4/13/22 (B)

LAST TIME, WE DISCUSSED

DEF: LET $g: I \to \mathbb{R}$ WITH I AN INTERVAL.
THEN

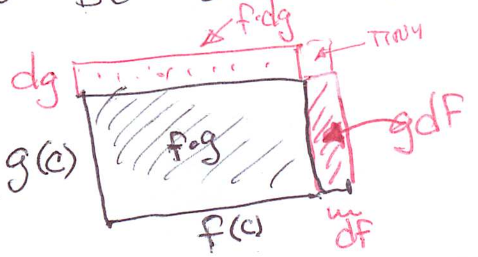
$$\frac{d}{dx} g(x)\Big|_c = g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c} = \lim_{h \to 0} \frac{g(c+h) - g(c)}{h}$$

~~WHICH GIV~~
THIS GIVES THE SLOPE OF THE GRAPH OF $g(x)$ AT THE POINT c.

- WE ALSO COVERED SEVERAL ALGEBRAIC RULES, PRODUCT RULE, QUOTIENT RULE, POWER RULE, ETC.

POWER RULE: $\frac{d}{dx}\left(f(x) \cdot g(x)\right) = f'(x) g(x) + f(x) g'(x)$

CAN BE "SEEN" IN THE FOLLOWING $= f\,dg + g\,df$



i.e. $d(f \cdot g) = f\,dg + g\,df$

CHAIN RULE: $f: A \to \mathbb{R}$, $g: B \to \mathbb{R}$ WITH $f(A) \subseteq B$

SO THAT $g \circ f: A \to \mathbb{R}$ WITH g DIFF AT $f(c)$, f DIFF AT c.

THEN $g \circ f$ IS DIFF AT c WITH

$$(g \circ f)'(c) = f'(c) \cdot g'(f(c))$$

IF $y = f(x)$, $z = g(y) = g(f(x))$ THEN $\frac{dg}{dz} = \frac{dg}{dy} \cdot \frac{dy}{dx}$

PROOF $g'(f(c)) = \lim_{y \to c} \frac{g(y) - f(c)}{y - f(c)}$. WRITE $d(y) = \frac{g(y) - g(f(c))}{y - f(c)}$

$d(f(c))$ IS NOT DEFINED, SO LET $d(f(c)) = g'(f(c))$ [THE LATTER IS DEFINED]

NOW WRITE

$$g(y) - g(f(c)) = d(y)[y - f(c)]$$

THIS IS DEFINED AT ALL $y \in B$,

SET $y = f(t)$ FOR ANY $t \in A$,

AND THEN

$$g(f(t)) - g(f(c)) = d(f(t)) \cdot [f(t) - f(c)]$$

DIVIDE BOTH SIDES BY $t - c$ AND TAKE LIMIT:

$$\lim_{t \to c} \left(\frac{\overbrace{g(f(t))}^{(g\circ f)(t)} - g(f(c))}{t - c} \right) = \lim_{t \to c} d(f(t)) \left[\frac{f(t) - f(c)}{t - c} \right]$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$(g \circ f)'(c) \qquad\qquad d(f(c)) \cdot f'(t)$$

$$= g'(f(c)) \cdot f'(t).$$

$\boxtimes$

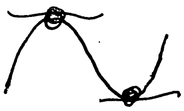
SAYS RATES OF CHANGE MULTIPLY.

THINK OF IT LIKE CONVERSION OF UNITES:

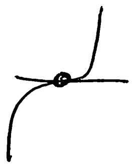
$$12 \frac{\text{MI}}{\text{HR}} = \left(12 \frac{\text{MI}}{\text{HR}}\right)\left(5280 \frac{\text{FT}}{\text{MI}}\right)\left(60 \frac{\text{MIN}}{\text{HR}}\right) = 12 \cdot 5280 \cdot 60 \frac{\text{MI}}{\text{MIN}}.$$

THM (EXTREMA) LET f BE DIFF ON (a,b).

IF f ATTAINS A (MAX OR MIN) AT $c \in (a,b)$ THEN $f'(c) = 0$.



BUT ALSO



SO CONVERSE FAILS.

Pf/ SINCE $c \in (a,b)$, THERE ARE SEQUENCES

$x_n \to c \leftarrow y_n$ WITH $x_n < c < y_n$ FOR ALL $n \in \mathbb{N}$

SINCE $f(c)$ IS A LOCAL MAX, $f(y_n) - f(c) \leq 0$

SO $$\lim_{n\to\infty} \frac{f(y_n) - f(c)}{y_n - c} \leq 0 \quad \left(\frac{-}{+}\right)$$

SIMILARLY $$\lim_{n\to\infty} \frac{f(x_n) - f(c)}{x_n - c} \geq 0 \quad \left(\frac{-}{-}\right)$$

BUT THEN $f'(c) = 0$ ∎

SO THIS ALLOWS US TO FIND MAXIMA/MINIMA BY CALCULUS.

COR (DARBOUX'S THM)

IF f IS DIFF. ON $[a,b]$ AND ~~$f'(a) < \alpha < f'(b)$~~ α IS BETWEEN $f'(a)$ AND $f'(b)$, THEN THERE IS $c \in (a,b)$ SO THAT $f'(c) = \alpha$.

