LAST TIME:

DEF: LET $f: A \to \mathbb{R}$ AND $c \in \bar{A}$ (IE, c IS A LIMIT POINT OF A)

WE SAY $\lim_{x \to c} f(x) = L$

IF <u>EVERY</u> SEQUENCE $\{x_n\}$ WITH $x_n \in A$ AND $\lim x_n \to c$

GIVES US $\lim f(x_n) \to L$

HARD TO CHECK EVERY SEQUENCE, SO LETS HAVE A THINK.

ASSUMING $x_n \to c$, WE WANT $f(x_n) \to L$.

THIS LAST MEANS FOR EVERY $\varepsilon > 0$, $|f(x_n) - L| < \varepsilon$ WHEN n IS LARGE ENOUGH.

BUT WE ALSO WANT $x_n \to c$, SO THAT MEANS ~~THE~~ GIVEN $\varepsilon > 0$, WE CAN MAKE x_n CLOSE TO c. BUT THE CLOSENESS MIGHT GO FASTER OR SLOWER THAN $f(x_n)$ TO L, SO THERE IS A δ WITH $0 < |x_n - c| < \delta$ (ADD "<0" BECAUSE WE DON'T WANT $x_n = c$).

NOTE THAT STATED THIS WAY, WE ACTUALLY HAVE AN INTERVAL INSTEAD OF A SEQUENCE.

THAT IS,

DEF LET $f : A \to \mathbb{R}$, $c \in \bar{A}$. THEN

$$\lim_{x \to c} f(x) = L \quad \text{WHENEVER}$$

FOR EVERY $\varepsilon > 0$, THERE IS A δ SO THAT

$$0 < |x - c| < \delta \ (\text{WITH } x \in A) \Rightarrow |f(x) - L| < \varepsilon.$$

STATED IN TERMS OF NEIGHBORHOODS,

DEF LET $f : A \to \mathbb{R}$, $c \in \bar{A}$ THEN

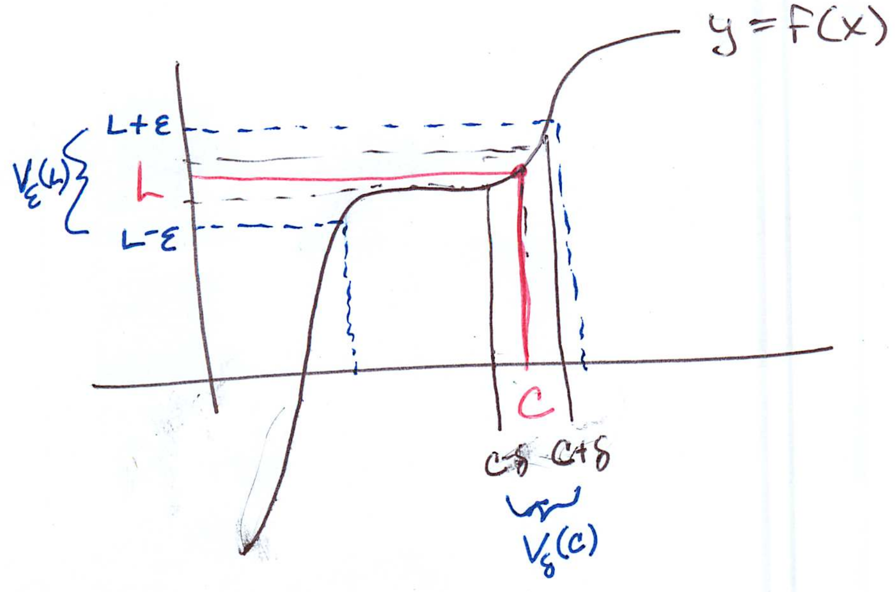
$$\lim_{x \to c} f(x) = L \quad \text{WHENEVER}$$

FOR EVERY NEIGHBORHOOD $V_\varepsilon(L)$, THERE IS A NEIGHBORHOOD $V_\delta(c)$ SO THAT

$$x \in V_\delta(c) \quad (\text{WITH } x \neq c, \ x \in A)$$

WE HAVE $f(x) \in V_\varepsilon(L)$

THESE ARE ALL EQUIVALENT. (THESE LAST TWO ARE OBVIOUS; EQUIVALENCE OF THE FIRST NEEDS A BIT OF WORK, BUT NOT MUCH.



