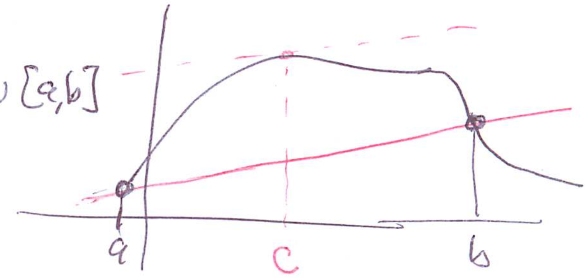
4/18/22 ①

LAST TIME:

MEAN VALUE THEOREM:

LET $f:[a,b]\to\mathbb{R}$ BE CONTINUOUS ON $[a,b]$ AND DIFF ON (a,b). THEN THERE IS A $c\in(a,b)$ WITH

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$



PROOF FOLLOWS FROM ROLLE'S THM.

COR: IF $g:A\to\mathbb{R}$ IS DIFF ON AN INTERVAL A AND $g'(x)=0$ FOR ALL $x\in A$, THEN g IS CONTANT ON A.

Pf/ TAKE ANY $x,y\in A$ WITH $x<y$.

MVT $\Rightarrow$ ~~FOR EVERY~~ THERE IS $c\in(x,y)$ WITH

$$g'(c)=\frac{g(x)-g(y)}{x-y}.$$

BUT $g'(c)=0$, SO $g(x)=g(y)$. SINCE x,y WERE ARBITRARY, g IS CONSTANT ON A. ∎

CAUCHY'S GENERALIZED MVT: LET f,g BE CONT ON $[a,b]$ DIFF ON (a,b). THEN THERE IS A $c\in(a,b)$ SO THAT

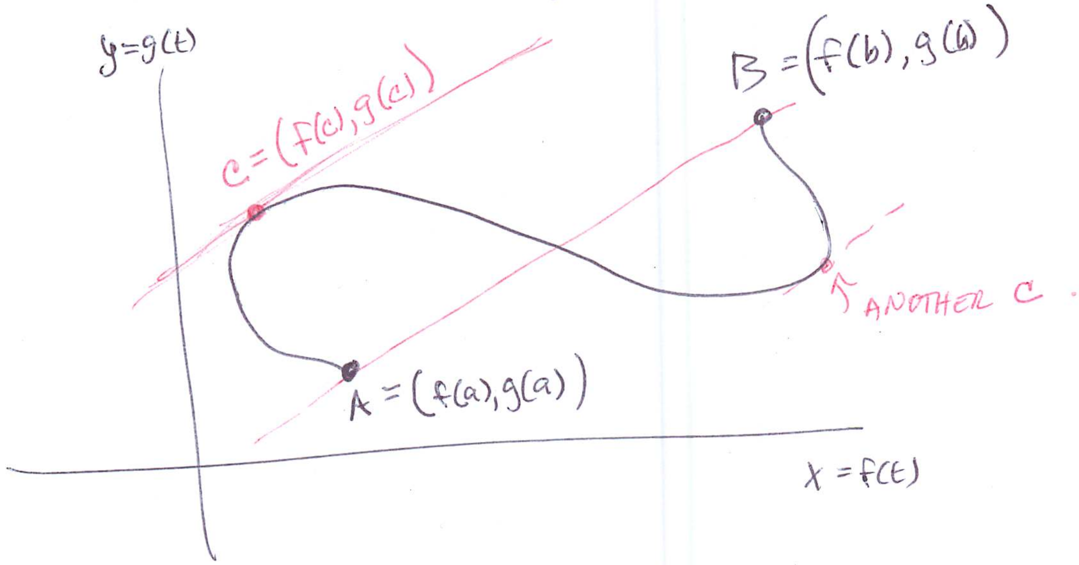
$$[f(b)-f(a)]\,g'(c)=[g(b)-g(a)]\,f'(c).$$

IF $g'(x)\neq 0$ FOR $x\in(a,b)$ THEN

$$\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$$

A GEOMETRIC ~~VERSION~~ INTERPRETATION.

LET $t \in [a,b]$, AND $\gamma(t)$ BE A CURVE $(f(t), g(t))$



PROOF: LET $h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$ FOR $x \in [a,b]$.

NOTE $h(a) = f(b)g(a) - \cancel{f(a)g(a)} - g(b)f(a) + \cancel{g(a)f(a)}$

$= f(b)g(a) - g(b)f(a)$

$= h(b)$.

SO WE CAN APPLY ROLLE'S THM TO GET $c \in (a,b)$ WITH $h'(c) = 0$, i.e.

$$h'(c) = [f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = 0$$

$\boxed{\Xi}$

THM: L'HOSPITAL'S RULE

AS ALWAYS, f, g CONT. ON $[a,b]$, DIFF ON (a,b).