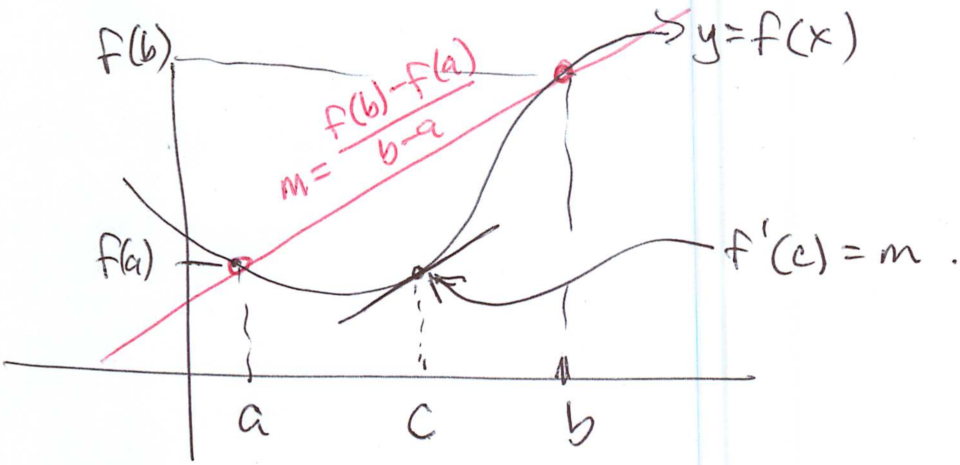
(IN OTHER WORDS, WHILE f' CAN BE UNDEFINED, IT CAN'T JUMP.

IS NOT THE GRAPH OF DERIVATIVE OF ANY FUNCTION,

ALTHOUGH

IS.)

THE MEAN VALUE THEOREM (MVT)



MVT: LET $f: [a,b] \to \mathbb{R}$ BE CONTINUOUS ON $[a,b]$ DIFF ON (a,b).

THEN THERE IS A $c \in (a,b)$ WITH

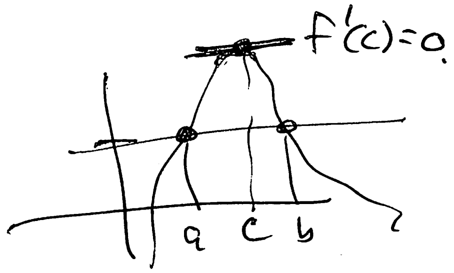
$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

EASIER VERSION

~~[illegible]~~

ROLLE'S THEOREM

LET $f:[a,b] \to \mathbb{R}$ BE CONTINUOUS ON $[a,b]$, DIFF ON (a,b).
IF $f(a)=f(b)$, THEN THERE EXISTS $c \in (a,b)$
WITH $f'(c)=0$



Pf/ f IS CONTINUOUS ON A COMPACT SET, SO HAS BOTH A MAX & MIN IN $[a,b]$.
~~IF $f(a)=f($~~ IF BOTH MAX AND MIN ARE AT THE ENDPOINTS, f IS CONSTANT.
ELSE THERE IS AN INTERIOR EXTREMUM, AT $c \in (a,b)$
SO $f'(c)=0$.

NOW MVT PROOF! ESSENTIALLY, CHANGE TO ROLLE'S THM.

LET $g(x) = f(x) - \left[\left(\frac{f(b)-f(a)}{b-a}\right)(x-a) + f(a)\right]$

NOW g IS CONTINUOUS ON $[a,b]$, DIFF., AND $g(a)=0=g(b)$

SO BY ROLLE'S, THERE IS A $c \in (a,b)$ WITH $g'(c)=0$.

BUT $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$

$0 = g'(c) = f'(x) - \frac{f(b)-f(a)}{b-a}$ $\boxtimes$

Cor Spoze $f : A \to \mathbb{R}$ diff on $\overset{\text{interval}}{A}$ with $f'(x) = 0$ for all $x \in A$. Then f is constant on A.

Pf/ Take $x, y \in A$ with $x < y$. By the MVT, for some $c \in (x,y)$, $\frac{f(y) - f(x)}{y - x} = f'(c)$.

But $f'(c) = 0$, so $f(x) = f(y)$. Since x, y arbitrary $\boxtimes$

Cor Spoze f & g are diff on some interval A with $f'(x) = g'(x)$ for $x \in A$.

Then for all $x \in A$ $f(x) = g(x) + k$ for some constant,

Cauchy's Generalized MVT:

Let f, g be cont on $[a,b]$, diff on (a,b), then there is a $c \in (a,b)$ with

$$\big(f(b) - f(a)\big) g'(c) = \big(g(b) - g(a)\big) f'(c).$$

If $g'(x) \neq 0$ for $x \in (a,b)$ then

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

PROOF APPLY THE MVT TO

$$h(x) = \big(f(b)-f(a)\big)g(x) - \big(g(b)-g(a)\big)f(x).$$

THM L'HÔPITAL'S RULE (0/0 CASE).

f, g CONT ON $[a,b]$, DIFF ON (a,b) EXCEPT MAYBE AT ~~AT~~ $c \in (a,b)$.

IF $f(c) = g(c) = 0$ AND $g'(x) \neq 0$ FOR $x \neq c$,

THEN $\displaystyle\lim_{x\to c} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x\to c} \frac{f(x)}{g(x)} = L$

Pf FOLLOWS FROM CAUCHY'S MVT,

EXTENDS EASILY TO $\frac{\infty}{\infty}$ CASE OR $L = \infty$.