EASY EXAMPLE $\quad f(x) = 2x+1$, SHOW $\lim_{x\to 3} f(x) = 7$

TAKE $\varepsilon > 0$, FIND δ SO THAT $0 < |x-3| < \delta \Rightarrow |f(x)-7| < \varepsilon$

NOTE $|f(x)-7| = |(2x+1)-7| = |2x-6| = 2|x-3|$.

ANY $\delta \le \varepsilon/2$ DOES THE JOB:

FIX $\varepsilon > 0$. IF $0 < |x-3| < \delta = \varepsilon/2$, THEN

$$|f(x)-L| = 2|x-3| < 2\delta = \varepsilon. \quad \boxed{\text{ß}}$$

- A BIT HARDER. ~~SHOW~~ $g(x) = x^2$, SHOW $\lim_{x\to 2} x^2 = 4$.

$|\overset{x^2}{g(x)} - 4| = |(x-2)(x+2)| = |x-2||x+2|$. HOW BIG CAN $x+2$ GET IF ~~FR~~ $|x-2|$ SMALL?

LETS AGREE $\delta < 1$ (WE JUST WANT IT SMALL) SO $1 < x < 3$.

SO ~~$|x+2||x+2|$~~ $|x+2| \le 5$ IF $\delta < 1$.

SO CHOOSE $\delta \le \min\{\varepsilon/5, 1\}$.

(SKIP IN CLASS)

TO SEE THE NEIGHBORHOOD DEF $\Leftrightarrow$ SEQ DEF.

ASSUME $\lim_{x \to c} f(x) = L$ BY NBHD DEF, IE IF $x \in V_\delta(c)$ $(x \neq c)$ THEN $f(x) \in V_\varepsilon(L)$

SO SPOZE HAVE ANY SEQ $\{x_n\} \to c$ WITH $x_n \neq c$, WANT TO SHOW $f(x_n) \to L$.

SINCE $x_n \to c$, THERE IS SOME NEIGHBORHOOD OF c CONTAINING ALL x_n FOR n LARGE. CALL IT $V_\delta(c)$. BUT BY ASSUMPTION THAT MEANS THERE IS A $V_\varepsilon(L)$ SO THAT $f(x) \in V_\delta(L)$ FOR ALL SUCH $x \in V_\delta(c)$ $x \neq c$.
IE $f(x_n) \to L$.

FOR THE OTHER DIRECTION, SUPPOSE THE CONTRARY. THAT IS, WE SUPPOSE FOR SOME $\varepsilon_0 > 0$, THERE IS NO $\delta > 0$ WITH THE REQUIRED PROPERTY, IE THERE IS AT LEAST ONE $x \in V_\delta(c)$ WITH $x \neq c$ AND $f(x) \notin V_\varepsilon(L)$.

TAKE A SEQUENCE OF $\delta \to 0$, IE $\delta = \frac{1}{n}$. FOR EACH $\delta = \frac{1}{n}$, THERE IS AN $x_n \neq c$ WITH $f(x_n) \notin V_{\varepsilon_0}(L)$.

BUT NOW WE HAVE A SEQ $\{x_n\} \to c$

BUT $f(x_n) \not\to L$.

ARITHMETIC ~~ALGEBRA~~ OF LIMITS WORKS AS EXPECTED.

ie ASSUME $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$ FOR SOME $c \in \bar{A}$

THEN

- $\forall k \in \mathbb{R}$, $\lim_{x\to c}(kf(x)) = kL$
- $\lim_{x\to c}(f(x) \pm g(x)) = L \pm M$
- $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$
- $\lim_{x\to c} f(x)/g(x) = L/M$ PROVIDED $M \neq 0$.

. THIS JUST FOLLOWS FROM ALGEBRA OF SEQUENCES EARLIER.

DIVERGENCE: LET $f: A \to \mathbb{R}$, $c \in \bar{A}$

SPOZE $x_n \in A$, $y_n \in A$ WITH $x_n \neq c$, $y_n \neq c$

AND $\lim x_n = c = \lim y_n$

BUT $\lim f(x_n) \neq \lim f(y_n)$.

THEN $\lim_{x\to c} f(x)$ DOES NOT EXIST.

