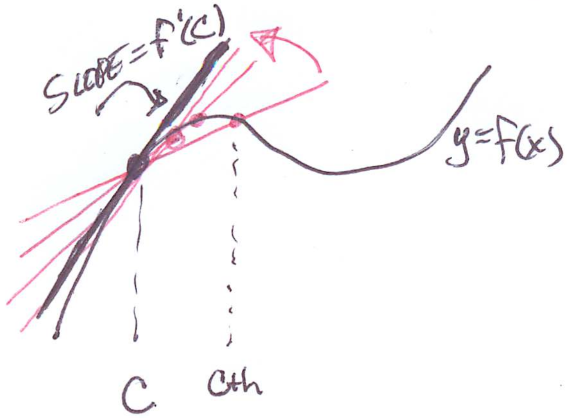
THE DERIVATIVE

MUCH OF THE FOUNDATIONS OF ANALYSIS AROSE FROM AN OUTGROWTH OF 16-17TH CENTURY INVESTIGATIONS (BARROW, DESCARTES, FERMAT, NEWTON, LEIBNITZ) OF NOTIONS OF SLOPE AND AREA.



THE IDEA OF LIMITING SLOPES OF SECANT LINES TENDING TO THE TANGENT LINE GIVES RISE TO

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

(NEWTON DESCRIBED THE DERIVATIVE IN TERMS OF FLUXIONS, LEIBNIZ VIA INFINITESIMALS, BUT NO FIRM FOUNDATION FOR THESE IDEAS EXISTED UNTIL A. ROBINSON IN THE 1960s. INSTEAD, LIMITS AND A FIRMER SETTING FOR $\mathbb{R}$ AROSE).

SOME QUESTIONS:
- HOW "NON DIFFERIANTABLE" CAN A CONTINUOUS FUNCTION BE?
- IF f IS DIFF, MUST IT BE CONT.?
- IF f' EXISTS AT ALL $x \in$ DOMAIN, WHAT CAN WE SAY ABOUT IT?

ETC.

DEF: LET $g: I \to \mathbb{R}$ BE DEFINED ON AN INTERVAL I.

GIVEN $c \in I$, THE DERIVATIVE OF g AT c IS

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c} = \lim_{h \to 0} \frac{g(c+h) - g(c)}{h}$$

PROVIDED THAT THE LIMIT EXISTS.

IF IT DOES, WE SAY <u>g IS DIFFERENTIABLE AT c</u>

AND IF g' EXISTS FOR ALL $c \in A$, IT IS <u>DIFFERENTIABLE ON A</u>.

EASY EXAMPLE: $f(x) = x^n$.

- NOTE $x^n - c^n = (x-c)(x^{n-1} + cx^{n-2} + c^2x^{n-3} + \dots + c^{n-1})$
  (DO LONG DIVISION)
  SO $f'(c) = \lim_{x \to c} \frac{x^n - c^n}{x - c} = \lim_{x \to c} (x^{n-1} + cx^{n-2} + c^2x^{n-3} + \dots + c^{n-1})$
  $= nc^{n-1}$.

- $g(x) = |x|$.

$$g'(c) = \begin{cases} 1 & \text{IF } c > 0 \\ -1 & \text{IF } c < 0 \\ \text{DNE} & \text{IF } c = 0 \end{cases}$$

THM: IF $g: I \to \mathbb{R}$ IS DIFFERENTIABLE AT $c \in I$, THEN g IS CONTINUOUS AT c.

Pf/ $$g'(c) = \lim \frac{g(x) - g(c)}{x - c}.$$

SO $$\lim_{x \to c} \big(g(x) - g(c)\big) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \cdot (x - c)$$
$$= g'(c) \cdot 0 = 0.$$

IE $\lim_{x \to c} g(x) = g(c)$ AND g IS CONT. AT c $\blacksquare$

THM ALGEBRA LET f, g DEFINED ON I, DIFF AT $c \in I$. THEN

(i) $(f+g)'(c) = f'(c) + g'(c)$

(ii) FOR ANY $k \in \mathbb{R}$, $(kf)'(c) = k \cdot f'(c)$

(iii) $(f \cdot g)'(c) = f'(c)g(c) + f(c)g'(c)$

(iv) $$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}$$

(i) & (ii) FOLLOW IMMEDIATELY FROM THE LIMITS.

(iii): WRITE $$\frac{(fg)(x) - (fg)(c)}{x - c} = \frac{f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)}{x - c}$$
$$= f(x)\left(\frac{g(x) - g(c)}{x - c}\right) + \left(\frac{f(x) - f(c)}{x - c}\right)g(c)$$ AND TAKE LIMIT

THM THE CHAIN RULE

LET $f: A \to \mathbb{R}$, $g: B \to \mathbb{R}$ SATISFY $f(A) \subseteq B$ SO THAT $(g \circ f): A \to \mathbb{R}$.

IF f IS DIFF AT c, g DIFF AT $f(c)$, THEN $g \circ f$ IS DIFF AT c WITH $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$

PF/

$$g'(f(c)) = \lim_{y \to c} \boxed{\frac{g(y) - g(f(c))}{y - f(c)}} = \lim_{y \to c} d(y)$$

~~LET $d(y) \ldots g(\ldots) \ldots$~~

JUST DECLARE $d(f(c)) = g'(f(c))$ [IT ISN'T DEFINED YET]

NOW WRITE

$$g(y) - g(f(c)) = d(y) \cdot [y - f(c)]$$

(JUST REWRITE DEF OF $d(y)$)

AND THIS HOLDS FOR ALL $y \in B$, EVEN AT $f(c)$

NOW LET $y = f(t)$ FOR ARB. t.

$$\frac{g(f(t)) - g(f(c))}{t - c} = \frac{d(f(t)) \cdot [f(t) - f(c)]}{t - c}$$

NOW DIVIDE BY $t-c$

AND TAKE THE LIMIT AS $t \to c$