LET $c \in [a,b]$ WITH $f(c) = g(c) = 0$, AND $g'(x) \neq 0$ ON $(a,c) \cup (c,b) = (a,b) - \{c\}$.

IF $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$, THEN $\lim_{x \to c} \frac{f(x)}{g(x)} = L$.

Pf/ LET $\{x_n\}$ SATISFY $x_n \neq c$, $x_n \to c$, $x_n \in (a,b)$. FOR ALL n.

BY THE CAUCHY MVT, FOR EACH x_n WE GET $\{c_n\}$ SO THAT

$$[f(x_n) - f(c)]g'(c_n) = [g(x_n) - g(c)]f'(c_n)$$

FOR EACH n.

SINCE $\underbrace{g'(x) \neq 0 \text{ AND}}_{\text{ON } (a,b) - \{c\}}$ $g(c) = 0$, WE HAVE $g(x_n) \neq 0$ FOR ALL n.

SINCE $f(c) = g(c) = 0$ SO $\frac{f(x_n)}{g(x_n)} = \frac{f(x_n) - f(c)}{g(x_n) - g(c)} = \frac{f'(c_n)}{g'(c_n)}$. $x_n \to c \Rightarrow c_n \to c$

SO WE GET RESULT.

IN THE ~~LATE~~ EARLY TO MID 19TH CENTURY, THERE WAS A LOT OF QUESTION ABOUT HOW NON-DIFFERENTIABLE A CONTINUOUS FUNCTION COULD BE.

IE $|x|$ IS CONTINUOUS BUT NOT DIFF AT $x=0$.

ITS EASY TO MAKE CONT. FUNCTIONS WHICH ARE NON-DIFF AT A FINITE # OF POINTS, BUT HOW NASTY CAN THEY BE?

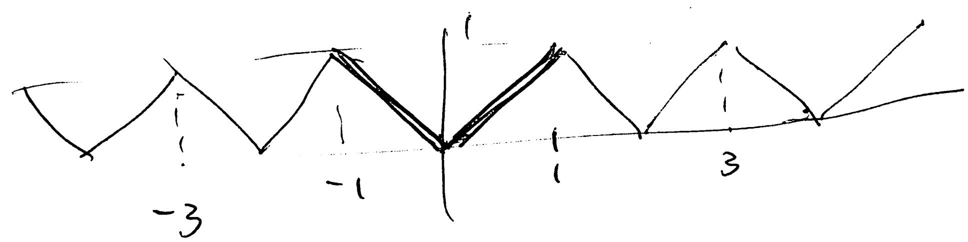
KARL WEIERSTRASS (1872) GAVE AN EXAMPLE OF A FUNCTION WHICH WAS CONTINUOUS BUT <u>NOWHERE</u> DIFFERENTIABLE.

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n x)$$

WITH $0<a<1$, b AN ODD INTEGER SO THAT $ab > 1 + 3\pi/2$

WE CAN DO AN "EASIER" VERSION WITH THE ABS VALUE.

LET $$h_0(x) = \begin{cases} |x| & \text{IF } -1 \le x \le 1 \\ h_0(x+2) = h_0(x) & \text{OTHERWISE} \end{cases}$$



SET $h_n(x) = \frac{1}{2^n} h_0(2^n x)$ FOR $n>0$

LET $g(x) = \sum_{n=0}^{\infty} h_n(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} h_0(2^n x)$

NOTE THAT FOR EACH x, THE SERIES

$$\sum_1^\infty \frac{1}{2^n} h_0(2^n x) \text{ CONV. ABSOLUTELY,}$$

~~SINCE~~ $\cancel{\left|\frac{1}{2^n}h(2^n x)\right| \le \frac{|x|^n}{2^n}}$ ~~FOR~~ $\cancel{|x|<1}$, ~~(AND~~ $\cancel{\ge 1}$ ~~FOR~~ $\cancel{|x|=1}$~~)~~

~~NOTE~~ FOR ANY $|x| \le 1$,

$$\left|\frac{1}{2^n} h(2^n x)\right| \le \frac{|x|^n}{2^n}$$

(...

$\sum_1^\infty h_n(x)$ HAS ∞-MANY VALLEYS.

CAN SHOW IT IS CONTINUOUS BUT NON-DIFF AT ANY ~~POINT~~ DYADIC POINT, IE A RATIONAL W/ DENOM OF FORM 2^k.

NOW IT CANT BE DIFF. BETWEEN $\frac{P_k}{2^k} < x < \frac{P_{m+1}}{2^k}$

~~SINCE THERE IS ANY ... OF D~~

SINCE ANY INTERVAL CONTAINS POINTS OF NON-DIFF..