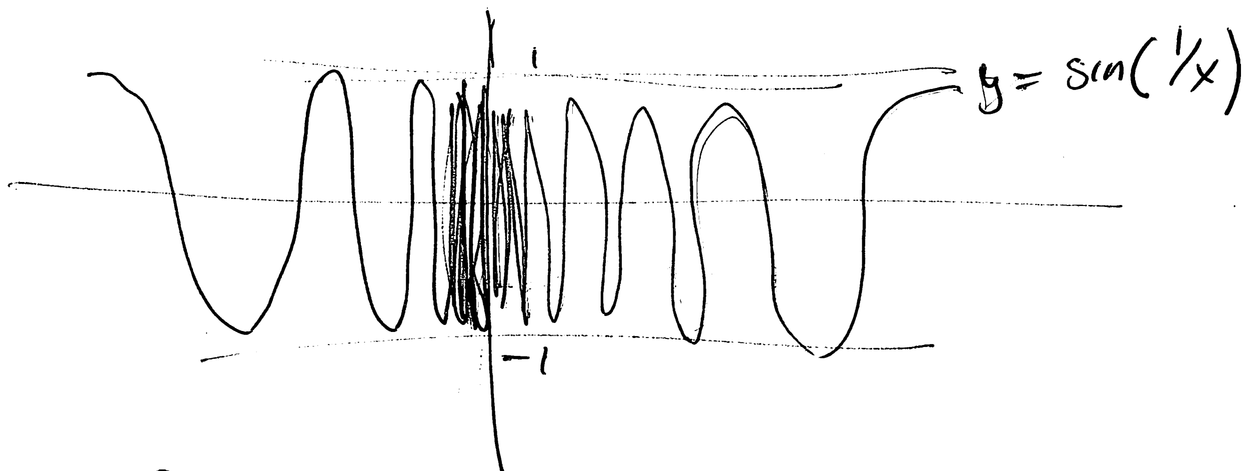
EXAMPLE: LET $f(x) = \sin(1/x)$ $\quad x \neq 0$.

TAKE $x_n = \frac{1}{2n\pi}$ $\quad y_n = \frac{1}{2n\pi + \pi/2}$

NOTE $\lim x_n = \lim y_n = 0$.

BUT $\sin\left(\frac{1}{2\pi n}\right) = 0$ FOR ALL $n \in \mathbb{N}$

$\sin\left(\frac{1}{2n\pi + \pi/2}\right) = 1$ FOR ALL $n \in \mathbb{N}$.



SO LIMIT DNE.

DEF: $\lim_{x \to c} f(x) = +\infty$ MEANS FOR ALL $M > 0$ WE CAN FIND $\delta > 0$ SO THAT $0 < |x - c| < \delta \Rightarrow f(x) > M$

$\lim_{x \to +\infty} f(x) = L$ MEANS FOR ALL $\varepsilon > 0$ THERE IS M SO THAT $x > M \Rightarrow |f(x) - L| < \varepsilon$

CONTINUITY

$f : A \to \mathbb{R}$ is <u>continuous</u> at $c \in A$ ← NOT CLOSURE!

IF $\lim_{x \to c} f(x) = f(c)$ (IF c IS A LIMIT POINT OF A)

IE, FOR ALL $\varepsilon > 0$, THERE IS $\delta > 0$ SO THAT

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon.$$

(AND $x \in A$)

IF c IS ISOLATED, THIS STILL WORKS, BUT $\lim_{x \to c} f(x)$ ISN'T DEFINED. SO WE JUST SAY $f(x)$ IS CONTINUOUS AT ISOLATED POINTS.

DEF IF $f(x)$ IS CONTINUOUS AT EVERY POINT OF A, THEN SAY "$f(x)$ IS CONTINUOUS ON A"

<u>EX:</u>

$$f(x) = \begin{cases} x \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

IS CONTINUOUS ON $\mathbb{R}$.

THOMAE'S FUNCTION (1875)

$$t(x) = \begin{cases} 1 & \text{IF } x=0 \\ 1/n & \text{IF } x = m/n \text{ IN LEAST TERMS AND } m \neq 0. \ (\text{IE } x \in \mathbb{Q} \setminus \{0\}. \\ 0 & \text{IF } x \notin \mathbb{Q}. \end{cases}$$

THIS IS CONTINUOUS AT ANY $x \notin \mathbb{Q}$,
BUT DISCONTINUOUS AT ALL $x \in \mathbb{Q}$!