- LAST TIME, SEVERAL EQUIVALENT DEFINITIONS OF THE LIMIT OF A FUNCTION $f: A \to \mathbb{R}$.

ONE VERY USEFUL PURPOSE HERE IS TO ~~DEF~~ MAKE CAREFUL SENSE OF WHAT IT MEANS FOR A FUNCTION TO BE CONTINUOUS

<u>DEF</u>: LET $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, ~~THEN~~ AND $c \in A$ THEN IF c IS A LIMIT POINT OF A, <u>f IS CONTINUOUS AT c</u> IF

$$\lim_{x \to c} f(x) = f(c).$$

(IF c IS AN ISOLATED POINT OF A, IT IS AUTOMATIC)

IF f IS CONTINUOUS AT EVERY POINT OF A, THEN <u>f IS CONTINUOUS ON A</u> (OR JUST "CONTINUOUS" IF A IS APPARENT)

~~NOTE THAT F~~ FROM THE PROPERTIES OF LIMITS, WE IMMEDIATELY HAVE

<u>THM</u>: SPOZE $f: A \to \mathbb{R}$, $g: A \to \mathbb{R}$ ARE BOTH CONTINUOUS AT $c \in A$. THEN,

- $\forall k \in \mathbb{R}$, $kf(x)$ IS CONTINUOUS AT c
- $f(x) + g(x)$ IS CONT AT c
- $f(x) \cdot g(x)$ " " " "
- $f(x)/g(x)$ " " " " " PROVIDED $f(x)/g(x)$ IS DEFINED

COR

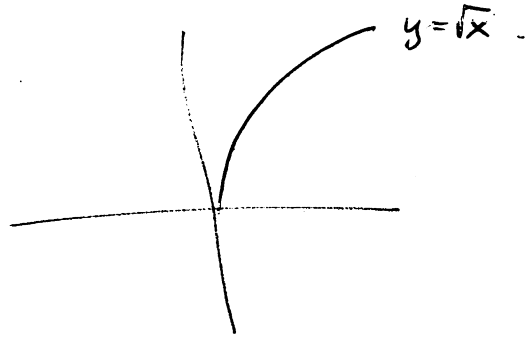
ALL POLYNOMIALS, AND RATIONAL FUNCTIONS ARE CONTINUOUS ON THEIR DOMAINS.

(SO, BE CAREFUL: $f(x) = \frac{x}{x^2-1}$ IS CONTINUOUS ON $(-\infty,-1) \cup (-1,1) \cup (1,\infty)$.

BUT NOT ON $\mathbb{R}$)

EX

LETS SHOW $f(x) = \sqrt{x}$ IS CONTINUOUS ON $\{x \mid x \geq 0\}$



LET $c \geq 0$, AND FIX $\varepsilon > 0$.

WE NEED TO SHOW THAT ~~BY CHOOSING~~ WE CAN MAKE

$|f(x) - f(c)| < \varepsilon$ FOR ALL $x \in (c-\delta, c+\delta)$ (AND $x \geq 0$)

- NOTE THAT IF $c = 0$, $\sqrt{0} = 0$, SO

  $|f(x) - f(c)| = \sqrt{x} < \varepsilon$, IE $x < \varepsilon^2$, SO LET $\delta = \varepsilon^2$

- FOR $c > 0$, WE NEED TO ESTIMATE

  $|\sqrt{x} - \sqrt{c}|$.

BUT

$$|\sqrt{x} - \sqrt{c}| = |\sqrt{x} - \sqrt{c}|\left(\frac{\sqrt{x}+\sqrt{c}}{\sqrt{x}+\sqrt{c}}\right) = \frac{|x-c|}{\sqrt{x}+\sqrt{c}} \leq \frac{|x-c|}{\sqrt{c}}$$

SO IF $|x-c| < \varepsilon$, WE CAN TAKE $\delta = \varepsilon\sqrt{c}$.

SO $|x-c| < \delta \Rightarrow |\sqrt{x} - \sqrt{c}| < \frac{\varepsilon\sqrt{c}}{\sqrt{c}} = \varepsilon$ ☺

Thm: Spoze $f: A \to \mathbb{R}$, $g: B \to \mathbb{R}$ and $f(A) \subseteq B$.

$\boxed{\text{If } f \text{ is continuous at } c \in A \text{ and } g \text{ is coninuous at } f(c) \in B, \text{ then } g \circ f \text{ is continuous at } c}$

Pf/ HW.

Recall we can describe $A \subseteq \mathbb{R}$ as open, bounded, closed, compact, perfect, connected....

Which (if any) are preserved by continuous functions?

ie: If A is open, is $f(A)$ open?

(No: consider ~~$f: \mathbb{R} \to \mathbb{R}$~~ $f(\mathbb{R})$ where $f(x) = x^2$)

Closed?

No! Let $g(x) = \dfrac{1}{1+x^2}$ on ~~$[0,\infty)$~~ $\mathbb{R}$

~~$g: [0,\infty) \to (0,1]$~~ $\quad g(\mathbb{R}) = (0,1]$

COMPACT? YES!

THM/ LET $f: A \to \mathbb{R}$ BE CONTINUOUS ON A.

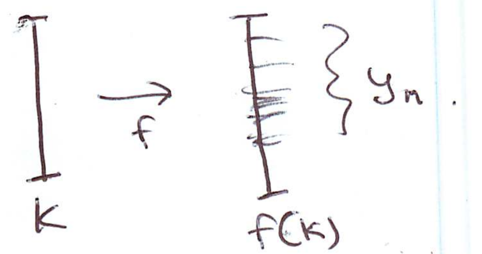
SPOZE $K \subseteq A$ IS COMPACT. THEN $f(K)$ IS COMPA

PF/ LET $\{y_n\}$ SATISFY $y_n \in f(K)$

~~WE NEED TO SHOW THAT~~

WE NEED TO FIND $\{y_{n_k}\}$ SO $y_{n_k} \to L \in f(K)$

TO DO SO, WE "PULL BACK" y_{n_k} TO K.



FOR EACH $y_n \in f(K)$, THERE IS ATLEAST ONE X_n

NOW, WE HAVE $\{x_n\}$ WITH $X_n \in K$.

SINCE K COMPACT, THIS GIVES $X_{n_k} \to a$

BUT THEN

$\cancel{\lim_{x \to a} f(x) = \lim_{n \to \infty}}$

$\lim f(x_{n_k}) = f(a)$ BY CONTINUITY OF f.

$\parallel$

$\lim y_{n_k}$ IE $f(K)$ IS COMPACT.