER
UPPER SUMS GET SMALLER)

Pf/ CONSIDER ADDING A POINT z TO $[x_{k-1}, x_k]$

$$m_k(x_k - x_{k-1}) = m_k(x_k - z) + m_k(z - x_{k-1})$$
$$\leq m_k'(x_k - z) + m_k''(z - x_{k-1}).$$



LEMMA FOR ANY TWO PARTITIONS P_1 & P_2 OF $[a,b]$,

$$L(f,P_1) \leq U(f,P_2).$$

Pf LET $Q = P_1 \cup P_2$ BE THE COMMON REFINEMENT.
THEN

$$L(f,P_1) \leq L(f,Q) \leq U(f,Q) \leq U(f,P_2)$$

<u>DEF</u> LET $\mathcal{P}$ BE THE COLLECTION OF ALL PARTITIONS OF $[a,b]$

~~DEF~~ THE <u>UPPER INTEGRAL OF F</u>

$$U(f) = \inf \{ U(f,P) \mid P \in \mathcal{P} \}$$

AND THE <u>LOWER INTEGRAL</u>

$$L(f) = \sup \{ L(f,P) \mid P \in \mathcal{P} \}.$$

<u>LEMMA</u> IF f IS BOUNDED ON $[a,b]$, $U(f) \geq L(f)$.

<u>DEF</u> A BOUNDED FUNCTION $f : [a,b] \to \mathbb{R}$ IS <u>RIEMANN INTEGRABLE</u> IF $U(f) = L(f)$. THEN

$$\int_a^b f = \int_a^b f(x)\,dx = U(f) = L(f)$$

<u>THM</u>: A BOUNDED FUNCTION f IS INTEGRABLE ON $[a,b]$

* $\iff$ FOR EVERY $\varepsilon > 0$, THERE IS A PARTITION P_ε OF $[a,b]$ SUCH THAT $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$.

<u>THM</u>: IF f IS CONTINUOUS ON $[a,b]$, IT IS INTEGRABLE ON $[a,b]$

Pf OF *

$\Leftarrow$/ LET $\varepsilon > 0$. IF SUCH A P_ε EXISTS,

$$U(f) - L(f) \leq U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon.$$

BUT SINCE ε IS ARBITRARY, $U(f) = L(f)$.

$\Rightarrow$/ SINCE $U(f)$ IS THE GREATEST LOWER BOUND OF UPPER SUMS, THERE MUST BE SOME PARTITION P_1

SO THAT $U(f, P_1) \leq U(f) + \varepsilon/2$.

SIMILARLY $L(f, P_2) \geq L(f) - \varepsilon/2$.

TAKE $P_\varepsilon = P_1 \cup P_2$.

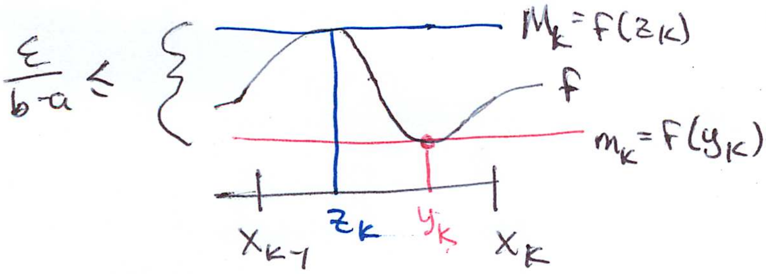
WRITE
$$\begin{aligned} U(f, P_\varepsilon) - L(f, P_\varepsilon) &\leq U(f, P_1) - L(f, P_2) \\ &\leq \left(U(f) + \varepsilon/2\right) - \left(L(f) - \varepsilon/2\right) \\ &= \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Pf OF INTEGRABILITY OF CONTINUOUS:

SINCE f IS CONTINUOUS ON A COMPACT SET, IT IS BOUNDED AND UNIFORMLY CONTINUOUS. SO GIVEN $\varepsilon > 0$, THERE IS δ SO THAT $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a}$

NOW LET P BE A PARTITION OF $[a, b]$ SO THAT $x_k - x_{k-1} < \delta$ FOR ALL k.

ON EACH INTERVAL, THE MAX M_k IS ATTAINED AT z_k,



MIN m_k IS ATTAINED AT y_k

WITH $M_k - m_k \leq \frac{\varepsilon}{b-a}$.

ADD EM UP! $< \varepsilon$.